Some properties of extensions of loops

The paper is a kind of sequel to the paper [3]. In the paper we shall analyse some aspects of the construction of extensions of loops given in [3]. Moreover, we shall consider this construction in the case when loops are groups.

Definitions of quasigroup, loop, subloop, normal subloop, coset, quotient loop are used according to Bruck [2].

DEFINITION 1. (cf. [3]). A loop $\Sigma$ is said to be an extension of a loop $K$ by a loop $L$ if the following conditions hold:

(i) $K$ is a normal subloop of the loop $\Sigma$,

(ii) the quotient loop $\Sigma/K$ and the loop $L$ are isomorphic.

Let $K$ be a normal subloop of a loop $\Sigma$. A mapping $s: \Sigma/K \to \Sigma$ is called a selector if it satisfies the following condition:
\[ \bigwedge_{M \in \sum/K} s(M) \in M. \]

Let \( L \) and \( K \) be loops. Let \( f, g : L \to K \) be arbitrary mappings. By a product \( fg \) of the mappings \( f \) and \( g \) we mean a mapping \( fg : L \to K \) defined as follows:

\[(fg)(l) = f(l)g(l)\]

for \( l \in L \).

**Theorem 1.** Let \( \sum \) and \( L \) be loops. Let \( K \) be a normal subloop of the loop \( \sum \). The loop \( \sum \) is an extension of the loop \( K \) by the loop \( L \) if and only if there exists a mapping \( \sigma : L \to \sum \) fulfilling the following conditions:

\[(w_1) \bigwedge_{x \in \sum} \bigvee l, k \in L \times K x = \sigma(l)k,\]
\[(w_2) \bigwedge_{l_1, l_2 \in L} [\sigma(l_1 l_2)K = (\sigma(l_1)K)(\sigma(l_2)K)].\]

**Proof.** If the loop \( \sum \) is an extension of the loop \( K \) by the loop \( L \), then the mapping \( \sigma = s \circ f \), where \( s : \sum/K \to \sum \) is a selector and \( f : L \to \sum/K \) is an isomorphism, satisfies conditions \((w_1)\) and \((w_2)\) (cf. [3]).

Let a mapping \( \sigma : L \to \sum \) satisfies conditions \((w_1)\) and \((w_2)\). We define a mapping \( f : L \to \sum/K \) as follows:

\[ f(l) = \sigma(l)K \]

for \( l \in L \).

As an easy consequence of conditions \((w_1)\) and \((w_2)\) we obtain that the mapping \( f \) is an isomorphism.

Then the loop \( \sum \) is an extension of the loop \( K \) by the loop \( L \).
THEOREM 2. Let a loop $\Sigma$ be an extension of a loop $K$ by a loop $L$. A mapping $\sigma : L \rightarrow \Sigma$ satisfies conditions $(w_1)$ and $(w_2)$ if and only if $\sigma = s \circ f$, where $s : \Sigma/K \rightarrow \Sigma$ is a selector and $f : L \rightarrow \Sigma/K$ is an isomorphism.

Proof. Let $\sigma : L \rightarrow \Sigma$ satisfies conditions $(w_1)$ and $(w_2)$. We define a mapping $f : L \rightarrow \Sigma/K$ as follows:

$$f(l) = \sigma(l)K$$

for $l \in L$. The mapping $f$ is an isomorphism of the loops $L$ and $\Sigma/K$. Define a selector $s : \Sigma/K \rightarrow \Sigma$ by the rule:

$$s(f(l)) = \sigma(l)$$

for $l \in L$. Hence $\sigma = s \circ f$.

If $\sigma = s \circ f$, where $s : \Sigma/K \rightarrow \Sigma$ is a selector and $f : L \rightarrow \Sigma/K$ is an isomorphism, then conditions $(w_1)$ and $(w_2)$ are fulfilled (cf. [3]).

LEMMA 1. Let a loop $\Sigma$ be an extension of a loop $K$ by a loop $L$. Let $\sigma_1 = s_1 \circ f_1$, where $s_1 : \Sigma/K \rightarrow \Sigma$ is a selector and $f_1 : L \rightarrow \Sigma/K$ is an isomorphism. Then for an arbitrary automorphism $\chi \in \text{Aut}(L)$ and an arbitrary mapping $\delta : L \rightarrow K$ the mapping $\sigma = (\sigma_1 \circ \chi)\delta$ satisfies conditions $(w_1)$ and $(w_2)$.

Proof. At first, we shall prove that the mapping $\sigma$ we can represent in the form $\sigma = s \circ f$, where $s : \Sigma/K \rightarrow \Sigma$ is a selector and $f : L \rightarrow \Sigma/K$ is an isomorphism.

The mapping $\sigma$ can be written as $\sigma = (s_1 \circ f_1 \circ \chi)\delta$. Put $f = f_1 \circ \chi$. Of course, $f : L \rightarrow \Sigma/K$ is an isomorphism and $\sigma = (s_1 \circ f)\delta$.  

A mapping $s: \Sigma/K \rightarrow \Sigma$ defined by the following rule:

$$s(f(l)) = s_1(f(l)) \delta(l)$$

for $l \in L$ is a selector.

Thus $s_1(l) = s_1(f(l)) \delta(l) = s(f(l)) = (s \circ f)(l)$ for an arbitrary $l \in L$, hence $s = s \circ f$. Applying Theorem 2 we get that the mapping $s$ satisfies conditions $(w_1)$ and $(w_2)$.

**THEOREM 3.** Let a loop $\Sigma$ be an extension of a loop $K$ by a loop $L$. Let $s_1 = s_1 \circ f_1$, where $s_1: \Sigma/K \rightarrow \Sigma$ is a selector and $f_1: L \rightarrow \Sigma/K$ is an isomorphism. Let $s: L \rightarrow \Sigma$ be an arbitrary mapping. The mapping $s$ satisfies conditions $(w_1)$ and $(w_2)$ if and only if there exist an automorphism $\alpha \in \text{Aut}(L)$ and a mapping $\delta: L \rightarrow K$ such that $s = (s_1 \circ \alpha) \delta$.

**Proof.** If a mapping $s: L \rightarrow \Sigma$ satisfies conditions $(w_1)$ and $(w_2)$, then according to Theorem 2 $s = s \circ f$, where $s: \Sigma/K \rightarrow \Sigma$ is a selector and $f: L \rightarrow \Sigma/K$ is an isomorphism. Notice that the mapping

$$(1) \quad \alpha = f_1^{-1} \circ f$$

is an automorphism of the loop $L$. And so we have $f = f_1 \circ \alpha$ and $s = s \circ f_1 \circ \alpha$. Moreover,

$$(2) \quad s((f_1 \circ \alpha)(l))K = s_1((f_1 \circ \alpha)(l))K$$

for $l \in L$. Using equality (2) one can define a mapping $\delta: L \rightarrow K$ as follows:

$$(3) \quad s((f_1 \circ \alpha)(l)) = s_1((f_1 \circ \alpha)(l)) \delta(l)$$

for $l \in L$. Since $s = s \circ f_1 \circ \alpha$ and so $\delta(l) =$

$= (s \circ f_1 \circ \alpha)(l) = ((s_1 \circ f_1 \circ \alpha)(l)) \delta(l) = (s_1 \circ \alpha)(l) \delta(l)$

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for \( l \in L \). Then \( \sigma = (\sigma_1 \circ \kappa) \delta \), where \( \kappa \in \text{Aut}(L) \) is an automorphism defined by rule (1) and \( \delta : L \to K \) is a mapping defined by formula (3).

If there exist an automorphism \( \kappa \in \text{Aut}(L) \) and a mapping \( \delta : L \to K \) such that \( \sigma = (\sigma_1 \circ \kappa) \delta \), then from Lemma 1 we get that \( \sigma \) satisfies conditions \( (w_1) \) and \( (w_2) \).

For ease of reference we now write the following definition (cf. [3]).

Let \( L \) and \( K \) be loops. Let \( \psi : L \times K \times L \times K \to K \) be any mapping fulfilling the following conditions:

1° \( \psi(l_1, k_1, 1, 1) = \psi(1, 1, 1, k) = k \),
2° \( \psi(1, k_1, 1, k_2) = k_1 k_2 \),
3° the mapping \( \psi(l_1, k_1, l_2, \cdot) : K \to K \) is a bijection,
4° the mapping \( \psi(l_1, \cdot, l_2, k_2) : K \to K \) is a bijection,
for \( 1, l_1, l_2 \in L \) and \( k, k_1, k_2 \in K \).

DEFINITION 2. An algebraic structure \((L \times K, \circ)\) with an operation \( \circ \) defined by the formula:

\[ \langle l_1, k_1 \rangle \circ \langle l_2, k_2 \rangle = \langle l_1 l_2, \psi(l_1, k_1, l_2, k_2) \rangle \]

for arbitrary pairs \( \langle l_1, k_1 \rangle, \langle l_2, k_2 \rangle \in L \times K \) is called a product \( \langle L \times K, \circ \rangle \).

A product \( \langle L \times K, \circ \rangle \) is a loop (cf. [3]).

Let a loop \( \Sigma \) be an extension of a loop \( K \) by a loop \( L \).

Let \( s, s_1 : \Sigma / K \to \Sigma \) be selectors such that \( s(K) = s_1(K) = 1 \) and let \( f, f_1 : L \to \Sigma / K \) be isomorphisms.

We define mappings \( \sigma, \sigma_1 : L \to \Sigma \) in the following way:

\[ \sigma = s \circ f \quad \text{and} \quad \sigma_1 = s_1 \circ f_1. \]
By means of the mappings \( \sigma \) and \( \sigma_1 \) we define mappings 
\[ \varphi, \varphi_1 : L \times K \times L \times K \to K \] 
by the formulas:
\[
(\sigma(l_1)k_1)(\sigma(l_2)k_2) = \sigma(l_1l_2) \varphi(l_1,k_1,l_2,k_2),
\]
\[
(\sigma_1(l_1)k_1)(\sigma_1(l_2)k_2) = \sigma_1(l_1l_2) \varphi_1(l_1,k_1,l_2,k_2)
\]
for \( l_1, l_2 \in L \) and \( k_1, k_2 \in K \).

The loops \( \langle L ; K \rangle \varphi \) and \( \langle L ; K \rangle \varphi_1 \) are extensions of the loop \( K^\# = \{ \langle 1 ; k \rangle : k \in K \} \) by the loop \( L \) (cf. [3]).

The extensions \( \langle L ; K \rangle \varphi \) and \( \langle L ; K \rangle \varphi_1 \) are isomorphic. Indeed, mappings \( F : \Sigma \to \langle L ; K \rangle \varphi \) and \( F_1 : \Sigma \to \langle L ; K \rangle \varphi_1 \) defined by the rules:
\[
F(x) = F(\sigma(l)k) = \langle l, k \rangle,
\]
\[
F_1(x) = F_1(\sigma_1(l_1)k_1) = \langle l_1, k_1 \rangle,
\]
for an arbitrary \( x = \sigma(l)k = \sigma_1(l_1)k_1 \in \Sigma \) are isomorphisms (cf. [3]). Then the mapping \( \psi = F_1 \circ F^{-1} \) is an isomorphism of the extensions \( \langle L ; K \rangle \varphi \) and \( \langle L ; K \rangle \varphi_1 \).

According to Theorem 2 the mappings \( \sigma \) and \( \sigma_1 \) satisfy conditions \( (w_1) \) and \( (w_2) \). It follows from Theorem 3 that \( \sigma = (\sigma_1 \circ \chi) \delta \) for some \( \chi \in \text{Aut}(L) \) and mapping \( \delta : L \to K \).

In the quotient loops \( \langle L ; K \rangle \varphi / K^\# \) and \( \langle L ; K \rangle \varphi_1 / K^\# \) all cosets have the same form \( \{1\} \times K \) for every \( l \in L \). It is easy to see that the loops \( \langle L ; K \rangle \varphi / K^\# \) and \( \langle L ; K \rangle \varphi_1 / K^\# \) are identical.

A mapping \( g : L \to \langle L, K \rangle \varphi / K^\# \) defined as follows:
\[
g(l) = \{1\} \times K
\]
for \( l \in L \) is an isomorphism (cf. [3]). By means of the mapping \( g \) and the automorphism \( \chi \in \text{Aut}(L) \) we define an
isomorphism \( g_1 : L \rightarrow \langle L, K \rangle^{/K} \) putting \( g_1 = g \circ \chi \).

Notice that \( g(1) = \{\langle 1, k \rangle : k \in K\} \) and \( g_1(1) = \{\langle 1_1, k \rangle : 1_1 = \chi(1) \wedge k \in K\} \) for every \( 1 \in L \). We shall prove that \( \psi(g(1)) = g_1(1) \) for \( 1 \in L \). If \( \langle 1, k \rangle \in g(1) \) then \( \psi(\langle 1, k \rangle) = F_1(F^{-1}(\langle 1, k \rangle)) = F_1(\xi(1)k) \). Since \( \xi(1) = (\xi_1 \circ \chi)(1) \xi(1) = \xi_1(1_1) \xi(1) \), where \( 1_1 = \chi(1) \) and so \( \psi(\langle 1, k \rangle) = F_1(\xi(1)k) = F_1((\xi_1(1_1) \xi(1))k) = F_1(\xi_1(1_1) \xi(1)) \circ F_1(k) = \langle 1_1, \xi(1) \rangle \circ \langle 1, k \rangle = \langle 1_1, g_1(1_1, \xi(1), 1, k) \rangle \in g_1(1) \).

Since the mappings \( \psi, g, g_1 \) are isomorphisms, then the inclusion \( \psi(g(1)) \subseteq g_1(1) \) implies the equality \( \psi(g(1)) = g_1(1) \) for every \( 1 \in L \). Notice that \( \psi(\langle 1, k \rangle) = F_1(F^{-1}(\langle 1, k \rangle)) = F_1(\xi(1)k) = F_1(\xi_1(1)k) = \langle 1, k \rangle \) for every \( k \in K \).

In the group theory is known the following definition of an extension of groups (cf. [4]).

**DEFINITION 3.** A group \( \Sigma \) is said to be an extension of a group \( K \) by a group \( L \) if the following conditions hold:

(i) \( K \) is a normal subgroup of the group \( \Sigma \),
(ii) the quotient group \( \Sigma/K \) and the group \( L \) are isomorphic.

Let a group \( \Sigma \) be an extension of a group \( K \) by a group \( L \). Let \( s: \Sigma/K \rightarrow \Sigma \) be a selector such that \( s(K) = 1 \) and let \( f: L \rightarrow \Sigma/K \) be an isomorphism. We define a mapping \( \xi : L \rightarrow \Sigma \) as follows:

\[ \xi = s \circ f. \]

Let \( \psi : L \times K \times L \times K \rightarrow K \) be a mapping defined by the
following rule:

\[(\sigma(l_1)k_1)(\sigma(l_2)k_2) = \sigma(l_1l_2)\varphi(l_1,k_1,l_2,k_2)\]

for arbitrary \(l_1,l_2 \in L\) and \(k_1,k_2 \in K\).

We shall prove that the mapping \(\varphi\) has the form:

\[\varphi(l_1,k_1,l_2,k_2) = \lambda(l_1,l_2)\mu(k_1,l_2)k_2\]

for arbitrary \(l_1,l_2 \in L\) and \(k_1,k_2 \in K\), where mappings

\[\lambda: L \times L \rightarrow K\]

and \(\mu: K \times L \rightarrow K\) satisfy the following conditions:

(a) \(\lambda(l,1) = \lambda(1,l) = 1\),

(b) \(\mu(k,1) = k\),

(c) \(\mu(1,l) = 1\),

(d) the mapping \(\mu(\cdot,1): K \rightarrow K\) is a bijection,

(e) \(\mu(k_1,k_2,k_1) = \mu(k_1,k_2)\mu(k_1,k_2,1)\mu(k_1,k_2,1)\lambda(l_1l_2,1)\)

(f) \(\lambda(l_1,l_2)\lambda(l_2,l_3) = \lambda(l_1l_2,l_3)\mu(\lambda(l_1,l_2),l_3)\)

for \(l_1,l_2,l_3 \in L\) and \(k,k_1,k_2 \in K\).

In view of Theorem 2 the mapping \(\sigma\) satisfies condition \((w_2)\) which may be written in the form:

\[(\sigma(l_1)\sigma(l_2))k = \sigma(l_1l_2)k\]

for arbitrary \(l_1,l_2 \in L\).

Using (5) we can define a mapping \(\lambda: L \times L \rightarrow K\) in the following way:

\[\sigma(l_1)\sigma(l_2) = \sigma(l_1l_2)\lambda(l_1,l_2)\]

for arbitrary \(l_1,l_2 \in L\).

A mapping \(\mu: K \times L \rightarrow K\) we define by the rule:

\[\mu(k, l) = \sigma(1)^{-1}k\sigma(l)\]

for arbitrary \(l \in L\) and \(k \in K\).
We shall prove that the mappings $\lambda$ and $\mu$ satisfy conditions (a_1) - (a_7).

It is easy to check that conditions (a_1) - (a_4) hold.

If $k_1, k_2 \in K$ and $l \in L$, then $\mu(k_1 k_2, l) = \sigma(l)^{-1} k_1 k_2 \sigma(l) = (\sigma(l)^{-1} k_1 \sigma(l)) (\sigma(l)^{-1} k_2 \sigma(l)) = \mu(k_1, l) \mu(k_2, l)$ and so condition (a_5) is fulfilled.

If $k \in K$ and $l_1, l_2 \in L$, then

$$\mu(k, l_1 l_2) = \sigma(l_1 l_2)^{-1} k \sigma(l_1 l_2) =$$

$$= [\sigma(l_1) \sigma(l_2) \lambda(l_1, l_2)^{-1}]^{-1} k [\sigma(l_1) \sigma(l_2) \lambda(l_1, l_2)^{-1}] =$$

$$= \lambda(l_1, l_2) \sigma(l_2)^{-1} (\sigma(l_1)^{-1} k \sigma(l_1)) \sigma(l_2) \lambda(l_1, l_2)^{-1} =$$

$$= \lambda(l_1, l_2) (\sigma(l_2)^{-1} \mu(k, l_1) \sigma(l_2)) \lambda(l_1, l_2)^{-1} =$$

$$= \lambda(l_1, l_2) \mu(\mu(k, l_1), l_2) \lambda(l_1, l_2)^{-1}$$

and so condition (a_6) is fulfilled.

If $l_1, l_2 \in L$ and $k_1, k_2 \in K$, then

$$(\sigma(l_1) k_1) (\sigma(l_2) k_2) = \sigma(l_1 l_2) \lambda(l_1, l_2) \mu(k_1, l_2) k_2.$$ 

Indeed, $(\sigma(l_1) k_1) (\sigma(l_2) k_2) =$

$$= \sigma(l_1) \sigma(l_2) \mu(k_1, l_2) k_2 =$$

$$= \sigma(l_1) \sigma(l_2) \mu(k_1, l_2) k_2 = \sigma(l_1 l_2) \lambda(l_1, l_2) \mu(k_1, l_2) k_2.$$ 

If $l_1, l_2, l_3 \in L$ and $k_1, k_2, k_3 \in K$, then

$$(\sigma(l_1) k_1) [(\sigma(l_2) k_2)(\sigma(l_3) k_3)] = [(\sigma(l_1) k_1)(\sigma(l_2) k_2)](\sigma(l_3) k_3).$$ 

Using (6) we have:

$$(\sigma(l_1) k_1) [(\sigma(l_2) k_2)(\sigma(l_3) k_3)] =$$

$$= (\sigma(l_1) k_1) [\sigma(l_2 l_3) \lambda(l_2, l_3) \mu(k_2, l_3) k_3] =$$

$$= \sigma(l_1 l_2 l_3) \lambda(l_1, l_2 l_3) \mu(k_1, l_2 l_3) \lambda(l_2, l_3) \mu(k_2, l_3) k_3;$$

$$[(\sigma(l_1) k_1) (\sigma(l_2) k_2)](\sigma(l_3) k_3) =$$

$$= [\sigma(l_1 l_2)(\lambda(l_1, l_2) \mu(k_1, l_2) k_2)](\sigma(l_3) k_3).$$
Hence, \( \lambda(1, 1, 1, 1) \mu(k_1, 1, 1, 1) = \lambda(1, 1, 1, 1) \mu(k_1, 1, 1, 1) \).

Applying conditions \((a_6)\) and \((a_7)\) to the left side and the right side of the above equality, respectively, we get:

\[
\lambda(1, 1, 1, 1) \mu(k_1, 1, 1, 1) \lambda(1, 1, 1, 1) = \lambda(1, 1, 1, 1) \mu(k_1, 1, 1, 1) \mu(1, 1, 1, 1) \mu(k_2, 1, 1, 1)
\]

and this means that condition \((a_7)\) holds.

Comparing equalities \((4)\) and \((6)\) we obtain

\[
\psi(l_1, k_1, l_2, k_2) = \lambda(l_1, l_2) \mu(k_1, l_2) k_2
\]

for \( l_1, l_2 \in L \) and \( k_1, k_2 \in K \).

It follows from \([3]\) that the group \( \Sigma \) and the loop \( <L; K>\psi \), where \( \psi \) is a mapping defined by formula \((4)\) are isomorphic, thus \( <L; K>\psi \) is a group.

If a mapping \( \psi : L \times K \times L \times K \rightarrow K \) has the form

\[
\psi(l_1, k_1, l_2, k_2) = \lambda(l_1, l_2) \mu(k_1, l_2) k_2
\]

for arbitrary \( l_1, l_2 \in L \) and \( k_1, k_2 \in K \), where mappings

\( \lambda : L \times L \rightarrow K \) and \( \mu : K \times L \rightarrow K \) satisfy conditions 

\((a_1) - (a_7)\), then \( <L; K>\psi \) is a group.

It is easy to check that conditions \(1^0 - 4^0\) of Definition 2 are fulfilled, then \( <L; K>\psi \) is a loop.

We shall show that the operation \( \circ \) in the loop

\( <L; K>\psi \) is associative.

If \( l_1, l_2, l_3 \in L \) and \( k_1, k_2, k_3 \in K \), then

\[
<l_1, k_1> \circ [<l_2, k_2> \circ <l_3, k_3>] =
\]

\[
= <l_1 l_2 l_3, \psi(l_1, k_1, l_2 l_3, \psi(l_2, k_2, l_3, k_3)>
\]
and \[ [<1_1,k_1> \circ <l_2,k_2>] \circ <l_3,k_3> = \]
\[ = <l_1l_2l_3, \psi(l_1l_2, \psi(l_1,k_1,l_2,k_2),l_3,k_3)> \]
Applying (a_6), (a_7) and (a_5) we obtain:

\[ \psi(l_1,k_1,l_2l_3, \psi(l_2,k_2,l_3,k_3)) = \]
\[ = \lambda(l_1,l_2l_3) \mu(k_1,l_2l_3) \lambda(l_2,l_3) \mu(k_2,l_3)k_3 = \]
\[ = \lambda(l_1,l_2l_3) \lambda(l_2,l_3) \mu(k_1,l_2,l_3) \lambda(l_2,l_3) \mu(k_2,l_3)k_3 = \]
\[ = \lambda(l_1,l_2l_3) \lambda(l_2,l_3) \mu(k_1,l_2,l_3) \mu(k_2,l_3)k_3 = \]
\[ = \lambda(l_1,l_2,l_3) \mu(k_1,l_2,l_3) \mu(k_2,l_3)k_3 = \]
\[ = \psi(l_1l_2, \psi(l_1,k_1,l_2,k_3),l_2,k_3). \]
Thus \( <L ; K \psi \) is a group.

It follows from the considerations in [3] that the group \( <L ; K \psi \), where the mapping \( \psi \) has form (7), is an extension up to isomorphism of the group \( K \) by the group \( L \).

In this way the problem of determination of all extensions of a group \( K \) by a group \( L \) can be reduced to the construction of all products \( <L ; K \psi > \), where the mapping \( \psi \) has form (7).

References
