On the generalized convex functions with respect to the three-parameters family of functions

1. INTRODUCTION

M.C. Peixoto gave in his paper [8] the definition of generalized convex function with respect to a three-parameters family of functions and proved that a function
\( \psi \in C^3(a,b) \) is a generalized convex function with respect to the family of solutions of the equation
\[ y'' = f(x,y,y',y'') \]
if and only if it satisfies the inequality
\[ \psi''(x) \geq f(x,\psi(x),\psi'(x),\psi''(x)). \]

In this paper we shall give some equivalent conditions for a two times differentiable function to be a generalized convex function with respect to a three-parameters family of functions of class \( C^2 \).

Similar results for the generalized convexity with respect to a two-parameters family of functions may be found in the papers [1], [2], [3], [5], [6] and [9].
2. SOME EQUIVALENT CONDITIONS OF CONVEXITY

We shall assume the following hypothesis:

H. Let $F$ be a three-parameter family of functions defined in an open interval $I = (a, b)$ and satisfying the following conditions:

(i) every function $f \in F$ belongs to $C^2(I)$;
(ii) for every $x_0 \in I$ and for every $y_0, y_1, y_2 \in \mathbb{R}$ there is a unique member $f$ of the family $F$ such that $f^{(i)}(x_0) = y_i$, $i = 0, 1, 2$;
(iii) for every three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, $a < x_1 < x_2 < x_3 < b$ there is a unique member $f$ of the family $F$ such that $f(x_i) = y_i$, $i = 1, 2, 3$.

DEFINITION. Let hypothesis H be fulfilled. The function $f$ is said to be strictly convex with respect to the family $F$ iff for all $x_1, x_2, x_3$ such that $a < x_1 < x_2 < x_3 < b$ the inequalities $\psi(x) > f(x)$ for $x \in (x_1, x_2)$ and $\psi(x) < f(x)$ for $x \in (x_2, x_3)$ hold, where $\psi \in F$ is determined by the conditions

(1) $\psi(x_i) = \psi(x_i)$, $i = 1, 2, 3$.

We say that $\psi$ is convex with respect to $F$ iff it satisfies weak inequalities ($\leq$) instead of the strong ones.

Similarly we define a strictly concave (concave) function with respect to $F$ by reversing the inequalities.
Remark 1. (see [8]) If a function $\psi$ is strictly convex with respect to $F$ then for all $x_1, x_2, x_3$ such that $a < x_1 < x_2 < x_3 < b$ the inequalities

$$\psi(x) < \varphi(x) \quad \text{for } x \in (a, x_1)$$
$$\psi(x) > \varphi(x) \quad \text{for } x \in (x_3, b)$$

hold, where $\varphi \in F$ is determined by the conditions (1).

**Theorem.** Let hypothesis $H$ be fulfilled and let a function $\psi : I \to \mathbb{R}$ be twice differentiable in $I$. Under these assumptions the following statements are equivalent:

(A) the function $\psi$ is strictly convex with respect to the family $F$;

(B) for every $x_1, x_2$, such that $a < x_1 < x_2 < b$ the inequalities

$$\psi(x) < \varphi(x) \quad \text{for } x \in (a, x_1),$$
$$\psi(x) > \varphi(x) \quad \text{for } x \in (x_1, b) \setminus \{x_2\}$$

hold, where $\varphi \in F$ is determined by the conditions

(2) $\varphi(x_1) = \psi(x_1), \varphi(x_2) = \psi(x_2), \varphi'(x_2) = \psi'(x_2)$;

(C) for every $x_0 \in (a, b)$ the inequalities

$$\psi(x) < \varphi(x) \quad \text{for } x \in (a, x_0),$$
$$\psi(x) > \varphi(x) \quad \text{for } x \in (x_0, b)$$

hold, where $\varphi \in F$ is determined by the conditions

(3) $\varphi^{(1)}(x_0) = \psi^{(1)}(x_0) \quad i = 0, 1, 2$;

(D) for every $x_1, x_2$, such that $a < x_1 < x_2 < b$ the inequalities

$$\psi(x) < \varphi(x) \quad \text{for } x \in (a, x_2) \setminus \{x_1\}.$$
\( \psi(x) > \varphi(x) \) for \( x \in (x_2, b) \)

hold, where \( \varphi \in F \) is determined by the conditions

(4) \( \psi(x_1) = \psi(x_1), \ \varphi'(x_1) = \psi'(x_1), \ \psi(x_2) = \psi(x_2). \)

First we shall give some lemmas needed in the proof of the theorem.

**Lemma 1.** (see [4], c.f. [7]) Let \( F \subset C^2(I) \), where \( I \) is an open interval, be a three-parameter family of functions and the initial value problem (see H(ii)), as well as the boundary value problem (see H(iii)) are uniquely solvable in \( F \). Then

(M1) for every \( x_1, x_2, y_0, y_1, y_2 \in \mathbb{R} \), such that \( a < x_1 < x_2 < b \) there is a unique member \( \varphi \) of \( F \) such that

\[ \varphi(x_1) = y_1, \ \varphi'(x_1) = y_0, \ \varphi(x_2) = y_2; \]

(M2) for every \( x_1, x_2, y_0, y_1, y_2 \in \mathbb{R} \), such that \( a < x_1 < x_2 < b \) there is a unique member \( \varphi \) of \( F \) such that

\[ \varphi(x_1) = y_1, \ \varphi(x_2) = y_2, \ \varphi'(x_2) = y_0 \]

(i.e. the mixed problems are also uniquely solvable in \( F \)).

The uniqueness of the functions \( \varphi \) determined by the conditions (2) and (4) follows from the above Lemma.

**Lemma 2.** (see [8]) Let \( a < x_1 < x_2 < b \) and \( \varphi_1 \) and \( \varphi_2 \) be two elements of the family \( F \) such that \( \varphi_1 \neq \varphi_2 \) and

\[ \varphi_1(x_1) = \varphi_2(x_1) \ i = 1, 2. \]

Then \( \varphi_1(x) > \varphi_2(x) \) for \( x \in (x_1, x_2) \) and

\( \varphi_1(x) < \varphi_2(x) \) for \( x \in (a, x_1) \cup (x_2, b) \) (or the reverse inequalities hold).
LEMMA 3. Let functions \( f, g \) be defined and two times differentiable in a neighbourhood of a point \( x_0 \) and let 
\[ f(x_0) = g(x_0). \]
Then

a) if \( f(x) > g(x) \) for \( x \neq x_0 \), then \( f'(x_0) = g'(x_0) \);

b) if \( f(x) > g(x) \) for \( x > x_0 \) (or for \( x < x_0 \)) and 
\[ f'(x_0) = g'(x_0), \] then \( f''(x_0) > g''(x_0) \);

c) if \( f(x) > g(x) > 0 \) for \( x > x_0 \) (or for \( x < x_0 \)) and 
\[ f(x_0) = f'(x_0) = f''(x_0) = 0, \] then \( g'(x_0) = g''(x_0) = 0 \).

Proof. Part a) is obvious.

We shall prove b) for \( x > x_0 \). For \( x < x_0 \) the proof is similar. Let us consider the function \( h(x) = f(x) - g(x) \).

From the assumptions we have \( h(x_0) = 0, h'(x_0) = 0 \) and 
\[ h(x) > 0 \] for \( x > x_0 \). Hence it follows that there exists a sequence \( \{x_n\} \) such that \( x_n \to x_0^+ \) and 
\[ h'(x_n) > 0, \] thus \( f'(x_0) > g'(x_0) \).

Part c). From a) we get \( g'(x_0) = 0 \), from b) 
\[ f''(x_0) > g''(x_0) \] and \( g''(x_0) > 0 \). Hence and from the equality 
\[ f''(x_0) = 0 \] we have \( g''(x_0) = 0 \).

LEMMA 4. Let \( x_0 \in I \) and let \( \varphi_1, \varphi_2 \in F \) be such that 
\[ \varphi_1 \neq \varphi_2, \varphi_1(x_0) = \varphi_2(x_0) \] and 
\[ \varphi_1'(x_0) = \varphi_2'(x_0). \] Then 
\[ \varphi_1(x) > \varphi_2(x) \] for \( x \in I \setminus \{x_0\} \) (or the reverse inequality holds).

Proof. It follows from Lemma 1 that \( \varphi_1(x) \neq \varphi_2(x) \) for \( x \neq x_0 \). Suppose that the statement does not hold.

Hence either \( \varphi_1(x) > \varphi_2(x) \) for \( x \in (a, x_0) \) and 
\[ \varphi_1(x) < \varphi_2(x) \] for \( x \in (x_0, b) \) or \( \varphi_1(x) < \varphi_2(x) \) for
\( x \in (a, x_0) \) and \( \phi_1(x) > \phi_2(x) \) for \( x \in (x_0, b) \). Let us consider the first case, the other is similar. Applying Lemma 3b) for \( f = \phi_1, g = \phi_2 \) and for \( f = \phi_2, g = \phi_1 \) we have \( \phi_1''(x_0) > \phi_2''(x_0) \) and \( \phi_2''(x_0) > \phi_1''(x_0) \), respectively. Hence \( \phi_1''(x_0) = \phi_2''(x_0) \). From (i) we get \( \phi_1 = \phi_2 \), a contradiction.

**Lemma 5.** Let the assumptions and condition (B) of the Theorem be fulfilled. Then for every \( x_1, x_2 \) such that \( a < x_1 < x_2 < b \) the inequality
\[
\psi(x) < \bar{\psi}(x) \quad \text{for } x \in (x_1, x_2)
\]
holds, where \( \bar{\psi} \in F \) is determined by conditions (4).

**Proof.** Let \( a < x_1 < x_2 < b \) and let \( \bar{\psi} \) be determined by conditions (4). Let us assume that the statement is false. Then two cases are possible

a) \( \psi(x) < \bar{\psi}(x) \) for all \( x \in (x_1, x_2) \) and
\[
\psi(c) = \bar{\psi}(c) \quad \text{for } c \in (x_1, x_2);
\]
b) there exists a \( c \in (x_1, x_2) \) such that \( \psi(c) > \bar{\psi}(c) \).

Case a). From Lemma 3a) we have \( \psi'(c) = \bar{\psi}'(c) \). Let us observe that \( \bar{\psi} \) is just the function determined by (2) with \( x_1 \) and \( x_2 = c \). By (B) we get \( \psi(x) > \bar{\psi}(x) \) for \( x \in (x_1, c) \), contrary to a).

Case b). Let us consider the function \( \psi_1 \) determined by (2) with \( \bar{x}, c \) in place of those \( x_1, x_2 \), where
\[
\bar{x} = \sup \{ x \in [x_1, c) : \psi(x) = \bar{\psi}(x) \}.
\]
By this definition
\[
\bar{x} \in [x_1, c) \quad \text{and}
\]
(5) \( \psi(\bar{x}) = \bar{\psi}(\bar{x}) = \psi_1(\bar{x}) \), \( \psi(x) > \bar{\psi}(x) \) for \( x \in (\bar{x}, c) \),
and from (B) we have
\[(6) \quad \psi(x) > \varphi_1(x) \text{ for } x \in (\bar{x}, b) \setminus \{c\}.\]

By (6) we have \(\bar{\varphi}(x_2) = \psi(x_2) > \varphi_1(x_2)\), but
\(\bar{\varphi}(c) < \psi(c) = \varphi_1(c)\), therefore there exists a point
d \in (c, x_2) such that \(\bar{\varphi}(d) = \varphi_1(d)\). The values of func-
tions \(\bar{\varphi}, \varphi_1\) are equal at the points \(\bar{x}\) and \(d\) and those
functions are not identically equal because \(\varphi_1(c) = \psi(c) >
> \bar{\varphi}(c)\). By Lemma 2 we have \(\varphi_1(x) > \bar{\varphi}(x)\) for \(x \in (\bar{x}, d)\)
and from (6) we get
\[(7) \quad \psi(x) > \varphi_1(x) > \bar{\varphi}(x) \text{ for } x \in (\bar{x}, c).\]

If \(\bar{x} = x_1\), then \(\psi(\bar{x}) = \bar{\varphi}(\bar{x})\), \(\varphi_1(\bar{x}) = \bar{\varphi}(\bar{x})\) and from
(7) we get \(\varphi_1'(\bar{x}) = \bar{\varphi}'(\bar{x})\). Thus the mixed problem
\[\varphi(\bar{x}) = \psi(\bar{x}), \quad \varphi'(\bar{x}) = \psi'(\bar{x}), \quad \varphi(d) = \bar{\varphi}(d)\]
has two different solutions \(\bar{\varphi}, \varphi_1\), what is impossible.

Let \(\bar{x} > x_1\). From (7) \(\varphi_1'(\bar{x}) > \bar{\varphi}'(\bar{x})\). If \(\varphi_1'(\bar{x}) = \bar{\varphi}'(\bar{x})\)
then we proceed as in the case \(\bar{x} = x_1\). Let \(\varphi_1'(\bar{x}) > \bar{\varphi}'(\bar{x})\).
Hence and from the equality \(\varphi_1(\bar{x}) = \bar{\varphi}(\bar{x})\) we get that
there is an \(l\) such that \(\varphi_1(x) < \bar{\varphi}(x)\) for \(x \in (1, \bar{x})\).

From (B) we have \(\varphi_1(x) > \psi(x)\) for \(x \in (a, \bar{x})\), in partic-
ular \(\varphi_1(x_1) > \psi(x_1) = \bar{\varphi}(x_1)\). From the continuity of \(\varphi_1\)
and \(\bar{\varphi}\) there is a \(p \in (x_1, \bar{x})\) such that \(\varphi_1(p) = \bar{\varphi}(p)\).
Therefore \(\varphi_1 = \bar{\varphi}\) as they are members of \(F\) passing
through the points \((p, \varphi_1(p)), (\bar{x}, \varphi_1(\bar{x})), (d, \varphi_1(d))\).

**Proof of Theorem.** (A) \(\implies\) (B).

Let \(a < x_1 < x_2 < b\) and let \(\bar{\varphi}\) be determined by (2). We
are going to prove the inequality
Let us assume that inequality (8) does not hold. Then two cases are possible

a) \( \psi(x) > \overline{\psi}(x) \) for all \( x \in (x_1, x_2) \) and \( \psi(c) = \overline{\psi}(c) \) for a \( c \in (x_1, x_2) \);

b) there exists a \( c \in (x_1, x_2) \) such that \( \psi(c) < \overline{\psi}(c) \).

Case a). Let us observe that \( \overline{\psi} \) is just the function determined by conditions (1) with \( x_1, c, x_2 \) in place of those \( x_1, x_2, x_3 \), therefore by (A) we have \( \psi(x) < \overline{\psi}(x) \) for \( x \in (c, x_2) \), contrary to a).

Case b). Let \( \bar{x} = \inf \{ x \in (c, x_2] : \psi(x) = \overline{\psi}(x) \} \). Hence \( \bar{x} \in (c, x_2] \), \( \psi(\bar{x}) = \overline{\psi}(\bar{x}) \) and \( \psi(x) < \overline{\psi}(x) \) for \( x \in [c, \bar{x}) \).

If \( \bar{x} < x_2 \), then since \( \psi \) is strictly convex with respect to \( F \), we have \( \psi(x) > \overline{\psi}(x) \) for \( x \in (x_1, \bar{x}) \) and \( \psi(x) < \overline{\psi}(x) \) for \( x \in (\bar{x}, x_2) \), because \( \overline{\psi}(x_1) = \psi(x_1) \), \( \overline{\psi}(\bar{x}) = \psi(\bar{x}) \) and \( \overline{\psi}(x_2) = \psi(x_2) \). Hence, in particular, \( \psi(x) > \overline{\psi}(x) \) for \( x \in [c, \bar{x}) \), a contradiction.

Let \( \bar{x} = x_2 \) and let us consider the function \( \psi_1 \) determined by (1) with the points \( x_1, c, \bar{x} \). From (A) we get the inequality \( \psi(x) < \psi_1(x) \) for \( x \in (c, \bar{x}) \) and just as in the proof of Lemma 4 (inequality (7)) we have

\[ \psi(x) < \psi_1(x) < \overline{\psi}(x) \] for \( x \in (c, \bar{x}) \). But \( \psi(\bar{x}) = \overline{\psi}(\bar{x}) \) and \( \psi'(\bar{x}) = \overline{\psi}'(\bar{x}) \), therefore \( \overline{\psi}'(\bar{x}) = \psi_1'(\bar{x}) \). In this way we obtain two different functions \( \overline{\psi}, \psi_1 \in F \) such that

\( \overline{\psi}(x_1) = \psi_1(x_1) \), \( \overline{\psi}(\bar{x}) = \psi_1(\bar{x}) \) and \( \overline{\psi}'(\bar{x}) = \psi_1'(\bar{x}) \), what by Lemma 1 is impossible. This concludes the proof of inequality (8).
The inequalities

\[ \psi(x) < \bar{\phi}(x) \quad \text{for } x \in (a, x_1) \]
\[ \psi(x) > \bar{\phi}(x) \quad \text{for } x \in (x_2, b) \]

follows from (8), (A) and from Remark 1.

(B) \implies (C).

Let \( x_0 \in I \) and let \( \bar{\phi} \) be determined by (3). First we are going to prove that

(9) \[ \psi(x) < \bar{\phi}(x) \quad \text{for } x \in (a, x_0). \]

Assume the contrary. Then either

a) \[ \psi(x) \leq \bar{\phi}(x) \quad \text{for } x \in (a, x_0) \quad \text{and} \]
\[ \psi(c) = \bar{\phi}(c) \]
for a \( c \in (a, x_0) \),
or

b) there exists a \( c \in (a, x_0) \) such that \[ \psi(c) > \bar{\phi}(c). \]

To disprove a) let us observe (Lemma 1) that \( \bar{\phi} \) is just the function determined by conditions (2) with \( x_1 = c \) and \( x_2 = x_0 \). By (B) we have \( \psi(x) > \bar{\phi}(x) \) for \( x \in (c, x_0) \), a contradiction.

Case b). Let \( \psi_1 \) be a function determined by (2) with \( x_1 = c \) and \( x_2 = x_0 \). From (B) we have

(10) \[ \psi(x) > \psi_1(x) \quad \text{for } x \in (c, x_0). \]

It follows from the definition of \( \psi_1 \) that \( \psi_1(x_0) = \bar{\phi}(x_0) \), \( \psi_1'(x_0) = \bar{\phi}'(x_0) \) and \( \psi_1(c) = \psi(c) > \bar{\phi}(c) \). Hence and from Lemma 4 we obtain the inequality

(11) \[ \psi_1(x) > \bar{\phi}(x) \quad \text{for } x \in (a, x_0). \]

By (10) and (11) we get

\[ \psi(x) > \psi_1(x) > \bar{\phi}(x) \quad \text{for } x \in (c, x_0). \]
From the definition of $\overline{\phi}$ and from Lemma 3c) used for the functions $f(x) = \psi(x) - \overline{\phi}(x)$ and $g(x) = \phi_1(x) - \overline{\phi}(x)$ we get $\phi_1''(x_0) = \overline{\phi}''(x_0)$, what contradicts H(ii).

Now we are going to prove that

$$\psi(x) > \overline{\phi}(x) \quad \text{for} \quad x \in (x_0, b).$$

Assume the contrary. Then two cases are possible

a) $\psi(x) \geq \overline{\phi}(x)$ for $x \in (x_0, b)$ and $\psi(c) = \overline{\phi}(c)$ for a $c \in (x_0, b)$;

b) there exists a $c \in (x_0, b)$ such that $\psi(c) < \overline{\phi}(c)$.

Case a). Let us observe (Lemma 1) that $\overline{\phi}$ is just the function determined by (4) with $x_1 = x_0$ and $x_2 = c$. From Lemma 5 we get $\psi(x) < \overline{\phi}(x)$ for $x \in (x_0, c)$, a contradiction.

Case b). Let $\phi_2$ be a function determined by (4) with $x_1 = x_0$ and $x_2 = c$. By Lemma 5 we have $\psi(x) < \phi_2(x)$ for $x \in (x_0, c)$. As in the proof of (9) (Case b) we obtain $\psi(x) < \phi_2(x) < \overline{\phi}(x)$ for $x \in (x_0, c)$ and $\phi_2''(x_0) = \overline{\phi}''(x_0)$. In this way we get two different functions $\phi_2, \overline{\phi} \in \mathcal{F}$ such that $\phi_2^{(i)}(x_0) = \overline{\phi}^{(i)}(x_0)$, $i = 0, 1, 2$, what by H(ii) is impossible.

$$(c) \implies (d).$$

Let $a < x_1 < x_2 < b$ and let $\overline{\phi}$ be determined by (4). First we shall show that

$$(12) \quad \psi(x) < \overline{\phi}(x) \quad \text{for} \quad x \in (x_1, x_2).$$

Let us assume that inequality (12) does not hold. Then either
a) \( \psi(x) \leq \varphi(x) \) for \( x \in (x_1, x_2) \) and \( \psi(c) = \varphi(c) \)
for a \( c \in (x_1, x_2) \)

or

b) there exists a \( c \in (x_1, x_2) \) such that \( \psi(c) > \varphi(c) \)

Case a). By Lemma 3a) we get \( \psi'(c) = \varphi'(c) \) and by
Lemma 3b) \( \psi''(c) < \varphi''(c) \). From (C) we obtain that the case
\( \psi''(c) = \varphi''(c) \) cannot occur, therefore \( \psi''(c) < \varphi''(c) \). Now
let us consider the function \( \psi_1 \) determined by (3) with
\( x_0 = c \), \( \psi_1 \neq \varphi \) because \( \psi''_1(c) = \psi''(c) < \varphi''(c) \). From (C)
we have

\[
(13) \quad \psi_1(x) > \psi(x) \quad \text{for } x \in (a, c)
\]

and

\[
(14) \quad \psi_1(x) < \psi(x) \quad \text{for } x \in (c, b).
\]

From (14) we get \( \psi_1(x) < \varphi(x) \) for \( x \in (c, x_2) \) and from
the definition of \( \psi_1 \) and from Lemma 4 it follows that
\( \psi_1(x) < \varphi(x) \) for \( x \in I \setminus \{c\} \). Hence, in particular,
\( \psi_1(x_1) < \varphi(x_1) \), but from (13) we have \( \psi_1(x_1) > \psi(x_1) = \varphi(x_1) \),
a contradiction.

Case b). Put \( \bar{x} = \sup \{ x \in [x_1, c) : \psi(x) = \varphi(x) \} \). Thus
\( \bar{x} \in [x_1, c) \) and \( \psi(x) > \varphi(x) \) for \( x \in (\bar{x}, c) \) and \( \psi(\bar{x}) = \varphi(\bar{x}) \). Hence \( \psi'(\bar{x}) > \varphi'(\bar{x}) \). Let \( \varphi_2 \in F \) be determined by
(3) with \( x_0 = \bar{x} \).

If \( \psi'(\bar{x}) > \varphi'(\bar{x}) \), then \( \bar{x} > x_1 \) and we proceed as in
the proof of Lemma 5 (case b) \( \bar{x} > x_1 \) and we get the con­
tradiction with the condition \( H(iii) \).
Let $\psi'(x) = \bar{\psi}'(x)$, then by Lemma 3b) we get $\psi''(x) > \bar{\psi}''(x)$ and it follows from (c) that $\psi''(x) \neq \bar{\psi}''(x)$, i.e.

(15) \[ \psi''(x) > \bar{\psi}''(x). \]

By (c) we have $\varphi_2(x) < \psi(x)$ for $x \in (\bar{x}, b)$ and in particular

(16) \[ \varphi_2(x_2) < \psi(x_2) = \bar{\psi}(x_2). \]

Lemma 4 and (16) yield $\varphi_2(x) < \bar{\psi}(x)$ for $x \in I \setminus \{ \bar{x} \}$. Thus

(17) \[ \varphi''_2(x) \leq \bar{\psi}''(x). \]

From (15) and from the definition of $\varphi_2$ we have $\varphi''_2(x) > \bar{\psi}''(x)$, what contradicts (17).

Now we shall show that

$\psi(x) < \bar{\psi}(x)$ \quad $x \in (a, x_1)$,

where $\bar{\psi}$ is determined by (4).

Let $\psi_3 \in P$ be determined by (3) with $x_0 = x_1$. From (c) we get $\psi(x) > \psi_3(x)$ for $x \in (x_1, b)$ and form (12)

$\psi_3(x) < \bar{\psi}(x)$ for $x \in (x_1, x_2)$. Hence and by Lemma 4 we obtain $\psi_3(x) < \bar{\psi}(x)$ for $x \in (a, x_1)$, from (c)

$\psi(x) < \psi_3(x)$ for $x \in (a, x_1)$. Thus $\psi(x) < \bar{\psi}(x)$ for $x \in (a, x_1)$.

The inequality

$\psi(x) > \bar{\psi}(x)$ \quad for $x \in (x_2, b)$

follows from (12), (A) and from lemmas 3 and 4.

(D) $\Rightarrow$ (A).

Let $a < x_1 < x_2 < x_3 < b$ and let $\bar{\psi}$ be determined by (1).

We shall prove that
\[ \psi(x) > \bar{\phi}(x) \quad \text{for } x \in (x_1, x_2). \]

Again, let us assume that inequality (18) does not hold. We shall consider two cases

a) \[ \psi(x) > \bar{\phi}(x) \quad \text{for } x \in (x_1, x_2) \quad \text{and} \quad \psi(c) = \bar{\phi}(c) \]

for a \( c \in (x_1, x_2) \);

b) there exists a \( c \in (x_1, x_2) \) such that \( \psi(c) < \bar{\phi}(c) \).

Case a). By Lemma 3a) we have \( \psi'(c) = \bar{\phi}'(c) \). Thus \( \bar{\phi} \) satisfies (4) with \( x_1 = c \) and \( x_2 \), so that, by (D), we get

\[ \psi(x) < \bar{\phi}(x) \quad \text{for } x \in (c, x_2), \]

contrary to the inequality in a).

Case b). Let \( \bar{x} = \sup \{ x \in [x_1, c] : \psi(x) = \bar{\phi}(x) \} \). By this definition \( \bar{x} \in [x_1, c] \), \( \psi(\bar{x}) = \bar{\phi}(\bar{x}) \) and \( \psi(x) < \bar{\phi}(x) \)

for \( x \in (\bar{x}, c) \), whence \( \psi'(\bar{x}) \leq \bar{\phi}'(\bar{x}) \). If we had \( \psi'(\bar{x}) = \bar{\phi}'(\bar{x}) \),

then \( \phi \) would satisfy (4) with \( x_1 = \bar{x}, \ x_2 = x_3 \) and the first inequality of (D) would contradict (1). Therefore

\[ \psi'(\bar{x}) < \bar{\phi}'(\bar{x}). \]

Now let us consider the function \( \phi_1 \) determined by (4) with \( x_1 = \bar{x} \) and \( x_2 \). Since \( \phi_1'(\bar{x}) = \psi'(\bar{x}) \) and \( \phi_1(\bar{x}) = \bar{\phi}(\bar{x}) \),

we see from (19) and from the continuity of \( \phi_1 \) and \( \bar{\phi} \) that

\[ \phi_1(x) < \bar{\phi}(x) \quad \text{for } x \text{ from a right neighbourhood of } \bar{x}. \]

From (D) we have the inequalities

\[ \psi(x) < \phi_1(x) \quad \text{for } x \in (\bar{x}, x_2) \]

and \( \psi(x) > \phi_1(x) \) for \( x \in (x_2, b) \). The latter yields

\[ \psi(x_3) > \phi_1(x_3). \]

Hence and from the equality \( \psi(x_3) = \bar{\phi}(x_3) \) we have

\[ \bar{\phi}(x_3) > \phi_1(x_3). \]
Applying Lemma 2 for \( \phi_1 \) and \( \tilde{\phi} \) with \( x_1 \) replaced by \( \bar{x} \), we get \( \phi_1(x) < \tilde{\phi}(x) \) for \( x \in (\bar{x}, x_2) \) and \( \phi_1(x) > \tilde{\phi}(x) \) for \( x \in (a, \bar{x}) \cup (x_2, b) \), in particular \( \phi_1(x_3) > \tilde{\phi}(x_3) \), what contradicts (21).

The inequality
\[
\psi(x) < \tilde{\psi}(x) \quad \text{for} \quad x \in (x_2, x_3)
\]
follows from (18) and from (D).

**Remark 2.** By a suitable change of inequalities that appear in the Theorem we get the conditions equivalent to the fact that \( \psi \) is convex, or concave or strictly concave with respect to \( F \).

**References**


