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Ring homomorphisms equation revisited

Dedicated to Professor Zenon Moszner on his 70-th birthday

Abstract. The functional equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{$*$}$$

has been studied by J. Dhombres (*Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186-212) for functions f mapping a given ring into another one. In this paper both rings were supposed to have unit elements; additionally the division by 2 had to be performable. Without these assumptions the study of equation $(*)$ becomes considerably more sophisticated (see author's paper *On an equation of ring homomorphisms*, Publ. Math. Debrecen **52** (1998), 397-417). At present, we deal with equation $(*)$ assuming that the domain is a unitary ring with no assumptions whatsoever upon the target ring (except for the second part of Theorem 5).

1. Introduction

Consider a map $f : X \longrightarrow Y$ between two rings X and Y satisfying the functional equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{1}$$

for all $x, y \in X$. Clearly, each homomorphism $f : X \longrightarrow Y$ yields a solution to equation (1). A natural question arises whether or not homomorphisms are the only solutions of (1). In 1988 the following result was obtained, among others, by J. Dhombres [1].

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Let X and Y be two unitary rings and let the division by 2 be uniquely performable in X. Then each solution $f : X \longrightarrow Y$ of equation (1) such that $f(0) = 0$ yields a ring homomorphism between X and Y, i.e. f yields a solution of the system

$$\begin{cases} f(x+y) = f(x) + f(y), \\ f(xy) = f(x)f(y) \end{cases}$$

of two Cauchy equations for every $x, y \in X$.

The crucial part of the proof was to get the oddness of a solution $f : X \longrightarrow Y$ of equation (1). However, even in the very simple case of unitary rings $X = \mathbb{Z}$ (the integers) and $Y = \mathbb{R}$ (the reals) equation (1) admits non-odd (actually even) and hence nonhomomorphic solutions of the form

$$f(x) = \begin{cases} 0 & \text{for } x \in 2\mathbb{Z}, \\ -1 & \text{for } x \in 2\mathbb{Z} + 1. \end{cases}$$

More generally, it is not hard to check that for any two elements c, d from Y such that $c = c^3$ and $cd = dc = d^2 = 0$ a map $f : \mathbb{Z} \longrightarrow Y$ given by the formula

$$f(x) = \begin{cases} \frac{1}{2}x(c + c^2) + d & \text{for } x \in 2\mathbb{Z}, \\ \frac{1}{2}(x-1)c^2 + \frac{1}{2}(x+1)c + d & \text{for } x \in 2\mathbb{Z} + 1. \end{cases}$$

yields a nonhomomorphic solution of equation (1) unless $c = c^2$ and $d = 0$.

Therefore it is most desirable to relax the assumptions upon the rings considered. However, an attempt to do that presented in [2] shows that omitting the divisibility hypothesis and/or the existence of unit elements, causes essential difficulties and requires some developed techniques. In the present paper, in a sense complementary to [2], we assume (except for Proposition 1 below) that a unitary ring stands for the domain of the solutions studied whereas the target ring is *quite arbitrary* (except for the last part of Theorem 5).

2. The results

We begin with a result that clarifies the role of the assumption that a solution f of equation (1) vanishes at zero.

PROPOSITION 1

Let X and Y be arbitrary rings and let $f : X \longrightarrow Y$ be a solution of equation (1). Denote by Y_0 the ring generated by $f(X)$ in Y and put $d := f(0)$. Then $f(x) = g(x) + d$, $x \in X$, where g stands for a solution of (1) with $g(0) = 0$ and $dY_0 = Y_0 d = \{0\}$; in particular, $d^2 = 0$.

Conversely, if Y is a ring admitting a nonzero element d such that $dY = Yd = \{0\}$ and $g : X \longrightarrow Y$ is a solution of (1) *then so is also $g + d$.*

Proof. Put $g := f - d$. Then $g(0) = 0$ whereas equation (1) gives the relationship

$$g(x+y) + g(xy) = g(x) + g(y) + g(x)g(y) + g(x)d + dg(y) + d^2 \tag{2}$$

valid for all $x, y \in X$. Now, setting subsequently $y = 0$ and $x = 0$ in (2) we arrive at

$$g(x)d + d^2 = 0, \quad x \in X,$$
$$dg(y) + d^2 = 0, \quad y \in X,$$

respectively. In particular, $d^2 = 0$ whence

$$g(x)d = dg(x) = 0$$

for all $x \in X$. Consequently, equation (2) reduces to (1). Moreover,

$$f(x)d = f(x)d - d^2 = g(x)d = 0$$

for all $x \in X$ and, similarly, $df(x) = 0$, $x \in X$.

Since Y_0 consists of finite sums of expressions of the form

$$\pm f(x_1)f(x_2)\dots f(x_n)$$

where $x_1, x_2, \dots, x_n \in X$, $n \in \mathbb{N}$, the assertion follows.

Conversely, assuming that there exists a $d \in Y \setminus \{0\}$ is such that $dY = Yd = \{0\}$ and $g : X \longrightarrow Y$ is a solution of (1), we infer that the map $f := g + d$ satisfies the equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) - f(x)d - df(y) + d^2$$

for all $x, y \in X$, which reduces itself to (1) because of the properties of the element d. This completes the proof.

So, without loss of generality, from now on, we will be assuming that jointly with equation (1) the equality $f(0) = 0$ is satisfied.

Lemma

Let X be a unitary ring with e as a unit and let Y be an arbitrary ring. Suppose that $f : X \longrightarrow Y$ is a solution of equation (1) *such that* $f(0) = 0$ *and put $c := f(e)$. Then for all $x, y \in X$ one has:*

$$cf(x) = f(x)c, \tag{3}$$

$$f(x+e) = cf(x) + c, \tag{4}$$

$$f(xy + x) = cf(xy) + f(x), \tag{5}$$

$$f(x) = c^2 f(x), \tag{6}$$

$$f(2x) = cf(x) + f(x) = (c + c^2)f(x) = cf(2x), \tag{7}$$

$$\begin{aligned} &(c^2 + c)\,[f(x+y) - f(x) - f(y)] = 0, \\ &(c^2 + c)\,[f(xy) - f(x)f(y)] = 0. \end{aligned} \tag{8}$$

Moreover,

$$c^3 = c \quad \textit{and} \quad f(2e) = c^2 + c. \tag{9}$$

Proof. Put subsequently $y = e$ and $x = e$ in (1) to get

$$f(x+e) = c + f(x)c \quad \text{and} \quad f(e+y) = c + cf(y)$$

for all $x, y \in X$, which implies (3) as well as (4).

Now, replace y by $y + e$ in (1) and apply (4) and (1) to the resulting equation to obtain (5).

Observe that on setting $x = y = -e$ in (5) we get the equalities

$$0 = f(0) = f(e - e) = cf(e) + f(-e) = c^2 + f(-e)$$

whence

$$f(-e) = -c^2. \tag{10}$$

On the other hand, equation (5) aplied for $x = -e$, gives

$$f(-y - e) = cf(-y) + f(-e) = cf(-y) - c^2$$

for every y from X, because of (10). Taking here $y = -x - e$, with the aid of (4) we conclude that

$$f(x) = cf(x+e) - c^2 = c^2 f(x)$$

for all $x \in X$, getting (6).

To prove (9), by setting $x = e$ and $y = -e$ in (1) we arrive at

$$c + cf(-e) = 0,$$

which jointly with (10) implies the first of equalities (9). The other results immediately from the substitution $x = y = e$ in (5).

Taking $y = e$ in (5), we infer that for every $x \in X$ one has $f(2x) = cf(x) + f(x)$ whence, by (6), $f(2x) = (c + c^2)f(x)$. Finally, by (9),

$$cf(2x) = c(c + c^2)f(x) = (c^2 + c^3)f(x) = (c^2 + c)f(x) = f(2x),$$

which finishes the proof of assertion (7).

To show the validity of (8), note that (7) implies the equality $f(4x) = (c + c^2)f(2x) = (c + c^2)^2 f(x)$ for all $x \in X$. Replace now x and y in (1) by $2x$ and $2y$, respectively, to get

$$(c+c^2)f(x+y)+(c+c^2)^2f(xy) = (c+c^2)f(x)+(c+c^2)f(y)+(c+c^2)^2f(x)f(y),$$

because of (3). Thus, by means of (9),

$$(c+c^2)\,[f(x+y)-f(x)-f(y)] = 2(c+c^2)\,[f(x)f(y)-f(xy)]$$

for every $x, y \in X$ which, on account of (1), is equivalent to the system (8).

The proof has been completed.

THEOREM 1

Let X be a unitary ring and let Y be an arbitrary ring. If $f : X \longrightarrow Y$ is a solution of equation (1) such that $f(0) = 0$, then the function $f|_{2X}$ yields a ring homomorphism between $2X$ and Y.

Proof. Denote by e the unit element in X and put $c := f(e)$. Let

$$g(x) := (c^2+c)f(x), \quad x \in X.$$

By means of (7) we have also $g(x) = f(2x)$, $x \in X$. Clearly, g is *additive* by virtue of the first of equations (8) whereas the other gives the relationship

$$g(xy) = g(x)f(y) \quad \text{for all } x, y \in X. \tag{11}$$

Now, (11) and (3) imply that

$$\begin{aligned} g(xy)+cg(xy) &= g(x)c^2f(y)+g(x)cf(y) = g(x)(c^2+c)f(y) \\ &= g(x)g(y) \end{aligned}$$

for all $x, y \in X$. This means that

$$(c+c^2)f(2xy) = f(2x)f(2y) \quad \text{for all } x, y \in X$$

whence, by (7),

$$f(2x\cdot 2y) = f(4xy) = (c+c^2)f(2xy) = f(2x)f(2y)$$

for all $x, y \in X$, which was to be shown.

THEOREM 2

Let X be a unitary ring with e as a unit and let Y be an arbitrary ring. If $f : X \longrightarrow Y$ is a solution of equation (1) such that $f(0) = 0$, then setting $c := f(e)$ we have:

(i) *if $f(2e) = 0$ then $c = -c^2$, f is even and $f|_{2X} = 0$;*

(ii) *if $f(2e) \neq 0$ and $f(2e)$ is not a zero divisor of the ring Y, then f yields a ring homomorphism between X and Y.*

Proof. Relations (8) and (9) immediately imply that

$$\begin{aligned} f(2e)\,[f(x+y)-f(x)-f(y)]&=0,\\ f(2e)\,[f(xy)-f(x)f(y)]&=0 \end{aligned} \tag{12}$$

for all $x,y \in X$. Now, the assertion (ii) is obvious.

If $f(2e)=0$ then the equality $c=-c^2$ follows from (9). Consequently, in view of (7), $f(2x)=0$ for every $x \in X$. The evenness of f results now from (5) on setting $y=-e$ and the use of the equality $c=-c^2$ jointly with (6).

This ends the proof.

THEOREM 3

Let X be a unitary ring with e as a unit and let Y be an arbitrary ring. If $f: X \longrightarrow Y$ is a solution of equation (1) such that $f(0)=0$, then the ring Y_0 generated by $f(X)$ in Y is unitary with c^2 as a unit, where $c := f(e)$. Moreover, $c^3=c$ and f satisfies the following system of functional equations

$$\begin{aligned} f(2x+y)&=f(2x)+f(y),\\ f(2xy)&=f(2x)f(y),\\ f(2z)\,[f(x+y)-f(x)-f(y)]&=0,\\ f(2z)\,[f(xy)-f(x)f(y)]&=0 \end{aligned} \tag{13}$$

for all $x,y,z \in X$.

In particular, if the ring X is either 2-divisible or $f(2a) \in \{c,c^2\}$ for some $a \in X$ or $f(2a) \neq 0$ is not a zero divisor for some $a \in X$, then f yields a ring homomorphism between X and Y.

Proof. Since the ring Y_0 consists of finite sums of expressions of the form $\pm f(x_1)f(x_2)\dots f(x_n)$ where $x_1,x_2,\dots,x_n \in X$, $n \in \mathbb{N}$, the fact that e^2 is a unit element of Y_0 results immediately from (6). The equality $c^3=c$ has already been observed in (9).

Setting $2x$ in place of x in equation (1) and applying (7) we obtain

$$f(2x+y)+cf(xy)+f(xy)=cf(x)+f(x)+f(y)+cf(x)f(y)+f(x)f(y)$$

whence, again by (1),

$$\begin{aligned} f(2x+y)-f(x+y)&=cf(x)+c\,[f(x)f(y)-f(xy)]\\ &=cf(x)+[f(xy)-f(x)f(y)] \end{aligned}$$

for all $x,y \in X$, because of (8) and (6). Applying (1) once again, we conclude that

$$f(2x+y)-f(x+y)=cf(x)+f(x)+f(y)-f(x+y)$$

stating that (see (7))

$$f(2x+y) = cf(x) + f(x) + f(y) = f(2x) + f(y)$$

for any $x, y \in X$. The second one from equations (13) results from the former by means of a direct application of (1).

To prove the third one from equations (13), fix arbitrarily points x, y, z from X and note that, in view of (7) and (3), one has

$$\begin{aligned} f(2z)\left[f(x+y) - f(x) - f(y)\right] &= (c+c^2)f(z)\left[f(x+y) - f(x) - f(y)\right] \\ &= f(z)(c+c^2)\left[f(x+y) - f(x) - f(y)\right] \\ &= 0 \end{aligned}$$

by means of (8). The last one from equations (13) may be derived likewise.

Now, from the remaining assertions of the theorem merely the statement that the existence of an element $a \in X$ such that $f(2a) \in \{c, c^2\}$ forces f to be a ring homomomorhism, requires a motivation. Putting $z = a$ in the third of equations (13) we see that either

$$c\left[f(x+y) - f(x) - f(y)\right] = 0 \quad \text{for all } x, y \in X, \tag{14}$$

or

$$c^2\left[f(x+y) - f(x) - f(y)\right] = 0 \quad \text{for all } x, y \in X. \tag{15}$$

On the other hand, the fact that c^2 serves as a unit in Y_0 implies that each of the equations (14) and (15) establishes the *additivity* of f. Indeed, this is trivial for (15) while (14) immediately implies (15). The *multiplicativity* of f results now directly from (1).

This ends the proof.

THEOREM 4

Under the assumptions and denotations of Theorem 3 the sets

$$I := \{x \in X : \ f(2x) = 0\} \quad \textit{and} \quad J := \{u \in Y_0 : \ uc = cu = -u\}$$

form 2-sided ideals in the rings X and Y_0, respectively. The quotient ring Y_0/J is unitary with the unit element $e_J := c + J$. Moreover,

$$\begin{aligned} &f(x+y) - f(x) - f(y) \in J, \\ &f(xy) - f(x)f(y) \in J \end{aligned} \tag{16}$$

for all $x, y \in X$. In other words, the map $X \ni x \longmapsto F(x) := f(x) + J \in Y_0/J$ establishes a homomorphism between the rings X and Y_0/J fulfilling the condition $F(e) = e_J$.

Proof. That I yields a left-sided ideal in X follows easily from Theorem 1 (stating that $f|_{2X}$ is additive) and from the second of the equations (13). To see that I is also right-sided note that, on account of (1) and the first one of equations (13) we have $f(y \cdot 2x) = f(y)f(2x)$ for all $x, y \in X$.

To show that J is a 2-sided ideal in Y_0 observe first that due to the fact that c^2 stands for the unit element in Y_0 (see Theorem 3) J may alternatively be written as

$$J = \{u \in Y_0 : \ (c + c^2)u = u(c + c^2) = 0\}. \tag{17}$$

Now, the relationships $J - J \subset J$ and $J \cdot Y_0 = Y_0 \cdot J \subset J$ become obvious once we realize that, in view of (3), each element of Y_0 commutes with c and hence also with $c + c^2$.

Observe that $e_J = c + J = c^2 + J$ because of (17) and the fact that $c^4 = c^2$ (see (9)). Therefore, for every $y \in Y_0$ one has

$$e_J(y + J) = c^2 y + J = y + J$$

and, similarly, $(y + J)e_J = y + J$ on account of the fact that c^2 is the unit of Y_0.

The validity of (16) becomes an immediate consequence of (8) and (17).

Finally. the equalities $F(x + y) = F(x) + F(y)$ and $F(xy) = F(x)F(y)$, $x, y \in X$, result now from (16), the definition of F and the fact that J is a 2-sided ideal in Y_0. To finish the proof, note that $F(e) = f(e) + J = c + J = e_J$.

Observe that for any homomorphism $h : X \longrightarrow Y$ and any function $j : X \longrightarrow J$ the map $f := h + j$ yields a solution to (16). Does it provide also a solution to (1)? Generally not; indeed, take $h = 0$ and note that, in general, a function f assuming its values in the ideal $J = \{u \in Y_0 : (c + c^2)u = u(c + c^2) = 0\}$ fails to be a solution of (1) unless the corresponding ideal $I := \{x \in X : \ f(2x) = 0\}$ coincides with the whole of X (recall that $(c + c^2)f(x) = f(2x)$, $x \in X$).

Nevertheless, it turns out that some kind of the splitting discussed may actually be performable. Namely, we have the following

Theorem 5

Under the assumptions and denotations of Theorems 3 and 4 there exist functions $h_0 : X \longrightarrow (c^2 + c)Y_0$ and $j_0 : X \longrightarrow (c^2 - c)Y_0 \subset J$ such that

(a) *for all $x, y \in X$ one has*

$$\begin{cases} h_0(x + y) = h_0(x) + h_0(y). \\ 2h_0(xy) = h_0(x)h_0(y); \end{cases}$$

(b) *j_0 is even, $j_0(0) = 0$ and $j_0(x + 2y) = j_0(x)$, $x, y \in X$;*

(c) *$j_0(x)h_0(y) = h_0(x)j_0(y) = 0$, $x, y \in X$;*

(d) *$2f(x) = h_0(x) + j_0(x)$, $x, y \in X$.*

In particular, if $c^2 + c \in 2Y$ and the additive group $(Y, +)$ admits no elements of order 2, then f uniquely splits up into the sum $h + j$ where $h : X \longrightarrow Y$ is a homomorphism and $j : X \longrightarrow Y$ is an even solution of equation (1); if that is the case then we have also
(c') $j(x)h(y) = h(x)j(y) = 0,\ x, y \in X$.

Proof. Put

$$h_0(x) := (c^2 + c)f(x) \quad \text{and} \quad j_0(x) := (c^2 - c)f(x), \quad x \in X.$$

By (7) we infer that $h_0(x) = f(2x)$, $x \in X$, whereas with the aid of the substitution $y = -x$ in the first one from equations (13) we get

$$h_0(x) = f(2x) = f(x) - f(-x), \quad x \in X. \tag{18}$$

Setting $y = -e$ in (5), jointly with (6), gives

$$j_0(x) = f(x) - cf(x) = f(x) + f(-x), \quad x \in X. \tag{19}$$

Clearly, $h_0(X) \subset (c^2+c)Y_0$ and $j_0(X) \subset (c^2-c)Y_0$; the inclusion $(c^2-c)Y_0 \subset J$ follows from (17), (3) and the fact that $c^4 = c^2$ (see (9)). To check the properties (a)-(d), note that the additivity of h_0 results from its definition and from (8). Moreover, applying subsequently the definition of h_0, relation (8), (9), (7), and the second one from equations (13), we deduce that

$$\begin{aligned} h_0(xy) - h_0(x)h_0(y) &= (c^2 + c)f(xy) - (c^2 + c)^2 f(x)f(y) \\ &= (c^2 + c)\left[f(xy) - f(x)f(y)\right] + \left[(c^2 + c) - (c^2 + c)^2\right] f(x)f(y) \\ &= -(c^2 + c)f(x)f(y) = -f(2x)f(y) = -f(2xy) \\ &= -h_0(xy), \end{aligned}$$

for all $x, y \in X$. The evenness of j_0 and its vanishing at 0 are obvious. The third one from properties (b) may be derived from the first one of equations (13) and from (9) as follows:

$$\begin{aligned} j_0(2x + y) &= (c^2 - c)f(2x + y) = (c^2 - c)\left[f(2x) + f(y)\right] \\ &= (c^2 - c)(c^2 + c)f(x) + j_0(y) = (c^4 - c^2)f(x) + j_0(y) \\ &= j_0(y). \end{aligned}$$

Equalities (c) result easily from the definitions of h_0 and j_0, jointly with (3) and (9).

Finally, for every $x \in X$ one has

$$h_0(x) + j_0(x) = 2c^2 f(x) = 2f(x),$$

because of (6).

Now, assuming that $c^2 + c = 2d$ and the equality $2u = 0$ forces u to be equal to 0 in the ring Y, we put $h(x) := df(x)$ and $j(x) := (d - c)f(x)$ for all $x \in X$, so that $2h = h_0$, $2j = j_0$ and, consequently, $f = h + j$. The additivity of h results from the same property of h_0 whereas the multiplicativity can be derived as follows:

$$4h(xy) = 2h_0(xy) = h_0(x)h_0(y) = 4h(x)h(y), \quad x, y \in X$$

whence, by double application of the fact that the group $(Y, +)$ contains no elements of order 2, we arrive at

$$h(xy) = h(x)h(y), \quad x, y \in X.$$

To see that j is even it suffices to apply (b), whereas the property (c') is a simple consequence of (c). Now, making use of the fact that both f and h are solutions of (1), one can easily check that j satisfies equation (1) as well.

Finally, if there were two decompositions $h + j = \bar{h} + \bar{j}$ of the map f, in virtue of the oddness of h and the evenness of j, we would get $2j = 2\bar{j}$ forcing j to coincide with $\bar{j}$ whence, consequently, $h = \bar{h}$.

Thus the proof has been completed.

3. Even solutions

Plainly, whatever has been told about solutions of equation (1) till now applies, in particular, for even solutions. Therefore, preserving the notation used in the previous sections, in the sequel we will freely apply the results obtained overthere.

THEOREM 6

Let X be a unitary ring with e as a unit and let Y be an arbitrary ring. If $j : X \longrightarrow Y$ is an even solution of equation (1) such that $j(0) = 0$, then the ring Y_0 generated by $j(X)$ in Y is unitary with c^2 as a unit, where $c := j(e)$. Moreover, $c^2 = -c$ and $j\,|_{2X} = 0$, j satisfies the functional equation of Hosszú:

$$j(x + y - xy) + j(xy) = j(x) + j(y), \quad x, y \in X \tag{20}$$

and

$$2j(x)\,(j(x) - c) = 0, \quad x \in X. \tag{21}$$

If, in addition, the cardinality of the quotient ring $X/2X$ does not exceed 2, then the set $Z := \{x \in X : \ j(x) = 0\}$ yields a two-sided ideal of the ring X, $2X \subset Z$ and

$$j(x) = \begin{cases} 0 & \text{for } x \in Z, \\ c & \text{for } x \in Z + e. \end{cases} \tag{22}$$

Conversely, in that case, each function $j : X \longrightarrow Y$ *of the form* (22) *with* $-c^2 = c = j(e)$ *yields an even solution of equation* (1), *vanishing at* 0.

Proof. Replacing y by $-y$ in (1) and using the evenness of j we infer that

$$j(x+y) = j(x-y) \quad \text{for all } x, y \in X \tag{23}$$

whence

$$j(x+2y) = j(x) \quad \text{for all } x, y \in X. \tag{24}$$

Consequently, $j(2x) = 0$ for every $x \in X$, i.e. $j\,|_{2X} = 0$, as claimed. In particular (cf. (9)), $c^2 + c = j(2e) = 0$, i.e. $c^2 = -c$.

Setting $x+e$ and $y+e$ in place of x and y, respectively, in (1) one obtains that

$$j(x+y+2e) + j(xy+x+y+e) = j(x+e) + j(y+e) + j(x+e)j(y+e)$$

for all $x, y \in X$. Hence, by means of (24) and the fact that (see (4) and (6))

$$j(x+e) = c + cj(x) = c - c^2 j(x) = c - j(x)$$

for every $x \in X$, we deduce that

$$j(x+y) + c - j(xy+x+y) = c - j(x) + c - j(y) + (c - j(x))(c - j(y))$$

i.e.

$$j(x+y) - j(x+y+xy) = j(x)j(y)$$

for any $x, y \in X$. The latter equality jointly with (1) and (23) implies now that

$$j(x+y-xy) + j(xy) = j(x) + j(y), \quad x, y \in X,$$

which states that j satisfies the celebrated Hosszú equation (20).

Setting $y = x$ in (20), by virtue of (23) and (24), we derive the equality

$$2j(x^2) = 2j(x)$$

valid for every x from X. On the other hand, equation (1) applied for $f = j$ and $y = x$ gives

$$j(x^2) = j(2x) + j(x^2) = 2j(x) + j(x)^2 = 2j(x^2) + j(x)^2$$

stating that

$$j(x)^2 = -j(x^2)$$

for all $x \in X$. Thus,

$$2j(x) = 2j(x^2) = -2j(x)^2,$$

i.e.

$$2(j(x)^2 + j(x)) = 0$$

or, equivalently, $2j(x)(j(x)+c^2)=0$ for all $x \in X$, which gives (21) because of the equality $c^2=-c$.

Now, assume that $c=j(e)\neq 0$ and

$$2X \cup (2X+e)=X. \tag{25}$$

Then, necessarily, for all $x, y \in X$ one has $x+y \in 2X$ or $xy \in 2X$ whence, with the aid of (20), (23) and (24), we infer that for any x, y from X one has

$$\text{either} \quad j(xy)=0 \quad \text{or} \quad 2j(xy)=j(x)+j(y). \tag{26}$$

On the other hand, by setting $x+e$ in place of x in (20), in view of the equality $j(x+e)=c-j(x)$ valid for every $x \in X$, we deduce that

$$j(x+xy)-j(y+xy)=j(x)-j(y)$$

for all $x, y \in X$. Replacing here x by $x+y$ one obtains

$$j(x+y+xy+y^2)-j(y+xy+y^2)=j(x+y)-j(y)$$

whence

$$j(x+xy)-j(xy)=j(x+y)-j(y)$$

for all $x, y \in X$ since, clearly, $y+y^2=y(e+y)$ belongs to $2X$ for every $y \in X$. Now, by means of (5) and the equality $cj=-j$, it follows that

$$-2j(xy)=j(x+y)-j(x)-j(y)=j(x)j(y)-j(xy)$$

for all $x, y \in X$ because of (1). In other words,

$$-j(xy)=j(x)j(y) \qquad x, y \in X,$$

which states that $-j$ is multiplicative. In particular, Z yields a two-sided ideal of the ring X. Relation (25) and the obvious inclusion $2X \subset Z$ imply that

$$Z \cup (Z+e)=X$$

whence, by (23), for every $x \in X$ one has

$$j(x)=0 \quad \text{or} \quad 0=j(x-e)=j(x+e)=c-j(x), \qquad x \in X,$$

i.e.

$$j(x) \in \{0, c\} \quad \text{for every } x \in X.$$

This ends the proof because the last statement of the assertion is a subject to a straightforward verification.

4. Final remarks and comments

As we have already pointed out, we had no assumptions whatsoever upon the target ring, except for the last part of Theorem 5 where we have required the element c^2+c to belong to $2Y$. One might conjecture that such a property is forced by equation (1) itself. However, it is not the case. To see that, let us first consider an arbitrary ring Y with a nonzero element c such that $c^3 = c$. Then the ring generated in Y by the singleton $\{c\}$ is equal to

$$Y_0 := \{kc^2 + \ell c : \ k, \ell \in \mathbb{Z}\}$$

and we have the following

Proposition 2

The general solution $f : \mathbb{Z} \longrightarrow Y_0$ of equation (1) *such that $f(0) = 0$ and $f(1) = c$ is given by the formula*

$$f(k) = \begin{cases} \frac{1}{2}k(c^2 + c) & \text{for } k \in 2\mathbb{Z}, \\ \frac{1}{2}(k-1)c^2 + \frac{1}{2}(k+1)c & \text{for } k \in 2\mathbb{Z}+1. \end{cases}$$

Proof. ($\Longrightarrow$) By induction we show that $f(2n) = n(c^2+c)$ for all nonnegative integers n. Indeed, we have $f(0) = 0$ and assuming that that the above formula is valid for some integer $n \geqslant 0$ we get by (1), (7) and (3) that

$$\begin{aligned} f(2(n+1)) &= f(2n) + f(2) + f(2n)f(2) - f(2n \cdot 2) \\ &= (c + f(2) - c^2 - c)f(2n) + f(2) = cn(c^2 + c) + (c^2 + c) \\ &= (n+1)(c^2 + c). \end{aligned}$$

Likewise, we have $f(1) = c$ and assuming that $f(2n+1) = nc^2 + c$ for some integer $n \geqslant 0$ we get by (4)

$$\begin{aligned} f(2(n+1)+1) &= cf(2(n+1)) + c = c(n+1)(c^2 + c) + c \\ &= (n+1)(c^2 + c) + c \end{aligned}$$

because of the equality $c^3 = c$.

Since $f(-k) = -cf(k)$ (which results from (5) by setting $x = k$ and $y = -1$ and from (6)), we see that the corresponding formulas carry over to negative integers as well.

($\Longleftarrow$) A simple calculation based on distiguishing the following three cases for a given pair of integer arguments: both are even, both are odd, and one is even and the other is odd.

This ends the proof.

Now, consider the ring $M_2(\mathbb{Z})$ of all (2×2)-matrices with integer entries and take

$$c := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $c^2 = \mathrm{id}$ — the unit matrix so that $c^3 = c$ and

$$c^2 + c = \mathrm{id} + c = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

whence

$$Y_0 = \left\{ \begin{bmatrix} k+\ell & 0 \\ 0 & k-\ell \end{bmatrix} : \ k, \ell \in \mathbb{Z} \right\}$$

and, according to Proposition 2, the function $f : \mathbb{Z} \longrightarrow Y_0$ given by the formula

$$f(k) = \begin{cases} \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} & \text{for } k \in 2\mathbb{Z}, \\ \begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix} & \text{for } k \in 2\mathbb{Z}+1 \end{cases} \tag{27}$$

is a (unique) solution of equation (1) satisfying the conditions $f(0) = 0$ and $f(1) = c$; moreover, the ring generated by the actual range $f(\mathbb{Z})$ coincides with Y_0. If we had $c^2 + c \in 2Y_0$, then there would exist integers k, ℓ such that

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 2(kc^2 + \ell c) = \begin{bmatrix} 2(\ell + k) & 0 \\ 0 & 2(\ell - k) \end{bmatrix}$$

implying that 4 divides 2, a contradiction.

Noteworthy is the fact that treating function (27) as a map from $\mathbb{Z}$ into $M_2(\mathbb{Z})$ we have

$$c^2 + c = 2d \quad \text{with } d := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin Y_0 \,.$$

Therefore, since the additive group $(M_2(\mathbb{Z}), +)$ admits no elements of order 2, Theorem 5 guarantees that f uniquely splits up into the sum of a homomorphism $h := df : \mathbb{Z} \longrightarrow M_2(\mathbb{Z})$ and of an even solution $j := (d - c)f : \mathbb{Z} \longrightarrow M_2(\mathbb{Z})$ of equation (1). Actually, we have

$$h(k) = df(k) = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$$

for all $k \in \mathbb{Z}$, and

$$j(k) = (d-c)f(k) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} f(k) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{for } k \in 2\mathbb{Z}, \\ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \text{for } k \in 2\mathbb{Z}+1. \end{cases}$$

This shows that while dealing with solutions of equation (1) in the class of maps f from a ring X with a unit element e into a ring Y whose additive group has no elements of order 2 and such that $f(2e) \notin 2Y$ one should look for possible embedding of Y into a ring $\bar{Y}$ with the property that $f(2e) \in \bar{Y}$ and $(\bar{Y}, +)$ admits no elements of order 2.

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