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Stability of a generalization of the Fréchet functional equation

Abstract. We prove some stability and hyperstability results for a generalization of the well known Fréchet functional equation, stemming from one of the characterizations of the inner product spaces. As the main tool we use a fixed point theorem for some function spaces. We end the paper with some new inequalities characterizing the inner product spaces.

1. Introduction

The following theorem has been proved in [2] ($\mathbb{N}$ and $\mathbb{Z}$ stand, as usual, for the sets of all positive integers and integers, respectively; moreover, $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$).

Theorem 1
Let $(X,+)$ be a commutative group, $X_0 := X \setminus \{0\}$, Y be a Banach space, and $f\colon X \to Y$, $c\colon \mathbb{Z}_0 \to [0,\infty)$ and $L\colon {X_0}^3 \to [0,\infty)$ satisfy the following three conditions:

$$\mathcal{M} := \{m \in \mathbb{Z}_0 :\ c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1) < 1\} \neq \emptyset,$$

$$L(kx,ky,kz) \le c(k)L(x,y,z), \qquad \begin{array}{l} x,y,z \in X_0,\ m \in \mathcal{M}, \\ k \in \{-2m, m+1, -m, 2m+1\}, \end{array} \tag{1}$$

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z)\| \le L(x,y,z),$$

for all $x,y,z \in X_0$. Then there is a unique function $F\colon X \to Y$ satisfying

$$F(x+y+z) + F(x) + F(y) + F(z) = F(x+y) + F(x+z) + F(y+z) \tag{2}$$

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for all $x, y, z \in X$ *and such that*

$$\|f(x) - F(x)\| \leq \rho_L(x), \qquad x \in X_0,$$

where

$$\rho_L(x) := \inf_{m \in \mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}, \qquad x \in X_0.$$

Equation (2) is sometimes called the Fréchet functional equation. The reason for this is that M. Fréchet [12] used it to characterize the inner product spaces in a similar way as Jordan and von Neumann [15] did using the parallelogram law. Namely, he proved that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if, for all $x, y, z \in X$,

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2. \quad (3)$$

For more information we refer to [1, 10, 21, 22, 23].

Theorem 1 yields the subsequent characterization of inner product spaces (see [2, Corollary 5(i)]).

Corollary 2
Let X *be a normed space and* $X_0 := X \setminus \{0\}$. *Write*

$$D(x,y,z) := \left| \|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 - \|x+y\|^2 - \|x+z\|^2 - \|y+z\|^2 \right|$$

for $x, y, z \in X$. *Assume that there exist* $w_0, \alpha_i, s_i \in \mathbb{R}$ *such that* $\alpha_i > 0$ *and* $w_0 s_i < 0$ *for* $i = 1, 2, 3$ *and*

$$\sup_{x,y,z \in X_0} \frac{D(x,y,z)}{(\alpha_1\|x\|^{s_1} + \alpha_2\|y\|^{s_2} + \alpha_3\|z\|^{s_3})^{w_0}} < \infty.$$

Then X *is an inner product space.*

Actually it is assumed in [2, Corollary 5(i)] that $w_0 s_i > 0$ for $i = 1, 2, 3$, but it is a mistake; the inequality should be as in Corollary 2.

Equation (2) can also be written in the form

$$\Delta_{x,y,z} f(0) = 0 \quad \text{and} \quad f(0) = 0, \quad (4)$$

where Δ denotes the Fréchet difference operator defined by

$$\Delta_y f(x) = \Delta_y^1 f(x) := f(x+y) - f(x), \qquad x, y \in S,$$
$$\Delta_{t,z} := \Delta_t \circ \Delta_z, \quad \Delta_t^2 := \Delta_{t,t}, \qquad t, z \in S,$$
$$\Delta_{t,u,z} := \Delta_t \circ \Delta_u \circ \Delta_z, \quad \Delta_t^3 := \Delta_{t,t,t}, \qquad t, u, z \in S$$

for functions mapping a commutative group $(S, +)$ into a group (see [2]). Moreover, (2) can be written as

$$C^2 f(x,y,z) = 0,$$

where

$$C^2f(x,y,z) = Cf(x,y+z) - Cf(x,y) - Cf(x,z)$$

and

$$Cf(x,y) = f(x+y) - f(x) - f(y),$$

i.e. C^2f is the Cauchy difference of f of the second order.

It is known (see [2, 18]) that every solution f of (2), mapping a commutative group $(G,+)$ into a real linear space X, has the form $f = a + q$ with an additive function $a\colon G \to X$ and a quadratic function $q\colon G \to X$.

In this paper we show that results analogous to Theorem 1 can be proved for the following more general functional equation

$$\begin{aligned} A_1f(x+y+z) + A_2f(x) + A_3f(y) + A_4f(z) \\ = A_5f(x+y) + A_6f(x+z) + A_7f(y+z), \end{aligned} \tag{5}$$

in the class of functions mapping a commutative group X into a Banach space Y over a field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, where $A_1,\ldots,A_7 \in \mathbb{K}$ are fixed ($\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively). It is easily seen that (5) becomes (2) with $A_1 = \ldots = A_7 = 1$.

The results we prove correspond also to some outcomes in [8, 11, 16, 19, 25].

The results in [2] as well as our main theorem have been motivated by the notion of hyperstability of functional equations (see, e.g., [3, 4, 5, 13, 20]), introduced in connection with the issue of stability of functional equations (for more details see, e.g., [14, 17]).

2. Auxiliary fixed point result

We need the subsequent fixed point theorem proved for function spaces in [6]; it will be the main tool in the proof of our main theorem ($\mathbb{R}_+$ stands for the set of nonnegative reals and A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$). For related outcomes we refer to [7, 9]; a similar approach to stability of functional equations has been already applied in [5, 24].

Theorem 3
Let the following three hypotheses be valid.

(H1) *S is a nonempty set, E is a Banach space, and functions $f_1,\ldots,f_k\colon S \to S$ and $L_1,\ldots,L_k\colon S \to \mathbb{R}_+$ are given.*

(H2) *$\mathcal{T}\colon E^S \to E^S$ is an operator satisfying the inequality*

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi,\mu \in E^S,\ x \in S.$$

(H3) *$\Lambda\colon {\mathbb{R}_+}^S \to {\mathbb{R}_+}^S$ is defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^S,\ x \in S.$$

Assume that functions $\varepsilon\colon S \to \mathbb{R}_+$ and $\varphi\colon S \to E$ fulfil the following two conditions

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \qquad x \in S,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in S.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in S.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in S.$$

3. The main result

The next theorem is the main result of the paper.

Theorem 4
Let $(X,+)$ be a commutative group, $\widehat{X} := X^3 \setminus \{(0,0,0)\}$, Y be a Banach space, and $A_1,\ldots,A_7 \in \mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ such that $A_1 \neq 0$ and

$$A_2 + A_3 + A_4 = A_5 + A_6 + A_7. \tag{6}$$

Assume that $f\colon X \to Y$, $c\colon \mathbb{Z}_0 \to [0,\infty)$ and $L\colon \widehat{X} \to [0,\infty)$ satisfy (1) and the following two conditions:

$$\begin{aligned}\mathcal{M} := \{m \in \mathbb{Z}_0 : |A_7|c(-2m) + |A_5+A_6|c(m+1)& \\ + |A_3+A_4|c(-m) + |A_2|c(2m+1) < |A_1|\} \neq \emptyset,&\end{aligned} \tag{7}$$

$$\begin{aligned}\|A_1 f(x+y+z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x+y)& \\ - A_6 f(x+z) - A_7 f(y+z)\| \leq L(x,y,x), \qquad (x,y,z) \in \widehat{X}.&\end{aligned} \tag{8}$$

Then there exists a unique function $F\colon X \to Y$ satisfying (5) for all $x,y,z \in X$ and such that

$$\|f(x) - F(x)\| \leq \rho_L(x), \qquad x \in X_0 := X \setminus \{0\}, \tag{9}$$

where

$$\rho_L(x) := \inf_{m\in\mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{|A_1| - \beta_m}, \tag{10}$$

$$\beta_m := |A_7|c(-2m) + |A_5+A_6|c(m+1) + |A_3+A_4|c(-m) + |A_2|c(2m+1).$$

Proof. First we consider the case $A_1 = 1$. Replacing x by $(2m+1)x$ and taking $y = z = -mx$ in (8) we obtain

$$\begin{aligned}\|f(x) &+ A_2 f((2m+1)x) + (A_3+A_4)f(-mx)\\ &- (A_5+A_6)f((m+1)x) - A_7 f(-2mx)\| \\ &\le L((2m+1)x, -mx, -mx) =: \varepsilon_m(x), \qquad x \in X_0,\ m \in \mathbb{Z}_0.\end{aligned} \tag{11}$$

Next put

$$\begin{aligned}\mathcal{T}_m\xi(x) &:= A_7\xi(-2mx) + (A_5+A_6)\xi((m+1)x)\\ &- (A_3+A_4)\xi(-mx) - A_2\xi((2m+1)x), \qquad \xi \in Y^X,\ x \in X,\ m \in \mathbb{Z}_0.\end{aligned}$$

It is easy to notice that, by (6),

$$\mathcal{T}_m^n f(0) = 0, \qquad n \in \mathbb{N},\ m \in \mathbb{Z}_0, \tag{12}$$

and inequality (11) can be written as

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \qquad x \in X_0,\ m \in \mathbb{Z}_0.$$

Define an operator $\Lambda_m\colon \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ for $m \in \mathbb{Z}_0$ by

$$\begin{aligned}\Lambda_m\eta(x) &:= |A_7|\eta(-2mx) + |A_5+A_6|\eta((m+1)x)\\ &+ |A_3+A_4|\eta(-mx) + |A_2|\eta((2m+1)x)\end{aligned}$$

for $\eta \in \mathbb{R}_+^{X_0}$ and $x \in X_0$. Notice that, for each $m \in \mathbb{Z}_0$, the operator $\Lambda := \Lambda_m$ has the form described in (H3) with $k = 4$, $S = X_0$, $E = Y$ and

$$\begin{aligned}&f_1(x) = -2mx, \quad f_2(x) = (m+1)x, \quad f_3(x) = -mx, \quad f_4(x) = (2m+1)x,\\ &L_1(x) = |A_7|, \quad L_2(x) = |A_5+A_6|, \quad L_3(x) = |A_3+A_4|, \quad L_4(x) = |A_2|, \quad x \in X_0.\end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X_0}$, $x \in X_0$, $m \in \mathbb{Z}_0$,

$$\begin{aligned}\|\mathcal{T}_m\xi(x) &- \mathcal{T}_m\mu(x)\|\\ &= \|A_7\xi(-2mx) + (A_5+A_6)\xi((m+1)x) - (A_3+A_4)\xi(-mx)\\ &\quad - A_2\xi((2m+1)x) - A_7\mu(-2mx) - (A_5+A_6)\mu((m+1)x)\\ &\quad + (A_3+A_4)\mu(-mx) + A_2\mu((2m+1)x)\|\\ &\le |A_7|\|(\xi-\mu)(-2mx)\| + |A_5+A_6|\|(\xi-\mu)((m+1)x)\|\\ &\quad + |A_3+A_4|\|(\xi-\mu)(-mx)\| + |A_2|\|(\xi-\mu)((2m+1)x)\|\\ &= \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|,\end{aligned}$$

where

$$(\xi-\mu)(y) := \xi(y) - \mu(y), \qquad y \in X_0.$$

Note that, in view of (1), we have

$$\Lambda_m\varepsilon_k(x) \le \beta_m\varepsilon_k(x), \qquad k, m \in \mathbb{Z}_0,\ x \in X_0. \tag{13}$$

By induction it is easy to show that the linearity of Λ_m implies

$$\Lambda_m^n \varepsilon_k(x) \le (\beta_m)^n \varepsilon_k(x) \tag{14}$$

for $x \in X_0$, $k, n \in \mathbb{N}$. So, we receive the following estimation

$$\varepsilon_m^*(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \le \sum_{n=0}^{\infty} (\beta_m)^n \varepsilon_m(x) = \frac{\varepsilon_m(x)}{1-\beta_m}, \qquad m \in \mathcal{M},\ x \in X_0.$$

By Theorem 3 (with $S = X_0$ and $E = Y$), for each $m \in \mathcal{M}$ there exists a function $F'_m\colon X_0 \to Y$ such that

$$\begin{aligned} F'_m(x) = A_7 F'_m(-2mx) &+ (A_5 + A_6) F'_m((m+1)x) \\ &- (A_3 + A_4) F'_m(-mx) - A_2 F'_m((2m+1)x), \qquad x \in X_0 \end{aligned}$$

and

$$\|f(x) - F'_m(x)\| \le \frac{\varepsilon_m(x)}{1-\beta_m}, \qquad x \in X_0.$$

Moreover,

$$F'_m(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X_0,\ m \in \mathcal{M}.$$

Now, define $F_m\colon X \to Y$ by

$$F_m(0) = 0, \quad F_m(x) := F'_m(x), \qquad x \in X_0,\ m \in \mathcal{M}.$$

Then it is easily seen that, by (12),

$$F_m(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x), \qquad x \in X,\ m \in \mathcal{M}.$$

Next, by induction we show that

$$\begin{aligned} \|\mathcal{T}_m^n f(x+y+z) &+ A_2 \mathcal{T}_m^n f(x) + A_3 \mathcal{T}_m^n f(y) + A_4 \mathcal{T}_m^n f(z) \\ &- A_5 \mathcal{T}_m^n f(x+y) - A_6 \mathcal{T}_m^n f(x+z) - A_7 \mathcal{T}_m^n f(y+z)\| \\ \le (\beta_m)^n & L(x,y,z) \end{aligned} \tag{15}$$

for every $(x,y,z) \in \widehat{X}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $m \in \mathcal{M}$.

Fix $m \in \mathcal{M}$. For $n = 0$ condition (15) becomes (8). So, take $l \in \mathbb{N}_0$ and suppose that (15) holds for $n = l$ and $(x,y,z) \in \widehat{X}$. Then we have

$$\begin{aligned} \|\mathcal{T}_m^{l+1} f(x+y+z) &+ A_2 \mathcal{T}_m^{l+1} f(x) + A_3 \mathcal{T}_m^{l+1} f(y) + A_4 \mathcal{T}_m^{l+1} f(z) \\ &- A_5 \mathcal{T}_m^{l+1} f(x+y) - A_6 \mathcal{T}_m^{l+1} f(x+z) - A_7 \mathcal{T}_m^{l+1} f(y+z)\| \\ = \|A_7 \mathcal{T}_m^l f(&-2m(x+y+z)) + (A_5+A_6)\mathcal{T}_m^l f((m+1)(x+y+z)) \\ &- (A_3+A_4)\mathcal{T}_m^l f(-m(x+y+z)) - A_2 \mathcal{T}_m^l f((2m+1)(x+y+z)) \\ &+ A_7 A_2 \mathcal{T}_m^l f(-2mx) + (A_5+A_6) A_2 \mathcal{T}_m^l f((m+1)x) \\ &- (A_3+A_4) A_2 \mathcal{T}_m^l f(-mx) - A_2 A_2 \mathcal{T}_m^l f((2m+1)x) \end{aligned}$$

$$\begin{aligned}
&+ A_7A_3\mathcal{T}_m^l f(-2my) + (A_5+A_6)A_3\mathcal{T}_m^l f((m+1)y)\\
&- (A_3+A_4)A_3\mathcal{T}_m^l f(-my) - A_2A_3\mathcal{T}_m^l f((2m+1)y)\\
&+ A_7A_4\mathcal{T}_m^l f(-2mz) + (A_5+A_6)A_4\mathcal{T}_m^l f((m+1)z)\\
&- (A_3+A_4)A_4\mathcal{T}_m^l f(-mz) - A_2A_4\mathcal{T}_m^l f((2m+1)z)\\
&- A_7A_5\mathcal{T}_m^l f(-2m(x+y)) - (A_5+A_6)A_5\mathcal{T}_m^l f((m+1)(x+y))\\
&+ (A_3+A_4)A_5\mathcal{T}_m^l f(-m(x+y)) + A_2A_5\mathcal{T}_m^l f((2m+1)(x+y))\\
&- A_7A_6\mathcal{T}_m^l f(-2m(x+z)) - (A_5+A_6)A_6\mathcal{T}_m^l f((m+1)(x+z))\\
&+ (A_3+A_4)A_6\mathcal{T}_m^l f(-m(x+z)) + A_2A_6\mathcal{T}_m^l f((2m+1)(x+z))\\
&- A_7A_7\mathcal{T}_m^l f(-2m(y+z)) - (A_5+A_6)A_7\mathcal{T}_m^l f((m+1)(y+z))\\
&+ (A_3+A_4)A_7\mathcal{T}_m^l f(-m(y+z)) + A_2A_7\mathcal{T}_m^l f((2m+1)(y+z))\|\\
\le{}& (\beta_m)^l\big(|A_7|L(-2mx,-2my,-2mz)\\
&+ |A_5+A_6|L((m+1)x,(m+1)y,(m+1)z)\\
&+ |A_3+A_4|L(-mx,-my,-mz) + |A_2|L((2m+1)x,(2m+1)y,(2m+1)z)\big)\\
\le{}& (\beta_m)^{l+1}L(x,y,z)
\end{aligned}$$

for every $(x,y,z)\in\widehat{X}$, which ends the proof of (15).

Letting $n\to\infty$ in (15), we obtain

$$\begin{aligned}
&F_m(x+y+z) + A_2F_m(x) + A_3F_m(y) + A_4F_m(z)\\
&\qquad = A_5F_m(x+y) + A_6F_m(x+z) + A_7F_m(y+z), \qquad (x,y,z)\in\widehat{X}.
\end{aligned} \tag{16}$$

So, we have proved that, for each $m\in\mathcal{M}$ there exists a function $F_m\colon X\to Y$ satisfying the equation (5) for $(x,y,z)\in\widehat{X}$ and such that

$$\|f(x)-F_m(x)\| \le \frac{\varepsilon_m(x)}{1-\beta_m}, \qquad x\in X_0. \tag{17}$$

Next, we show that $F_m=F_k$ for all $m,k\in\mathcal{M}$. So, fix $m,k\in\mathcal{M}$. Note that F_k satisfies (16) with m replaced by k. Hence, replacing x by $(2m+1)x$ and taking $y=z=-mx$ in (16), we obtain that $\mathcal{T}_mF_j=F_j$ for $j=m,k$ and

$$\|F_m(x)-F_k(x)\| \le \frac{\varepsilon_m(x)}{1-\beta_m} + \frac{\varepsilon_k(x)}{1-\beta_k}, \qquad x\in X_0,$$

whence, by the linearity of Λ and (14),

$$\begin{aligned}
\|F_m(x)-F_k(x)\| = \|\mathcal{T}_m^nF_m(x)-\mathcal{T}_m^nF_k(x)\| &\le \frac{\Lambda_m^n\varepsilon_m(x)}{1-\beta_m} + \frac{\Lambda_m^n\varepsilon_k(x)}{1-\beta_k}\\
&\le \frac{(\beta_m)^n\varepsilon_m(x)}{1-\beta_m} + \frac{(\beta_k)^n\varepsilon_k(x)}{1-\beta_k}
\end{aligned}$$

for every $x\in X_0$ and $n\in\mathbb{N}$. Therefore, letting $n\to\infty$ we get $F_m=F_k=:F$. Thus, in view of (17), we have proved that

$$\|f(x)-F(x)\| \le \frac{\varepsilon_m(x)}{1-\beta_m}, \qquad x\in X,\ x\neq 0,\ m\in\mathcal{M},$$

whence we derive (9).

Since (in view of (16)) it is easy to notice that F is a solution to (5) (i.e. (5) holds for all $x, y, z \in X$), it remains to prove the statement concerning the uniqueness of F. So, let $G\colon X \to Y$ be also a solution of (5) and $\|f(x) - G(x)\| \le \rho_L(x)$ for $x \in X$, $x \neq 0$. Then

$$\|G(x) - F(x)\| \le 2\rho_L(x), \qquad x \in X,\, x \neq 0. \tag{18}$$

Further, $\mathcal{T}_m G = G$ for each $m \in \mathbb{Z}_0$. Hence, with a fixed $m \in \mathcal{M}$, by (14) we get

$$\begin{aligned}\|G(x) - F(x)\| = \|\mathcal{T}_m^n G(x) - \mathcal{T}_m^n F(x)\| \le 2\Lambda_m^n \rho_L(x) \le \frac{2\Lambda_m^n \varepsilon_m(x)}{1-\beta_m}\\ \le \frac{2(\beta_m)^n \varepsilon_m(x)}{1-\beta_m}\end{aligned}$$

for $x \in X_0$ and $n \in \mathbb{N}$. Consequently, letting $n \to \infty$ we obtain that $G = F$, and that ends the proof in the case $A_1 = 1$.

If $A_1 \neq 1$, then (8) can be rewritten in the form

$$\begin{aligned}\|f(x+y+z) + A_2' f(x) + A_3' f(y) + A_4' f(z) - A_5' f(x+y)\\ - A_6' f(x+z) - A_7' f(y+z)\| \le L'(x,y,z), \qquad (x,y,z) \in \widehat{X},\end{aligned} \tag{19}$$

where

$$A_i' := \frac{A_i}{A_1}, \qquad i = 2, \ldots, 7,$$

$$L'(x,y,z) := \frac{L(x,y,z)}{|A_1|}, \qquad (x,y,z) \in \widehat{X},$$

and it is easily seen that the statement can be easily deduced from the case $A_1 = 1$.

The following hyperstability result can be deduced from Theorem 4. It corresponds to the recent hyperstability outcomes in [5, 24] and some classical stability results concerning the Cauchy equation (see, e.g., [3, p.3], [14, p.15,16] and [17, p.2]).

Corollary 5
Let $(X,+)$ be a commutative group, $\widehat{X} := X^3 \setminus \{(0,0,0)\}$, Y be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A_1, \ldots, A_7 \in \mathbb{K}$, $A_1 \neq 0$ and (6) *be valid. Assume that $f\colon X \to Y$, $c\colon \mathbb{Z}_0 \to [0,\infty)$ and $L\colon \widehat{X} \to [0,\infty)$ satisfy conditions* (1), (7), (8) *and*

$$\sup_{m \in \mathcal{M}} \beta_m < |A_1|, \qquad \inf_{m \in \mathcal{M}} L((2m+1)x, -mx, -mx) = 0, \qquad x \in X,\, x \neq 0.$$

Then there exists a function $F\colon X \to Y$ satisfying (5) *for all $x, y, z \in X_0$ such that $f(x) = F(x)$ for $x \in X_0$.*

Proof. It is easily seen that $\rho_L(x) = 0$ for each $x \in X_0$, where ρ_L is defined by (10). Hence Theorem 4 implies the statement.

Remark 6
If, in Theorem 4,

$$L(x,y,z) = (\alpha_1\|x\|^p + \alpha_2\|y\|^p + \alpha_3\|z\|^p)^w, \qquad (x,y,z) \in \widehat{X}, \tag{20}$$

with some $\alpha_i, p, w \in \mathbb{R}$ such that $\alpha_i > 0$ for $i = 1,2,3$, $p > 0$ and $w < 0$, then it is easily seen that the function c can, for instance, have the form $c(m) = |m|^{pw}$.

The next corollary generalizes Corollary 2 to some extent; it shows possible application of the main result of this paper.

Corollary 7
Let X be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $X_0 := X\setminus\{0\}$, $A_1, \ldots, A_7 \in \mathbb{K}$, $A_1 \neq 0$, and (6) be valid. Write

$$\begin{aligned}\widehat{D}(x,y,z) := \big|A_1\|x+y+z\|^2 + A_2\|x\|^2 + A_3\|y\|^2 + A_4\|z\|^2 \\ - A_5\|x+y\|^2 - A_6\|x+z\|^2 - A_7\|y+z\|^2\|\big|\end{aligned}$$

for $x,y,z \in X$. Assume that there exist $\alpha_i, w, p \in \mathbb{R}$ such that $p > 0$, $w < 0$, $\alpha_i > 0$ for $i = 1,2,3$ and

$$\sup_{(x,y,z)\in\widehat{X}} \frac{\widehat{D}(x,y,z)}{(\alpha_1\|x\|^p + \alpha_2\|y\|^p + \alpha_3\|z\|^p)^w} < \infty.$$

Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$ for $x \in X$. Then, with L and c of the forms described in Remark 6, from Corollary 5 and (6) we easily derive that f is a solution to equation (5).

We show that $A_1 = \ldots = A_7$. Replacing x by αx, y by βx and z by γx in (5), where $\alpha, \beta, \gamma \in \mathbb{K}$, we obtain

$$\begin{aligned}&(A_1(\alpha+\beta+\gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2)\|x\|^2 \\ &\quad = (A_5(\alpha+\beta)^2 + A_6(\alpha+\gamma)^2 + A_7(\beta+\gamma)^2)\|x\|^2, \qquad x \in X,\end{aligned}$$

whence

$$\begin{aligned}&A_1(\alpha+\beta+\gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2 \\ &\quad = A_5(\alpha+\beta)^2 + A_6(\alpha+\gamma)^2 + A_7(\beta+\gamma)^2, \qquad \alpha,\beta,\gamma \in \mathbb{K}.\end{aligned} \tag{21}$$

Taking $\alpha = 1$, $\beta = \gamma = 0$ in (21) we have $A_1 + A_2 = A_5 + A_6$, and next, with $\beta = -\alpha = 1$ and $\gamma = 0$ in (21) we obtain the equality $A_2 + A_3 = A_6 + A_7$ and consequently

$$A_1 - A_3 = A_5 - A_7. \tag{22}$$

Analogously, with $\beta = 1$, $\alpha = \gamma = 0$ and $\beta = -\gamma = 1$, $\alpha = 0$ we obtain

$$A_1 - A_4 = A_7 - A_6, \tag{23}$$

and $\gamma = 1$, $\alpha = \beta = 0$ and $\alpha = -\gamma = 1$, $\beta = 0$ gives

$$A_1 - A_2 = A_6 - A_5. \tag{24}$$

Further, inserting, $1 = \alpha = -\beta = -\gamma$, $1 = -\alpha = \beta = -\gamma$ and $1 = -\alpha = -\beta = \gamma$ into (21), we respectively get

$$A_1 + A_2 + A_3 + A_4 = 4A_7,$$
$$A_1 + A_2 + A_3 + A_4 = 4A_6,$$
$$A_1 + A_2 + A_3 + A_4 = 4A_5,$$

whence $A_5 = A_6 = A_7$ and consequently, by (22)–(24), $A_1 = A_2 = A_3 = A_4$. This and (6) finally yield $A_1 = \ldots = A_7$.

Thus we have proved that (3) holds, which implies the statement.

References

[1] C. Alsina, J. Sikorska, M.S. Tomás, *Norm derivatives and characterizations of inner product spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010. Cited on 70.

[2] A. Bahyrycz, J. Brzdęk, M. Piszczek, J. Sikorska, *Hyperstability of the Fréchet equation and a characterization of inner product spaces*, J. Funct. Spaces Appl. 2013, Art. ID 496361, 6 pp. Cited on 69, 70 and 71.

[3] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal. 2012, Art. ID 716936, 41 pp. Cited on 71 and 76.

[4] J. Brzdęk, *Remarks on hyperstability of the Cauchy functional equation*, Aequationes Math. **86** (2013), no. 3, 255–267. Cited on 71.

[5] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141** (2013), no. 1-2, 58–67. Cited on 71 and 76.

[6] J. Brzdęk, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal. **74** (2011), no. 17, 6728–6732. Cited on 71.

[7] J. Brzdęk, K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal. **74** (2011), no. 18, 6861–6867. Cited on 71.

[8] M. Chudziak, *On solutions and stability of functional equations connected to the Popoviciu inequality*, Ph.D. Thesis (in Polish), Pedagogical University of Cracow (Poland), Cracow 2012. Cited on 71.

[9] L. Cădariu, L. Găvruţa, P. Găvruţa, *Fixed points and generalized Hyers-Ulam stability*, Abstr. Appl. Anal. 2012, Art. ID 712743, 10 pp. Cited on 71.

[10] S.S. Dragomir, *Some characterizations of inner product spaces and applications*, Studia Univ. Babeş-Bolyai Math. **34** (1989), no. 1, 50–55. Cited on 70.

[11] W. Fechner, *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. **322** (2006), no. 2, 774–786. Cited on 71.

[12] M. Fréchet, *Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert*, Ann. of Math. (2) **36** (1935), no. 3, 705–718. Cited on 70.

[13] E. Gselmann, *Hyperstability of a functional equation*, Acta Math. Hungar. **124** (2009), no. 1-2, 179–188. Cited on 71.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables. Progress in Nonlinear Differential Equations and their Applications*, 34. Birkhäuser Boston, Inc., Boston, MA, 1998. Cited on 71 and 76.

[15] P. Jordan, J. Von Neumann, *On inner products in linear, metric spaces*, Ann. of Math. (2) **36** (1935), no. 3, 719–723. Cited on 70.

[16] S.-M. Jung, *On the Hyers-Ulam stability of the functional equation that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), no. 1, 126–137. Cited on 71.

[17] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer Optimization and Its Applications, 48, Springer, New York, 2011. Cited on 71 and 76.

[18] P. Kannappan, *Functional equations and inequalities with applications*, Springer Monographs in Mathematics, Springer, New York, 2009. Cited on 71.

[19] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, Journal of the Chungcheong Mathematical Society **22** (2009), no. 2, 201–209. Cited on 71.

[20] G. Maksa, Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17** (2001), no. 2, 107–112. Cited on 71.

[21] M.S. Moslehian, J.M. Rassias, *A characterization of inner product spaces concerning an Euler-Lagrange identity*, Commun. Math. Anal. **8** (2010), no. 2, 16–21. Cited on 70.

[22] K. Nikodem, Z. Pales, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (2011), no. 1, 83–87. Cited on 70.

[23] Th. M. Rassias, *New characterizations of inner product spaces*, Bull. Sci. Math. (2) **108** (1984), no. 1, 95–99. Cited on 70.

[24] M. Piszczek, *Remark on hyperstability of the general linear equation*, Aequationes Math. **88** (2014), no. 1-2, 163–168. Cited on 71 and 76.

[25] J. Sikorska, *On a direct method for proving the Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl. **372** (2010), no. 1, 99–109. Cited on 71.

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