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On the stability of the squares of some functional equations

Dedicated to Professors Roman Ger and Zygfryd Kominek on their 70th anniversary.

Abstract. We consider the stability, the superstability and the inverse stability of the functional equations with squares of Cauchy's, of Jensen's and of isometry equations and the stability in Ulam-Hyers sense of the alternation of functional equations and of the equation of isometry.

1. Introduction

The Cauchy's equation of an additive function

$$f(x+y) = f(x) + f(y),$$

being Ulam-Hyers stable on the adequate suppositions [5], is not superstable, i.e. it is not true that if for every unbounded function f the function $f(x+y)-f(x)-f(y)$ is bounded, then f is additive (in fact, for the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x + c$ with $c \neq 0$, the function $f(x+y) - f(x) - f(y)$ is bounded and the function f is not additive). The equations

$$f(x+y) - f(x) = f(y)$$

and

$$f(x+y) - f(x) - f(y) = 0$$

have the same property.

The square of the last equation

$$[f(x+y) - f(x) - f(y)]^2 = 0$$

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is evidently not superstable. For the function f from an Abelian semigroup to a finite-dimensional normed algebra without divisors of zero, B. Batko has formulated ([3]) that, if the function $[f(x+y)]^2 - [f(x)+f(y)]^2$ is bounded, then f is bounded or it is an additive function, thus the equation

$$[f(x+y)]^2 = [f(x)+f(y)]^2 \tag{1}$$

is superstable.

The proof of this theorem is based on the observation that if the function $[f(x+y)]^2 - [f(x)+f(y)]^2$ is bounded, then the function $[f(x+y)+f(x)+f(y)][f(x+y)-f(x)-f(y)]$ is bounded too, which is not proved in [3]. The proof in [3] is valid for the function

$$[f(x+y)+f(x)+f(y)][f(x+y)-f(x)-f(y)]$$

or if the algebra is commutative.

Therefore, the following question arises:

Question 1.1
Is the Batko's result true for the algebra of quaternions?

Notice that such an algebra satisfies all the assumptions of the Batko's theorem.

We consider below the stability, the superstability and the inverse stability of the functional equations with squares of Cauchy's, of Jensen's and of isometry equations and the stability in Ulam-Hyers sense of the alternation of functional equations and of the equation of isometry.

2. Superstability and stability

In connection with the Batko's result we begin with

Theorem 2.1
For the function f from an Abelian semigroup G to a Banach algebra $\mathcal{A}$ satisfying the assumption

$$\forall a_n, b_n \in \mathcal{A}^{\mathbb{N}}: \ [a_n b_n \to 0 \ \textit{and} \ |b_n| = 1 \implies a_n \to 0] \tag{C}$$

if the function $[f(x+y)-f(x)-f(y)][f(x+y)+f(x)+f(y)]$ is bounded, then f is bounded or it is an additive function.

Before the proof of this theorem we prove the following result.

Lemma 2.2
Condition (C) *implies*

$$\forall a_n, b_n \in \mathcal{A}^{\mathbb{N}}: \ [a_n b_n \to 0 \ \textit{and} \ b_n \to b \neq 0 \implies a_n \to 0]. \tag{C$_1$}$$

Proof. Since $b_n \to b \neq 0$ it is possible to assume that $|b_n| \neq 0$, thus $|b_n|^{-1} \to |b|^{-1}$. We have $a_n b_n |b_n|^{-1} \to 0$, hence $a_n \to 0$, by (C).

Proof of Theorem 2.1. Assume that

$$\big|[f(x+y) - f(x) - f(y)][f(x+y) + f(x) + f(y)]\big| \leq \delta \tag{2}$$

for a $\delta \geq 0$ and for all $x, y \in G$.

Let us observe that there exist $\gamma, \varepsilon > 0$ such that

$$|f(x+y) - f(x) - f(y)| \leq \gamma \quad \text{or} \quad |f(x+y) + f(x) + f(y)| \leq \varepsilon$$

since, in the contrary case, there would exist sequences x_n and y_n such that

$$|f(x_n+y_n) - f(x_n) - f(y_n)| \to +\infty \quad \text{and} \quad |f(x_n+y_n) + f(x_n) + f(y_n)| \to +\infty.$$

Putting $\alpha_n := f(x_n+y_n) + f(x_n) + f(y_n)$ (we may assume that $\alpha_n \neq 0$) we obtain by (2),

$$|[f(x_n+y_n) - f(x_n) - f(y_n)]\alpha_n||\alpha_n|^{-1} \leq \delta|\alpha_n|^{-1},$$

which for $n \to \infty$ contradicts (C).

By Theorem 1 in [2] there exists an additive function $a\colon G \to \mathcal{A}$ such that

$$|f(x) - a(x)| \leq \max(\gamma, \varepsilon)$$

for $x \in G$.

If f is unbounded, then the function a is unbounded too, thus there exists $x_0 \in G$ such that $a(x_0) \neq 0$. We obtain $f(x) = a(x) + b(x)$, where $b\colon G \to \mathcal{A}$ is bounded. By (2) we have

$$\big|[b(x+y) - b(x) - b(y)][2a(x+y) + b(x+y) + b(x) + b(y)]\big| \leq \delta. \tag{3}$$

Thus for $x = nx_0$,

$$\big|[b(nx_0+y) - b(nx_0) - b(y)][2a(nx_0+y) + b(nx_0+y) + b(nx_0) + b(y)]\big| \leq \delta.$$

Dividing this inequality by n, since

$$\lim_{n\to+\infty} \frac{2a(nx_0+y) + b(nx_0+y) + b(nx_0) + b(y)}{n} = 2a(x_0) \neq 0,$$

we get, by Lemma 2.2,

$$b(y) = \lim_{n\to+\infty} [b(nx_0+y) - b(nx_0)].$$

Thus

$$b(x+y) = \lim_{n\to+\infty} [b(nx_0+x+y) - b(nx_0)].$$

Putting $y + nx_0$ in place of y in (3) we have, by the same method,

$$b(x) = \lim_{n\to+\infty} [b(x+y+nx_0) - b(y+nx_0)]$$

hence $b(x+y) = b(x) + b(y)$. As a consequence $b(x) \equiv 0$, as an additive and bounded function, which yields that $f = a + b$ is additive.

Remark 2.3
If the norm in $\mathcal{A}$ is super-multiplicative, then (C) is true since

$$0 \le |a_n| = |a_n||b_n| \le |a_n b_n|.$$

Notice that if (C) holds true, then (2) implies

$$\exists \gamma, \varepsilon : \ \big[|f(x+y) - f(x) - f(y)| \le \gamma \quad \text{or} \quad |f(x+y) + f(x) + f(y)| \le \varepsilon \big],$$

(see the beginning of the proof of Theorem 2.1). The inverse implication is not true, e.g., for $G = \mathcal{A} = \mathbb{R}$ and $f(x) = x + \gamma$.

The algebra $\mathcal{A}$, which satisfies (C), has no topological zero divisors. Indeed, if there existed $x \ne 0$ and x_n, $n \in \mathbb{N}$, in $\mathcal{A}$ such that $|x_n| = 1$ and $xx_n \to 0$, then (C) would be not true for $a_n = x$ and $b_n = x_n$. Thus $\mathcal{A} = \mathbb{R}$ or $\mathcal{A} = \mathbb{C}$ or $\mathcal{A}$ is the field of quaternions ([12] p. 62).

From now on we assume, unless it is not stated otherwise, that $\mathcal{A} = \mathbb{R}$ or $\mathcal{A} = \mathbb{C}$. The condition (C) is evident in this case.

It is possible to prove the above theorem by the method used in [9].

Theorem 2.4
Let $(G, +)$ be an Abelian semigroup.

I. *Then for the function $f \colon G \to \mathcal{A}$ equation*

$$[f(x+y) - f(x)]^2 = [f(y)]^2 \tag{4}$$

is superstable, i.e. if $[g(x+y) - g(x)]^2 - [g(y)]^2$ is bounded, then g is bounded or it is a solution of (4).

II. *Moreover, if G is a group, then the function f is bounded or additive.*

Lemma 2.5
If $a, a_n \in \mathcal{A}$ and $a_n^2 \to a^2$, then a_n is bounded.

Proof. If a_n is unbounded, then there exist a subsequence a_{k_n} such that $|a_{k_n}| \to +\infty$. Since $|a_n - a| \ge |a_n| - |a|$ we obtain $|a_{k_n} - a| \to +\infty$, thus it is possibly to assume that $|a_{k_n} - a| \ne 0$. We have

$$\lim_{n \to +\infty} (a_{k_n} + a)(a_{k_n} - a)|a_{k_n} - a|^{-1} = 0,$$

thus $a_{k_n} \to a$, by (C). The sequence a_{k_n} is bounded, a contradiction.

Proof of Theorem 2.4. Part I. Assume that

$$\begin{aligned} &\big|[f(x+y)-f(x)]^2-[f(y)]^2\big| \\ &\qquad = \big|[f(x+y)-f(x)-f(y)][f(x+y)-f(x)+f(y)]\big| \\ &\qquad \le \delta \end{aligned} \tag{5}$$

for a $\delta > 0$ and $x, y \in G$, and that $f\colon G \to \mathcal{A}$ is unbounded. Then there exists a sequence $x_n \in G$ such that $|f(x_n)| \to +\infty$ and $|f(x_n)| \neq 0$. The sequence $\frac{f(x_n)}{|f(x_n)|}$ is bounded and the algebra $\mathcal{A}$ is finite-dimensional, thus there exists a subsequence x_{k_n} of x_n such that there exists

$$\lim_{n\to+\infty} \frac{f(x_{k_n})}{|f(x_{k_n})|}) =: \alpha_0.$$

Since $|\alpha_0| = 1$, thus $\alpha_0 \neq 0$. For fixed x putting $y = x_{k_n}$ in (5) and dividing by $|f(x_{k_n})|^2$ we have

$$\lim_{n\to+\infty} \Big[\frac{f(x+x_{k_n})}{|f(x_{k_n})|}\Big]^2 = [\alpha_0]^2 \qquad \text{for } x \in G. \tag{6}$$

The sequence $\frac{f(x+x_{k_n})}{|f(x_{k_n})|}$ is bounded by Lemmas 2.2 and 2.5. There exists a subsequence of this sequence (designed for simplification by $\frac{f(x+x_n)}{|f(x_n)|}$) that there exists

$$\lim_{n\to+\infty} \frac{f(x+x_n)}{|f(x_n)|} =: \alpha(x).$$

By (6) we have $[\alpha(x)]^2 = [\alpha_0]^2$, thus $\alpha(x) = \alpha_0$ or $\alpha(x) = -\alpha_0$.

Similarly, for a fixed y in the sequence $\frac{f(x+y+x_n)}{|f(x_n)|}$ there exists a subsequence (designed for simplification by $\frac{f(x+y+x_n)}{|f(x_n)|}$) such that there exists

$$\lim_{n\to+\infty} \frac{f(x+y+x_n)}{|f(x_n)|} =: \alpha(x+y)$$

and $\alpha(x+y) = \alpha_0$ or $\alpha(x+y) = -\alpha_0$.

Assume that $\alpha(x) = \alpha(x+y) = \alpha_0$. Dividing inequality (5) for $y = x_n$ by $|f(x_n)|$ we have

$$\Big|[f(x+x_n)-f(x)-f(x_n)]\frac{f(x+x_n)-f(x)+f(x_n)}{|f(x_n)|}\Big| \le \frac{\delta}{|f(x_n)|},$$

thus

$$\lim_{n\to+\infty} [f(x+x_n)-f(x)-f(x_n)]\frac{f(x+x_n)-f(x)+f(x_n)}{|f(x_n)|} = 0.$$

Since

$$\lim_{n\to+\infty} \frac{f(x+x_n)-f(x)+f(x_n)}{|f(x_n)|} = \alpha(x) + \alpha_0 = 2\alpha_0 \neq 0,$$

by Lemma 2.2,

$$\lim_{n\to+\infty} |f(x+x_n) - f(x) - f(x_n)| = 0$$

and

$$f(x) = \lim_{n\to+\infty} [f(x+x_n) - f(x_n)]. \tag{7}$$

By (5) we have

$$\big|[f(x+y+x_n) - f(x+y) - f(x_n)][f(x+y+x_n) - f(x+y) + f(x_n)]\big| \le \delta,$$

thus dividing this inequality by $|f(x_n)|$ we obtain

$$f(x+y) = \lim_{n\to+\infty} [f(x+y+x_n) - f(x_n)]. \tag{8}$$

Again by (5) we get

$$\big|[f(x+y+x_n) - f(y) - f(x+x_n)][f(x+y+x_n) - f(y) + f(x+x_n)]\big| \le \delta,$$

which after dividing by $|f(x_n)|$ yields

$$f(y) = \lim_{n\to+\infty} [f(x+y+x_n) - f(x+x_n)]. \tag{9}$$

The conditions (7),(8) and (9) imply that

$$f(x+y) = f(x) + f(y). \tag{10}$$

By the similar argument we get the same result if $\alpha(x) = \alpha(x+y) = -\alpha_0$ and we obtain

$$f(x+y) = f(y) - f(x)$$

for $\alpha(x) = \alpha_0$, $\alpha(x+y) = -\alpha_0$ or $\alpha(x) = -\alpha_0$, $\alpha(x+y) = \alpha_0$. The function f is a solution of

$$[f(x+y) - f(y)]^2 = [f(x)]^2,$$

hence it satisfies (4).

Part II. From (5) we have

$$\begin{aligned} &|2f(x+y)[f(x+y) - f(x) - f(y)]| \\ &\qquad = |f^2(x+y) - 2f(x+y)f(x) + f^2(x) - f^2(y) + f^2(x+y) \\ &\qquad\quad - 2f(x+y)f(y) + f^2(y) - f^2(x)| \\ &\qquad = [[f(x+y) - f(x)]^2 - f^2(y)] + [[f(x+y) - f(y)]^2 - f^2(x)] \\ &\qquad \le \delta + \delta = 2\delta, \end{aligned}$$

thus

$$|f(x+y)[f(x+y) - f(x) - f(y)]| \le \delta.$$

The function f, if it is unbounded, is additive by the proofs of Theorems 2.6.1 and 2.5.2 in [9].

Remark 2.6

If $\mathcal{A} = \mathbb{R}$ there exist a bounded solutions of the inequality (5) which are not additives, e.g. $f(x) = \sqrt{\delta}$.

Theorem 2.7

If G is an Abelian group divisible by 2 and $f\colon G \to A$, then equations (4) and (10) are equivalent.

Lemma 2.8

Let G be a groupoid divisible by 2. If the function $f\colon G \to A$ is the solution of the equation

$$[f(2x) - 2f(x)]f(2x) = 0,$$

then $f(2x) = 2f(x)$. Thus if f is bounded, then $f = 0$.

Proof. Observe that the relation $xRy := \exists k \in \mathbb{Z} : y = 2^k x$ for $x, y \in G$ is the equivalence relation. Putting $a(k) = f(2^k x_0)$ for $k \in \mathbb{Z}$ and a fixed $x_0 \in G$ we have $[a(k+1) - 2a(k)]a(k+1) = 0$, thus $a(k+1) = 2a(k)$ or $a(k+1) = 0$. If there exists k_0 such that $a(k_0) = 0$, then $a(k_0 + 1) = 0$, hence $a(k) = 0$ for $k \geq k_0$, and $a(k_0) = 2a(k_0 - 1)$ or $a(k_0) = 0$. Therefore, $a(k) = 0$ for $k < k_0$. As a consequence, $a(k) = 0$ for every $k \in \mathbb{Z}$ or $a(k) \neq 0$ for every $k \in \mathbb{Z}$. It follows that $a(k+1) = 2a(k)$ for $k \in \mathbb{Z}$ and that f is a solution of $f(2x) - 2f(x) = 0$ on $\{2^k x_0 : k \in \mathbb{Z}\}$. Since this set is an arbitrary equivalence class of the relation R we obtain the first assertion. By induction we get $f(2^k x) = 2^k f(x)$ for $k \in \mathbb{N}$, thus if f is bounded, we get $f = 0$, which completes the proof.

Remark 2.9

The assumption that the groupoid G is divisible by 2 is essential in Lemma 2.8. Indeed, suppose that G is an Abelian group not divisible by 2, then $2G = \{2x : x \in G\}$ is also a group and $2G$ does not generate G, since $G \neq 2G$. In this case G is orientable, i.e. there exists a subgroup G^* with index 2 ([8]). Moreover, the function $f\colon G \to A$ which vanishes on G^* and is not identical on $G \setminus G^*$ is a solution of (4) and $f(2x) \neq 2f(x)$ for some x.

Proof of Theorem 2.7. If f is an unbounded solution of (4), then f is additive by Theorem 2.4. On the other hand, if f is bounded and satisfies (4), then $f(x) = 0$ for $x \in G$, which follows from Lemma 2.8 as f is the solution of $[f(2x) - 2f(x)]f(2x) = 0$.

Remark 2.10

If G is a semigroup and $\mathcal{A}$ is an integral domain of characteristic different from 3, then (1) is equivalent to (10) (see [7] p. 337–339), which is not superstable. For such sets G and $\mathcal{A}$ equations (4) and (10) are not equivalent. In fact, the function $f(2n) = 0$ and $f(2n-1) = 1$ for $n \in \mathbb{Z}$, is a solution of (4) and it is not additive. This shows that the assumption that G is 2-divisible is essential in Lemma 2.8.

Remark 2.11
Let $(G,+)$ be a groupoid and $\mathcal{A}=\mathbb{R}$. The inequalities (5) and

$$\big|[f(x+y)-f(x)-f(y)][f(x+y)+f(x)+f(y)]\big|\leq\delta$$

are not equivalent as $f(x)=\sqrt{\delta}$ is not a solution of the above inequality and it satisfies (5).

Theorem 2.12
Let G and $\mathcal{A}$ be as in Theorem 2.4. Then the equation

$$f(x+y)=f(x)f(y) \tag{11}$$

and his square

$$[f(x+y)]^2=[f(x)f(y)]^2 \tag{12}$$

are superstable.

Proof. Assume that

$$|f(x+y)-f(x)f(y)|\leq\delta \tag{13}$$

for a $\delta>0$ and $x,y\in G$ and let x_n be a sequence as at the beginning of the proof of Theorem 2.4. Dividing (13) with $y=x_n$ by $|f(x_n)|$ we have

$$\lim_{n\to+\infty}\frac{f(x+x_n)}{|f(x_n)|}=f(x)\varepsilon,$$

where $\varepsilon:=\lim_{n\to+\infty}\frac{f(x_n)}{|f(x_n)|}$. Similarly,

$$\lim_{n\to+\infty}\frac{f(x+y+x_n)}{|f(x_n)|}=f(x+y)\varepsilon$$

and

$$\lim_{n\to+\infty}\frac{f(y+x+x_n)}{|f(x_n)|}=f(y)\lim_{n\to+\infty}\frac{f(x+x_n)}{|f(x_n)|}=f(y)f(x)\varepsilon,$$

thus

$$f(x+y)\varepsilon=f(y)f(x)\varepsilon.$$

Since $\mathcal{A}$ has no zero divisors, we get $f(x+y)=f(x)f(y)$, thus equation (1) is superstable.

Now, assume that

$$\big|[f(x+y)]^2-[f(x)f(y)]^2\big|\leq\delta \tag{14}$$

for a $\delta>0$ and $x,y\in G$ and let x_n be a sequence as at the beginning of the proof of Theorem 2.4. Substituting $y=x_n$ into (14) and dividing it by $|f(x_n)|^2$ we have

$$\lim_{n\to+\infty}\frac{[f(x+x_n)]^2}{|f(x_n)|^2}=[f(x)\varepsilon]^2,$$

where $\varepsilon = \lim_{n\to+\infty} \frac{f(x_n)}{|f(x_n)|}$. Thus

$$\lim_{n\to+\infty} \frac{f(x+x_n)}{|f(x_n)|} = \pm f(x)\varepsilon$$

for a subsequence of x_n (denoted as x_n, too). Similarly, we obtain

$$\lim_{n\to+\infty} \frac{f(x+y+x_n)}{|f(x_n)|} = \pm f(x+y)\varepsilon$$

and

$$\lim_{n\to+\infty} \frac{f(y+x+x_n)}{|f(x_n)|} = \pm f(y) \lim_{n\to+\infty} \frac{f(x+x_n)}{|f(x_n)|} = \pm f(y)f(x)\varepsilon.$$

Hence

$$\pm f(x+y)\varepsilon = \pm f(y)f(x)\varepsilon$$

and, as a consequence, $\pm f(x+y) = \pm f(x)f(y)$. Namely, $f(x+y) = f(x)f(y)$ or $f(x+y) = -f(x)f(y)$. Thus f is a solution of (12), which proves the superstability of (12).

Remark 2.13
Equation

$$f(xy) = f(x)f(y), \tag{15}$$

where $f\colon G \to \mathcal{A}$ and G and $\mathcal{A}$ are as in the Theorem 2.7, and equation (11) are of the same type. Namely, the binary operation in G in (11) is $(x,y) \mapsto x+y$, whereas in (15) we set $(x,y) \mapsto xy$.

If in G is an absorbing element 0, i.e. $0 \cdot x = 0$ for $x \in G$, then the equations

$$f(xy) = f(x) + f(y)$$

and

$$[f(xy)]^2 = [f(x) + f(y)]^2 \tag{16}$$

are evidently superstable, which follows from the fact that if $f(xy) - f(x) - f(y)$ ($[f(xy)]^2 - [f(x)+f(y)]^2$) is bounded, then f is bounded (this is evident for $x = 0$).

Theorem 2.14
Let G and $\mathcal{A}$ be as in Theorem 2.7. The second power of Jensen's equation

$$\Big[f\Big(\frac{x+y}{2}\Big)\Big]^2 = \Big[\frac{f(x)+f(y)}{2}\Big]^2 \tag{17}$$

is superstable.

Proof. Assume that for an unbounded function $f\colon G \to \mathcal{A}$,

$$\Big\|\Big[f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2}\Big]\Big[f\Big(\frac{x+y}{2}\Big) + \frac{f(x)+f(y)}{2}\Big]\Big\| \le \delta$$

for a $\delta > 0$ and $x, y \in G$. Putting here $h(x) = f(x) - f(0)$ we have

$$\Big\|\Big[h\Big(\frac{x+y}{2}\Big) - \frac{h(x)+h(y)}{2}\Big]\Big[h\Big(\frac{x+y}{2}\Big) + \frac{h(x)+h(y)}{2} + a\Big]\Big\| \le \delta, \tag{18}$$

where $a = 2f(0)$, h is unbounded and $h(0) = 0$. Then there exists a sequence $x_n \in G$ such that $|h(x_n)| \to +\infty$ and $|h(x_n)| \neq 0$. The sequence $\frac{h(x_n)}{|h(x_n)|}$ is bounded and the algebra $\mathcal{A}$ is finite-dimensional, thus there exists a subsequence x_{k_n} of x_n such that

$$\lim_{n\to+\infty} \frac{h(x_{k_n})}{|h(x_{k_n})|} =: 2\alpha_0. \tag{19}$$

Since $|2\alpha_0| = 1$, we get $\alpha_0 \neq 0$. By (18) for $y = x_n$ we obtain

$$\Big|\Big[h\Big(\frac{x+x_n}{2}\Big) - \frac{h(x)+h(x_n)}{2}\Big]\Big[h\Big(\frac{x+x_n}{2}\Big) + \frac{h(x)+h(x_n)}{2} + a\Big]\Big| \leq \delta, \tag{20}$$

Dividing this inequality by $|h(x_n)|^2$ (dividing each factor on the left side of this inequality by $|h(x_n)|$) we obtain

$$\lim_{n\to+\infty} \Big[h\Big(\frac{x+x_n}{2}\Big)|h(x_n)|^{-1}\Big]^2 = \alpha_0^2,$$

thus $[h(\frac{x+x_n}{2})|h(x_n)|^{-1}]^2$ is bounded and so is $h(\frac{x+x_n}{2})|h(x_n)|^{-1}$. There exists a convergent subsequence of this sequence – designed for simplification by $h(\frac{x+x_n}{2})|h(x_n)|^{-1}$. Putting $\alpha(x) := \lim_{n\to+\infty} h(\frac{x+x_n}{2})|h(x_n)|^{-1}$, we have $\alpha(x) = \pm\alpha_0$. Similarly we obtain $\alpha(x+y) = \pm\alpha_0$.

From (20) it follows that

$$\begin{aligned}\Big|\Big[h\Big(\frac{x+y+x_n}{2}\Big) - \frac{h(y)+h(x+x_n)}{2}\Big]& \\ \times \Big[h\Big(\frac{x+y+x_n}{2}\Big) + \frac{h(y)+h(x+x_n)}{2} + a\Big]\Big| &\leq \delta,\end{aligned} \tag{21}$$

and for $y = 0$ ($h(0) = 0$) we get

$$\Big|\Big[h\Big(\frac{x+x_n}{2}\Big) - \frac{h(x+x_n)}{2}\Big]\Big[h\Big(\frac{x+x_n}{2}\Big) + \frac{h(x+x_n)}{2} + a\Big]\Big| \leq \delta.$$

Dividing this inequality by $|h(x_n)|^2$ we have

$$\lim_{n\to+\infty} \Big[\frac{h(x+x_n)}{2}|h(x_n)|^{-1}\Big]^2 = \alpha_0^2,$$

and by the adequate choice we obtain

$$\beta(x) := \lim_{n\to+\infty} \frac{h(x+x_n)}{2}|h(x_n)|^{-1} = \pm\alpha_0.$$

We consider the cases.

1. $\alpha(x) = \alpha(x+y) = \beta(x) = \alpha_0$. Dividing (18) by $|h(x_n)|$ we obtain by Lemma 2.2,

$$h(x) = \lim_{n\to+\infty} \Big[2h\Big(\frac{x+x_n}{2}\Big) - h(x_n)\Big].$$

Putting $x+y$ in place of x in (20) we have

$$\begin{aligned}\Big|\Big[h\Big(\frac{x+y+x_n}{2}\Big) - \frac{h(x+y)+h(x_n)}{2}\Big]\\ \times \Big[h\Big(\frac{x+y+x_n}{2}\Big) + \frac{h(x+y)+h(x_n)}{2} + a\Big]\Big| \le \delta,\end{aligned} \tag{22}$$

hence

$$h(x+y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - h(x_n)\Big].$$

From (21) after division by $|h(x_n)|$ we obtain

$$h(y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - h(x+x_n)\Big],$$

which for $y=0$ gives

$$0 = h(0) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+x_n}{2}\Big) - h(x+x_n)\Big].$$

Thus

$$h(y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - 2h\Big(\frac{x+x_n}{2}\Big)\Big].$$

Finally, $h(x+y) = h(x)+h(y)$. The same conclusion can be drown in the case $\alpha(x) = \alpha(x+y) = -\alpha_0$, $\beta(x) = \alpha_0$.

2. $\alpha(x) = \alpha(x+y) = \beta(x) = -\alpha_0$. Similarly as above we obtain

$$h(x) = \lim_{n\to+\infty}\Big[-2h\Big(\frac{x+x_n}{2}\Big) - h(x_n) - 2a\Big],$$

$$h(x+y) = \lim_{n\to+\infty}\Big[-2h\Big(\frac{x+y+x_n}{2}\Big) - h(x_n) - 2a\Big],$$

$$h(y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - h(x+x_n)\Big].$$

Since

$$0 = h(0) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+x_n}{2}\Big) - h(x+x_n)\Big]$$

we get

$$h(y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - 2h\Big(\frac{x+x_n}{2}\Big)\Big]$$

and $h(x) - h(y) = h(x+y)$.

The same result is in the case if $\alpha(x) = \alpha(x+y) = \alpha_0$, $\beta(x) = -\alpha_0$.

3. The remaining cases are impossible. Indeed, if $\alpha(x) = -\alpha_0$, $\alpha(x+y) = \alpha_0$, then

$$h(y) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+y+x_n}{2}\Big) - h(x+x_n)\Big],$$

thus

$$0 = h(0) = \lim_{n\to+\infty}\Big[2h\Big(\frac{x+x_n}{2}\Big) - h(x+x_n)\Big].$$

Dividing this equality by $|h(x_n)|$ we obtain $2\alpha(x) - 2\beta(x) = -2\alpha_0 - 2\alpha_0 \neq 0$.

The function h is a solution of

$$[h(x+y)-h(x)-h(y)][h(x+y)-h(x)+h(y)]=0$$

and it is is additive by Theorem 2.4, thus $f(x)=h(x)+f(0)$ is the solution of Jensen's equation

$$f\Big(\frac{x+y}{2}\Big)=\frac{f(x)+f(y)}{2}, \tag{23}$$

which proves that (17) is superstable.

Remark 2.15
The Jensen's equation (23) is not superstable. In fact, the function $f(x)=x+D(x)$, where $D(x)=0$ for $x\in\mathbb{Q}$ and $D(x)=1$ for $x\in\mathbb{R}\setminus\mathbb{Q}$, is unbounded, $f(\frac{x+y}{2})-\frac{f(x)+f(y)}{2}$ is bounded and f is not a solution of (23).

Remark 2.16
Let $\mathcal{M}$ be the algebra of diagonal real 2×2-matrices with the ordinary matrix addition and multiplication and with the norm

$$\left\|\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\right\| := \max\{|a|,|b|\} \qquad \text{for } a,b\in\mathbb{R}.$$

If $G=(\mathbb{R},+)$ and the function $f\colon G\to\mathcal{M}$ is given by

$$f(x)=\begin{bmatrix} x & 0\\ 0 & D(x)\end{bmatrix},$$

where $D(x)$ is as above, then f is unbounded, the functions

$$[f(x+y)-f(x)-f(y)][f(x+y)-f(x)+f(y)]$$

and

$$\Big[f\Big(\frac{x+y}{2}\Big)-\frac{f(x)+f(y)}{2}\Big]\Big[f\Big(\frac{x+y}{2}\Big)+\frac{f(x)+f(y)}{2}\Big]$$

are bounded and f is not a solution of (4) and (17).

For equations (11) and (12) the unbounded function

$$f(x)=\begin{bmatrix} \exp x & 0\\ 0 & 2\end{bmatrix},$$

is not a solution of these equations and the functions

$$f(x+y)-f(x)f(y) \quad \text{and} \quad [f(x+y)]^2-[f(x)f(y)]^2$$

are bounded, thus these equations are not superstable.

Question 2.17
Are the Theorems 2.4, 2.7, 2.12 and 2.14 true for the infinite-dimensional algebra $\mathcal{A}$? The same problem for the algebra of quaternions.

Remark 2.18
J.A. Baker proved in [1, Theorem 1] that if S is a semigroup, then for every function $f\colon S \to \mathbb{C}$ such that $|f(xy) - f(x)f(y)| \le \delta$ for $x, y \in S$ and for some positive δ we have

$$|f(x)| \le \frac{1+\sqrt{1+4\delta}}{2} \quad \text{for } x \in S \quad \text{or} \quad f(xy) = f(x)f(y) \ \text{ for } x, y \in S.$$

In this theorem the constant (i.e. $\frac{1+\sqrt{1+4\delta}}{2}$) is such that it bounds commonly the bounded solutions of the inequality $|f(xy) - f(x)f(y)| \le \delta$.

Notice that the methods of the proof of the superstability of functional equations used in this paper does not provide such constants.

We have here

Theorem 2.19
Let G be a semigroup divisible by 2 and let $\mathcal{A}$ be an algebra with a multiplicative norm. Then for every bounded solution $f\colon G \to \mathcal{A}$ of

$$\big|[f(x+y)]^2 - [f(x)+f(y)]^2\big| \le \delta, \tag{24}$$

where $\delta \ge 0$, we have

$$|f(x)| \le \sqrt{\frac{\delta}{3}} \qquad \text{for } x \in G. \tag{25}$$

We say that (1) is *uniformly superstable* if the bounded solutions of the inequality (24) are commonly bounded. Notice that the sine functional equation is superstable, however it is not uniformly superstable (see [4]).

Proof. From (24) for $y = x$ we obtain

$$|g(2x) - 4g(x)| \le \delta, \tag{26}$$

where $g(x) := f^2(x)$, thus

$$\Big|\frac{g(x)}{4} - g\Big(\frac{x}{2}\Big)\Big| \le \frac{\delta}{4} < \delta\Big(\frac{1}{3} - 4^{-2}\Big). \tag{27}$$

We prove by induction that

$$|4^{-n}g(x) - g(2^{-n}x)| \le \delta\Big(\frac{1}{3} - 4^{-n-1}\Big). \tag{28}$$

For $n = 1$ we get (28) by (27). Fix n and assume that (28) holds true. By (26) it follows that

$$|4^{-1}g(2^{-n}x) - g(2^{-n-1}x)| \le \frac{\delta}{4}$$

and by (28) that

$$|4^{-n-1}g(x) - 4^{-1}g(2^{-n}x)| \le \delta\Big(\frac{1}{12} - 4^{-n-2}\Big),$$

thus

$$|4^{-n-1}g(x) - g(2^{-n-1}x)| \leq \delta\Big(\frac{1}{3} - 4^{-n-2}\Big)$$

and (28) is proved. If f is bounded, then so is g, thus $\lim_{n\to+\infty} 4^{-n}g(2^n x) = 0$. Putting $2^n x$ in place of x in (28) and letting $n \to +\infty$ yields

$$|g(x)| = |f^2(x)| = |f(x)|^2 \leq \frac{\delta}{3},$$

hence (25) holds true.

Remark 2.20
If the norm in $\mathcal{A}$ is multiplicative, then $\mathcal{A} = \mathbb{R}$ or $\mathcal{A} = \mathbb{C}$ or $\mathcal{A}$ is the algebra of quaternions ([12, p. 30]).

Remark 2.21
For $\mathcal{A}$ and G as in Theorems 2.1 and 2.14 the equations (1) and (10) are equivalent, since it is possible to take $\delta = 0$ (compare with Remark 2.10).

Conclusion 2.22
Let G and $\mathcal{A}$ be as in Theorem 2.19. Equation (1) *is stable in the Ulam-Hyers sense, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function $f\colon G \to \mathcal{A}$ such that if* (24) *is fulfilled there exists a solution g of* (1) *for which $|f(x) - g(x)| \leq \varepsilon$. The function g is unique.*

Proof. It is sufficient to put $\delta = 3\varepsilon^2$. Let g_1 and g_2 be the solutions of (1) such that $|f(x) - g_i(x)| \leq \varepsilon$ for $i = 1, 2$ and let there exist an x_0 such that $g_1(x_0) \neq g_2(x_0)$. Since (1) is equivalent to (10) we have for $n \in \mathbb{N}$,

$$\begin{aligned} n|g_1(x_0) - g_2(x_0)| &= |g_1(nx_0) - g_2(nx_0)| \\ &= |g_1(nx_0) - f(nx_0) + f(nx_0) - g_2(nx_0)| \\ &\leq 2\varepsilon, \end{aligned}$$

which yields a contradiction. Thus $g_1 = g_2$.

Remark 2.23
Not all functions, which fulfil (25), are the solutions of (24). Indeed, for $G = \mathcal{A} = \mathbb{R}$ the function $f(0)$ =0 and $f(x) = \sqrt{\frac{\delta}{3}}$ for $x \neq 0$ is not a solution of (24) $(|[f(-1+1)]^2 - [f(-1) + f(1)]^2| = 4\frac{\delta}{3} > \delta)$. The estimation (25) of the solutions f of (24) is the best possible since the function $f(x) = \sqrt{\frac{\delta}{3}}$ is a solution of (24).

Question 2.24
Are equations (4), (12) and (17) uniformly superstable?

Remark 2.25
If the estimation for the bounded solutions of the inequality

$$\big|[f(x+y)-f(x)]^2-f^2(y)\big|\le\delta$$

exists for a $\delta_0>0$, then it exists for every $\delta>0$. In fact, assuming that this estimation M exists for a $\delta_0>0$. For a bounded function g such that

$$\big|[g(x+y)-g(x)]^2-g^2(y)\big|\le\delta=\delta_0\frac{\delta}{\delta_0}$$

the function $h=\sqrt{\frac{\delta_0}{\delta}}g$ satisfies $|[h(x+y)-h(x)]^2-h^2(y)|\le\delta_0$, thus $|h(x)|\le M$ which implies $|g(x)|\le M\sqrt{\frac{\delta}{\delta_0}}$.

The same situation occurs for the square of Jensen equation (17). Let G and $\mathcal{A}$ are as in Theorem 2.4 and let G be divisible by 2. The conditions

(i) there exists a $\delta_0>0$ such that there exists the estimation M as above,

(ii) equation (4) is Ulam-Hyers stable

are equivalent.

Since it is possible to assume that $M>0$ for the proof that (i)⇒(ii) it is sufficient to set in the definition of Ulam-Hyers stability $\delta=\delta_0(\frac{\varepsilon}{M})^2$ for $\varepsilon>0$. Indeed, suppose that $g\colon G\to\mathcal{A}$ satisfies (5), then by Theorem 2.4 this function is a solution of (4) or it is bounded. In the first case we have $|g(x)-g(x)|\le\varepsilon$, whereas in the second case $|g(x)-0|=|g(x)|\le M\sqrt{\frac{\delta}{\delta_0}}\le\varepsilon$, thus (ii) is proved.

The proof of (ii)⇒(i). By the definition of Ulam-Hyers stability for $\varepsilon=1$ there exists a $\delta_0>0$ such that for every solution g of $|[g(x+y)-g(x)]^2-g^2(y)|\le\delta_0$ there exists a solution f of (4) such that $|g(x)-f(x)|\le 1$. If g is bounded, then so is f. Since f is the solution of (4), we get $f(2x)=2f(x)$ by Lemma 2.8. Next by induction, $f(2^kx)=2^kf(x)$ for every $k\in\mathbb{N}$, thus $f=0$, as f is bounded. Therefore $|g(x)|\le 1$ and (i) holds true.

The estimation as in the Question 2 is not (25) since the function $f(x)=\sqrt{\delta}$ is a solution of the above inequality. This estimation exists not, e.g., for the orientable group G, e.g., $G=\mathbb{Z}$, and an algebra $\mathcal{A}$ with the unity e. Really, the function $f_n(x)=0$ for $x\in 2G$ and $f_n(x)=ne$ for $x\in G\setminus 2G$ and $n\in\mathbb{N}_0$ is a bounded solution of the above inequality and the family of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ is not equi-bounded. Each f_n is a solution of (4). Observe that f_n satisfies the implication

$$\forall x,y\in\mathbb{Z}:\ |f_n(x+y)-f_n(x)+f_n(y)|>\delta\implies|f_n(x+y)-f_n(x)-f_n(y)|\le\delta.$$

This shows that the following implication is not true: if G is an abelian semigroup and $\mathcal{A}$ is a Banach space and if for some $\varepsilon_1,\varepsilon_2\ge 0$ and all $x,y\in G$ a function $f\colon G\to\mathcal{A}$ satisfies

$$|f(x+y)-f(x)+f(y)|>\varepsilon_1\implies|f(x+y)-f(x)-f(y)|\le\varepsilon_2,$$

then there exists a unique additive function $a\colon G \to \mathcal{A}$ such that

$$|f(x) - a(x)| \le \max\{\varepsilon_1, \varepsilon_2\}$$

for all $x \in G$ (the modification of Batko's Theorem 1 in [2], where the Cauchy conditional equation

$$|f(x+y) + f(x) + f(y)| > \varepsilon_1 \implies |f(x+y) - f(x) - f(y)| \le \varepsilon_2$$

is considered). Indeed, since f_n is bounded, then so is an additive function a_n such that $|f_n(x) - a_n(x)| \le \delta$. Thus $a_n(x) = 0$ and $|f_n(x)| \le \delta$. We have a contradiction since the family $\{f_n(x)\}_{n\in\mathbb{N}}$ is not equi-bounded.

The bounded solutions of the inequality

$$|f^2(2x) - 4f^2(x)| \le \delta$$

(it is obtained by $y = x$ in the inequality (24)) are commonly bounded by $\sqrt{\delta}$. On the contrary, the bounded solutions of the inequality

$$\big|[f(2x) - 2f(x)]f(2x)\big| \le \delta \tag{29}$$

(received by setting $y = x$ in (5)) are not equi-bounded. Namely, for $f_{n\delta}\colon \mathbb{R} \to \mathbb{R}$, where $n \in \mathbb{N}$, defined by

$$f_{n\delta}(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \{2^1, \dots, 2^n\}, \\ x\frac{\sqrt{\delta}}{2}, & \text{if } x \in \{2^1, \dots, 2^n\}, \end{cases}$$

we have (29), this function is bounded and the family $\{f_{n\delta}\}_{n\in\mathbb{N}}$ is not equi-bounded.

Proposition 2.26
The equation

$$[f(2x) - 2f(x)]f(2x) = 0$$

for $f\colon \mathbb{R} \to \mathbb{R}$, is not Ulam-Hyers stable and it is unsuperstable.

Proof. For the indirect proof assume that the equation in consideration is Ulam-Hyers stable. Thus for $\varepsilon = 1$ there exists a $\delta > 0$ such that for every solution g of (29) there exists a solution f of our equation such that $|g(x) - f(x)| \le 1$. The function $g = f_{n\delta}$, where n is such that $2^{n-1}\sqrt{\delta} > 1$, is a solution of (29), thus there exists a solution f of our equation such that $|g(x) - f(x)| \le 1$. Function g is bounded and so is f, thus by Lemma 2.8 we obtain $f = 0$. We have a contradiction with the fact that $|g(2^n)| = 2^{n-1}\sqrt{\delta} > 1$.

The function obtained from $f_{n\delta}$ by substituting $\{2^1, \dots, 2^n\}$ by $\{2^k :\ k \in \mathbb{N}\}$ proves that our equation is not superstable.

Proposition 2.27
The equation

$$f^2(2x) = 4f^2(x)$$

for $f\colon \mathbb{R} \to \mathbb{R}$, is Ulam-Hyers stable and it is not superstable.

Proof. Let $\delta = 3\varepsilon^2$ for $\varepsilon > 0$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be such that

$$|g^2(2x) - 4g^2(x)| \le \delta \qquad \text{for } x \in \mathbb{R}.$$

Thus for $h = g^2$ we have

$$|h(2x) - 4h(x)| \le \delta.$$

By the procedure as in the "direct method" of Hyers (see [4]) there exists a solution

$$k(x) = \lim_{n\to+\infty} \frac{h(2^n x)}{4^n} = \lim_{n\to+\infty} \frac{g^2(2x)}{4^n} \ge 0$$

of the equation $k(2x) - 4k(x) = 0$ such that

$$|k(x) - h(x)| = |(\sqrt{k(x)})^2 - g^2(x)| \le \frac{\delta}{3}.$$

We have thus for every $x \in \mathbb{R}$,

$$|\sqrt{k(x)} - g(x)| \le \sqrt{\frac{\delta}{3}} \quad \text{or} \quad |\sqrt{k(x)} + g(x)| \le \sqrt{\frac{\delta}{3}}.$$

For the function l such that $l(x) = \sqrt{k(x)}$, if $|\sqrt{k(x)} - g(x)| \le \sqrt{\frac{\delta}{3}}$ and $l(x) = -\sqrt{k(x)}$, if $|\sqrt{k(x)} - g(x)| > \sqrt{\frac{\delta}{3}}$ we have $|l(x) - g(x)| \le \sqrt{\frac{\delta}{3}} = \varepsilon$. The proof of the stability of our equation is finished. The function $f(x) = \sqrt{x^2+1}$ proves that our equation is not superstable.

3. Stability of the alternation of functional equations

The equations

$$f(x+y) - f(x) - f(y) = 0 \quad \text{and} \quad f(x+y) + f(x) + f(y) = 0, \tag{30}$$

for f from an Abelian group G to a Banach space $\mathcal{A}$, are stables in Ulam-Hyers sense. For the first equation the stability result is known. For the second one, if $|g(x+y) + g(x) + g(y)| \le \delta$, then for $x = y = 0$ we have $|g(0)| \le \frac{\delta}{3}$ and for $y = 0$

$$|2g(x)| - |g(0)| \le |2g(x) + g(0)| \le \delta,$$

thus $|g(x) - 0| \le 2\frac{\delta}{3}$. For a $\varepsilon > 0$ it is sufficient to take $\delta = 3\frac{\varepsilon}{2}$ in the definition of Ulam-Hyers stability.

The following alternation

$$f(x+y) - f(x) - f(y) = 0 \quad \text{or} \quad f(x+y) + f(x) + f(y) = 0 \tag{31}$$

is stable in Ulam-Hyers sense ([2, Theorem 1]), i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function $g\colon G \to \mathcal{A}$ if

$$|g(x+y) - g(x) - g(y)| \le \delta \quad \text{or} \quad |g(x+y) + g(x) + g(y)| \le \delta,$$

then there exists a solution f of (31) such that $|g(x) - f(x)| \leq \varepsilon$.

The equations

$$f(x+y) - f(x) - f(y) = 0 \quad \text{and} \quad f(x+y) - f(x) + f(y) = 0 \tag{32}$$

are also stable (for the second equation the proof is analogical as above). However, the alternation of these equations is not stable, since the result analogous to Batko's result (Remark 2.25) is not true and the equations (4) and (10) are equivalent.

The alternation: "$E_1(f) = 0$ or $E_2(f) = 0$" is said to be superstable if for every function f if there exists a $\delta > 0$ such that $|E_1(f)| \leq \delta$ or $|E_2(f)| \leq \delta$, then f is bounded or f is a solution of this alternation. The function $f(x) = x + \delta$ proves that the alternation of equations from (30) (of equations from (32)) for $f\colon \mathbb{R} \to \mathbb{R}$, is not superstable according to this definition. On the other hand, the equation

$$[f(x+y) - f(x) - f(y)][f(x+y) + f(x) + f(y)] = 0$$

(resp. $[f(x+y) - f(x) - f(y)][f(x+y) - f(x) + f(y)] = 0$) is superstable though it is equivalent to the alternation of equations from (30) (resp. (32)).

Similarly, the equations

$$f(2x) - 2f(x) = 0 \quad \text{and} \quad f(2x) = 0$$

are stable for $f\colon \mathbb{R} \to \mathbb{R}$ (the first by the "direct method" of Hyers, the second is stable evidently) and its alternation is unstable. In fact, assume that $h = f_{n\delta^2}$, where n is such that $2^n\delta > 1$. This function satisfies

$$\big|[h(2x) - 2h(x)]h(2x)\big| \leq \delta^2,$$

thus we have

$$|h(2x) - 2h(x)| \leq \delta \quad \text{or} \quad |h(2x)| \leq \delta.$$

The rest of the proof runs similarly as the proof of the Proposition 2.26, as $[f(2x) - 2f(x)]f(2x) = 0$ and the alternation

$$f(2x) - 2f(x) = 0 \quad \text{or} \quad f(2x) = 0$$

are equivalent.

4. Inverse stability

The Cauchy's equation (10) is inversely stable ([10]), i.e. the function, approximated by a solution of the Cauchy's equation, is an approximate solution of this equation. More exact, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every f, for which there exists a solution g of (10) such that

$$|f(x) - g(x)| \leq \delta,$$

we have

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon.$$

On the contrary, the squares of the Cauchy's equation

$$[f(x+y)]^2 = [f(x)+f(y)]^2 \quad \text{and} \quad [f(x+y)-f(x)]^2 = [f(y)]^2$$

are not inversely stable. In fact, if $G = \mathcal{A} = \mathbb{R}$, then for $f(x) = x+\delta$ there exists an additive function $g(x) = x$ for which $|f(x)-g(x)| = \delta$ and $[f(x+y)]^2-[f(x)+f(y)]^2$ and $[f(x+y)-f(x)]^2 - [f(y)]^2$ are unbounded. We have the same situation for the remaining of the Cauchy's equations (e.g., for equation (16) if $G = \mathbb{R}\setminus\{0\}$; if $G = \mathbb{R}$ equation (16) is inversely stable) and for the Jensen's equation (23). The square of this equation is not inversely stable since for $f(x) = x + h(x)$, where $h(0) = 0$ and $h(x) = \delta$ for $x \neq 0$, there exists a solution $g(x) = x$ of this square such that $|f(x) - g(x)| \leq \delta$ and the function

$$\Big[f\Big(\frac{x+y}{2}\Big)\Big]^2 - \Big[\frac{f(x)+f(y)}{2}\Big]^2$$

is unbounded.

Since for $\varepsilon, \delta > 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x) = x + \delta$ there exists $x_0 \in \mathbb{R}$ such that

$$\big|[g(2x_0) - 2g(x_0)][g(2x_0) + 2g(x_0)]\big| \geq \big|[g(2x_0) - 2g(x_0)]g(2x_0)\big| = \delta(2x_0+\delta) > \varepsilon,$$

thus equations

$$[f(2x) - 2f(x)][f(2x) + 2f(x)] = 0 \quad \text{and} \quad [f(2x) - 2f(x)]f(2x) = 0$$

are not inversely stable.

The isometry equation

$$|f(x) - f(y)| = |x - y|$$

is inversely stable ([10]) and its square not (proof for $f\colon \mathbb{R} \to \mathbb{R}$ is as in the above case of square of Jensen's equation).

Comparison 4.1

Under the adequate assumptions the Cauchy's equation (10) *is Ulam-Hyers stable and inversely stable being not superstable and his square* (1)*, equivalent to* (10)*, is Ulam-Hyers stable and superstable being not inversely stable.*

Let $(S, +)$ be a groupoid and let G be an Abelian group with the metric $|\cdot|$. The alternation

$$f(x+y) - f(x) - f(y) = 0 \quad \text{or} \quad f(x+y) + f(x) + f(y) = 0, \tag{33}$$

for $f\colon S \to G$, is inversely stable. In fact, for $\varepsilon > 0$ set $\delta = \frac{\varepsilon}{3}$. Assume that for the function $g\colon S \to G$ there exists a solution f of (33) such that $|g(x) - f(x)| \leq \delta$. Putting

$$A := \{(x,y) \in S \times S : \ f(x+y) - f(x) - f(y) = 0\}$$

and

$$B := \{(x,y) \in S \times S : \ f(x+y) + f(x) + f(y) = 0\}$$

we have $A \cup B = S \times S$. We obtain for $(x, y) \in A$

$$\begin{aligned}|g(x+y) - g(x) - g(y)| &= \big|g(x+y) - g(x) - g(y) - [f(x+y) - f(x) - f(y)]\big| \\ &= \big|[g(x+y) - f(x+y)] - [g(x) - f(x)] - [g(y) - f(y)]\big| \\ &\leq 3\delta = \varepsilon\end{aligned}$$

and analogously for $(x, y) \in B$. Similarly, alternations

$$f(x+y) - f(x) - f(y) = 0 \quad \text{or} \quad f(x+y) - f(x) + f(y) = 0$$

and

$$f(2x) - 2f(x) = 0 \quad \text{or} \quad f(2x) = 0$$

are inversely stable.

5. Stability of the isometry equation and of isometry equation of the square

D.H. Hyers and S.M. Ulam have proved in [6] that the isometry equation

$$|f(x) - f(y)| = |x - y|$$

is stable in the class C of functions from a complete Euclidean vector space E onto E, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function $f\colon E \to E$ for which

$$\big||f(x) - f(y)| - |x - y|\big| \leq \delta \qquad \text{for } x, y \in E$$

there exists an isometry g such that

$$|f(x) - g(x)| \leq \varepsilon \qquad \text{for } x \in E$$

(in [6] $\delta = \frac{\varepsilon}{10}$). They have showed that the above statement is not true for the functions from one Euclidean space into the part of another.

Question 5.1
Is this situation possibly for the functions from an Euclidean space into the same (see theorem 5.4)?

M. Omladič and P. Šemrl have generalized in [11] this result for the class of functions from a Banach space onto a Banach space. We have a mentioned below theorem without the supposition "onto".

Theorem 5.2
The isometry equation

$$\|f(x) - f(y)\| = |x - y| \tag{34}$$

for $f\colon X_1 \to X_2$, where X_1, X_2 are the normed spaces of dimension one, normed by the norms $|\cdot|$ and $\|\cdot\|$, is stable in the sense of Ulam-Hyers.

Proof. Assume

$$\big|\|g(x)-g(y)\|-|x-y|\big|\le\varepsilon \tag{35}$$

for $g\colon X_1\to X_2$ and $\varepsilon>0$ and put $h(x)=g(x)-g(0)$. Thus

$$\big|\|h(x)-h(y)\|-|x-y|\big|\le\varepsilon \tag{36}$$

and $h(0)=0$. Set $h(x)=H(x)e_2$ for $H\colon X_1\to\mathbb{R}$ and $e_2\in X_2$ such that $\|e_2\|=1$. Moreover, for $e_1\in X_1$ such that $|e_1|=1$ we put $]x[=|x|$ if $x=|x|e_1$ and $]x[=-|x|$ if $x=-|x|e_1$. Then $H(0)=0$ and by (36) we obtain

$$\big||H(x)-H(y)|-|]x[-]y[|\big|\le\varepsilon,$$

and from here

$$\big||H(x)|-|]x[|\big|\le\varepsilon \qquad \text{for } x\in X_1,$$

thus for every $x\in X_1$

$$\text{either } |H(x){-}]x[|\le\varepsilon \text{ (if } H(x)]x[\ge 0) \quad\text{or}\quad |H(x){+}]x[|\le\varepsilon \text{ (if } H(x)]x[<0).$$

We will show by the indirect proof that

$$\text{either } \forall\,|x|>\tfrac{3}{2}\varepsilon:\ |H(x){-}]x[|\le\varepsilon \quad\text{or}\quad \forall\,|x|>\tfrac{3}{2}\varepsilon:\ |H(x){+}]x[|\le\varepsilon. \tag{37}$$

Assume that there exists a u such that $|u|>\frac{3}{2}\varepsilon$ and $|H(u){-}]u[|>\varepsilon$, thus $|H(u){+}]u[|\le\varepsilon$. Moreover, assume that there exists a v such that $|v|>\frac{3}{2}\varepsilon$ and $|H(v){+}]v[|>\varepsilon$, thus $|H(v){-}]v[|\le\varepsilon$. This gives

$$|-H(u){-}]u[{+}H(v){-}]v[|\le|-H(u){-}]u[|+|H(v){-}]v[|\le\varepsilon+\varepsilon=2\varepsilon. \tag{38}$$

We have by (34),

$$|H(u)-H(v){-}]u[{+}]v[|\le\varepsilon \quad\text{or}\quad |H(u)-H(v){+}]u[{-}]v[|\le\varepsilon. \tag{39}$$

We obtain by the first of these inequalities and by (38),

$$|-2]u[|\le|-H(u){-}]u[{+}H(v){-}]v[|+|H(u)-H(v){-}]u[{+}]v[|\le 3\varepsilon.$$

From here $|u|\le\frac{3}{2}\varepsilon$, which gives a contradiction. We have analogous contradiction in the case of the second inequality in (39) ($|v|\le\frac{3}{2}\varepsilon$). Thus the condition (37) is proved.

Assume that for every x such that $|x|>\frac{3}{2}\varepsilon$ we have $|H(x){-}]x[|\le\varepsilon$. Either this inequality is valid for every $x\in X_1$ or there exists $x_0\in X_1$ such that $|H(x_0){-}]x_0[|>\varepsilon$. In this case we must have $|x_0|\le\frac{3}{2}\varepsilon$ and $|H(x_0){+}]x_0[|\le\varepsilon$ and from here

$$|H(x_0){-}]x_0[|=|H(x_0){+}]x_0[{-}2]x_0[|\le|H(x_0){+}]x_0[|+|-2]x_0[|\le\varepsilon+3\varepsilon=4\varepsilon.$$

We thus get $|H(x){-}]x[|\le 4\varepsilon$ for every $x\in X_1$ and from here $\|h(x){-}]x[e_2\|\le 4\varepsilon$, thus $\|g(x)-(]x[e_2+g(0))\|\le 4\varepsilon$. Since $]x[e_2+g(0)$ is an isometry, the proof of the stability is completed in his case. We state in the second case in (36) that $\|g(x)-(-]x[e_2+g(0))\|\le 4\varepsilon$, where $-]x[e_2+g(0)$ is also an isometry. The proof of Theorem 5.2 is completed ($\delta=\frac{\varepsilon}{4}$ in the definition of Ulam-Hyers stability).

Example 5.3
The function $g\colon \mathbb{R} \to \mathbb{R}$, defined as

$$\text{for } \varepsilon_1 \le \varepsilon \text{ and } \varepsilon_1 \in \mathbb{Q}: \ g(x) = \varepsilon_1 \Big[\frac{x}{\varepsilon_1}\Big],$$

where $[\cdot]$ denotes the entire part, is not “onto” (it has the values in $\mathbb{Q}$) and fulfils (36).

This function has the range dense in $\mathbb{R}$, the function $g(x) = x$ for $x \notin K(0, \frac{\varepsilon}{2}) \subset \mathbb{R}^n$ and $g(x) = 0$ for $x \in K(0, \frac{\varepsilon}{2})$ fulfills (35) and its range is not dense in $\mathbb{R}^n$. This example shows that the equation of isometry is not superstable.

Theorem 5.4
The function $g\colon \mathbb{R} \to \mathbb{R}$, for which the function $|g(x) - g(y)| - |x - y|$ is bounded, is a surjection if and only if its range is an interval.

Proof. Assume that

$$\big||g(x) - g(y)| - |x - y|\big| \le \varepsilon$$

for some $\varepsilon > 0$. “Only if” is evident. For “if” we will show the indirect proof. Assume that a is not in the range E of the function g. Since E is an interval, thus either $E \cap [a, +\infty) = \emptyset$ or $E \cap (-\infty, a] = \emptyset$. Assume that the first case holds (the proof in the second case is analogous). Hence, $g(x) < a$ for $x \in \mathbb{R}$. By the proof of Theorem 5.2 there exists an isometry $i\colon \mathbb{R} \to \mathbb{R}$ such that $|g(x) - i(x)| \le 4\varepsilon$ for $x \in \mathbb{R}$. The isometry i is a surjection, thus there exists x_0 such that $i(x_0) = a + 4\varepsilon$. From here

$$4\varepsilon \ge |g(x_0) - i(x_0)| = |a + 4\varepsilon - g(x_0)| = a - g(x_0) + 4\varepsilon > 4\varepsilon,$$

which gives a contradiction.

Conclusion 5.5
A function from $\mathbb{R}$ to $\mathbb{R}$ with the Darboux property (also a continuous function from $\mathbb{R}$ to $\mathbb{R}$) and for which the function $|g(x) - g(y)| - |x - y|$ is bounded, is a surjection.

On the other hand, there exist surjections fulfilling (35) for some $\varepsilon > 0$ which do not have the Darboux property, e.g.: $g(x) = x$ for $x \in \mathbb{R}\backslash(-1, 1)$ and $g(x) = -x$ for $x \in (-1, 1)$ (here $\varepsilon = 2$).

Theorem 5.6
The square of isometry equation

$$\|f(x) - f(y)\|^2 = |x - y|^2$$

is Ulam-Hyers stable.

Proof. For $g\colon X_1 \to X_2$ such that

$$\big|\|g(x)-g(y)\|^2-|x-y|^2\big| = \big|\|g(x)-g(y)\|-|x-y|\big|\cdot\big|\|g(x)-g(y)\|+|x-y|\big| \le \varepsilon$$

we have for $x, y \in X_1$,

$$\big|\|g(x)-g(y)\|-|x-y|\big| \le \sqrt{\varepsilon} \quad \text{or} \quad \big|\|g(x)-g(y)\|+|x-y|\big| \le \sqrt{\varepsilon}$$

and since $|a-b| \le |a|+|b|$ thus for $x, y \in X_1$

$$\big|\|g(x)-g(y)\|-|x-y|\big| \le \sqrt{\varepsilon}.$$

By Theorem 5.2 the proof is finished.

Remark 5.7
By the same method as in the proof of Theorem 2.14 we can prove that the square of the isometry equation

$$|f(x)-f(y)|^2 = |x-y|^2$$

for $f\colon \mathbb{R} \to \mathbb{R}$ is superstable.

Conclusion 5.8
If for a function $f\colon \mathbb{R} \to \mathbb{R}$ the function $|f(x)-f(y)|^2-|x-y|^2$ is bounded, then f is an isometry.

It follows from the fact that in this case f is unbounded (for f bounded $|f(x)-f(y)|^2$ is also bounded and since $|x-y|^2$ is unbounded we deduce that $|f(x)-f(y)|^2-|x-y|^2$ is unbounded).

Remark 5.9
The isometry equation is inversely stable ([10]) and its square is not (proof for $f\colon \mathbb{R} \to \mathbb{R}$ is as in the above case of the square of Jensen's equation).

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