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Marek Karaś and Anna Serwatka

Discrete-time market models from the small investor point of view and the first fundamental-type theorem

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Abstract. In this paper, we discuss the no-arbitrage condition in a discrete financial market model which does not hold the same interest rate assumptions. Our research was based on, essentially, one of the most important results in mathematical finance, called the Fundamental Theorem of Asset Pricing. For the standard approach a risk-free bank account process is used as numeraire. In those models it is assumed that the interest rates for borrowing and saving money are the same. In our paper we consider the model of a market (with d risky assets), which does not hold the same interest rate assumptions. We introduce two predictable processes for modelling deposits and loans. We propose a new concept of a martingale pair for the market and prove that if there exists a martingale pair for the considered market, then there is no arbitrage opportunity. We also consider special cases in which the existence of a martingale pair is necessary and the sufficient conditions for these markets to be arbitrage free.

1. Introduction

In this paper, we will discuss the no-arbitrage condition in a discrete financial market model which does not hold the same interest rate assumptions. Our research was based on, essentially, one of the most important results in mathematical finance, called the Fundamental Theorem of Asset Pricing or the Dalang-Morton-Willinger theorem [1]. It states that for the standard discrete-time, finite horizon market model there is no arbitrage opportunity if and only if the price process is a martingale, with respect to an equivalent probability measure. There are various

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proofs of this theorem in existence, which use different areas of mathematics, for more detail see [1, 6, 3]. The Fundamental Theorem of Asset Pricing for a model with finite Ω was proven by M. Harrison and D. Kreps in 1979 [4]. Harrison and Pliska [5] proved a more general version of this theorem. In [2] Delbaen and Schachermayer show a concept which characterizes the existence of an equivalent martingale measure for a general class of processes in terms of the no free lunch with vanishing risk. The main theorem of that paper is the general version of the Dalang-Morton-Willinger theorem for real valued semi-martingales.

The Fundamental Theorem of Asset Pricing was studied in detail by many mathematicians, who checked various aspects and focused on additional equivalent conditions of this theorem. Many new theorems were proved through the investigation of different aspects of the Dalang-Morton-Willinger theorem. Among them are theorems which include taxes like in [7], where the author gives the necessary and sufficient conditions for a linear taxation system to be neutral-within the multi-period discrete time in a no arbitrage model. Kabanov and Safarian worked on multi-asset discrete-time models with friction and gave in [10] conditions equivalent to the absence of arbitrage in markets with friction. The theory goes further and there are papers with equivalent conditions for the absence of so-called weak arbitrage [9] and robust no-arbitrage opportunities [12].

It is also an interesting concept to consider the models with bid and ask price processes in [8] and [11]. Rola in [11] considers a market with a multi-dimensional bid and ask processes and with a money account, introduces the notion of an equivalent bid-ask martingale measure and proves that the existence of such a measure is equivalent to no-arbitrage in this model.

In many papers on arbitrage in discrete market models the authors consider models containing $d+1$ financial assets: one risk-free asset $\{B_t\}_{t=0,1,\ldots,T}$ (which is interpreted as a bank account) and d risky assets $\{S_t^i\}_{t=0,1,\ldots,T}$ for $i \in \{1,\ldots,d\}$ (say i.e. stocks). For the standard approach the risk-free bank account process is used as numéraire. In those models it is assumed that the interest rates for borrowing and saving money are the same. In our paper we consider the model of a market (with d risky assets), which does not hold the same interest rate assumptions. We introduce two predictable processes $\{B_t^+\}_{t=0,1,\ldots,T}$ and $\{B_t^-\}_{t=0,1,\ldots,T}$ for modelling deposits and loans. We propose a new concept of a martingale pair $(\{B_t\}_{t=0,\ldots,T}, P^*)$ for the market $\mathcal{M} = (S, \mathcal{P})$ and prove that if a martingale pair for the considered market exists, then there is no arbitrage opportunity. We also consider special cases in which the existence of a martingale pair is necessary and the sufficient conditions for these markets to be arbitrage free.

2. Model description

We assume that there is a given probability space $(\Omega, \mathcal{F}, P)$, a finite number $T \in \mathbb{N}_+ = \mathbb{N} \cup \{0\}$ called the time horizon, and a filtration $\{\mathcal{F}_t\}_{t=0,1,\ldots,T}$ of the measurable space $(\Omega, \mathcal{F})$.

We propose the following model of a market which does not satisfy the assumption that the interest rates for borrowing and saving money are the same.

Definition 2.1

By the model of a market (with d risky assets), which does not hold the same interest rates assumption, we mean, in general, the triple $\mathcal{M} = (S, \mathcal{P}, \varphi)$, where

- $S = \{(B_t^+, B_t^-, S_t^1, \ldots, S_t^d)\}_{t=0,1,\ldots,T}$ is the adapted stochastic process, with values in $(0, +\infty)^{d+2}$, such that the processes $\{B_t^+\}_{t=0,1,\ldots,T}$ and $\{B_t^-\}_{t=0,1,\ldots,T}$ are predictable, and

$$B_0^+ = B_0^- = 1,$$

$$\frac{B_{t+1}^+}{B_t^+} \leq \frac{B_{t+1}^-}{B_t^-} \qquad \text{for all } t \in \{0, \ldots, T-1\},$$

- $\mathcal{P}$ is a subset of the set of all predictable stochastic processes $\mathbf{\Theta} = \{\mathbf{\Theta}_t\}_{t=0,\ldots,T} = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$, with values in $[0, +\infty)^2 \times \mathbb{R}^d$,
- φ is the adapted process $\{(\varphi_t^1, \ldots, \varphi_t^d)\}_{t=0,1,\ldots,T}$, with values in the space $([0, +\infty)^{\mathbb{R}})^d$ (i.e. for each $t \in \{0, \ldots, T\}$, each $i \in \{0, \ldots, d\}$ and each $\omega \in \Omega$, $\varphi_t^i(\omega)$ is a function $\mathbb{R} \to [0, +\infty)$).

We assume that the process $\{B_t^+\}_{t=0,1,\ldots,T}$ is modelling the changes, in time, of the value of one unit of money given in the bank deposit at time $t = 0$, and is called *deposit process*, while the process $\{B_t^-\}_{t=0,1,\ldots,T}$, called *loan process*, is modelling the amount of money that must be given back to the bank at time t, if one has borrowed one unit of money from the bank at time $t = 0$.

We also assume, as usual, that the processes $\{S_t^i\}_{t=0,1,\ldots,T}$, for $i = 1, \ldots, d$, are modelling the prices of the risky assets, say stocks.

Definition 2.2

Assume that the market $\mathcal{M} = (S, \mathcal{P}, \varphi)$ is given. By a *portfolio* on the market $\mathcal{M}$ we mean any vector $\mathbf{\Theta} = (\Theta^+, \Theta^-, \Theta^1, \ldots, \Theta^d) \in [0, +\infty)^2 \times \mathbb{R}^d$. The value of the portfolio $\mathbf{\Theta} = (\Theta^+, \Theta^-, \Theta^1, \ldots, \Theta^d)$ at time t is defined by

$$V_t^{\mathbf{\Theta}} = \Theta^+ B_t^+ - \Theta^- B_t^- + \Theta^1 S_t^1 + \ldots + \Theta^d S_t^d.$$

Definition 2.3

By the *strategy* (or *trading strategy*) on the market $\mathcal{M}$ we mean any predictable process $\mathbf{\Theta} = \{\mathbf{\Theta}_t\}_{t=0,\ldots,T} = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$ with values in $[0, +\infty)^2 \times \mathbb{R}^d$, which is an element of the set $\mathcal{P}$, called the *set of all strategies.*

In our consideration, we will assume that $\mathcal{P}$ consists of all predictable processes $\mathbf{\Theta} = \{\mathbf{\Theta}_t\}_{t=0,\ldots,T} = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$, with values in $[0, +\infty)^2 \times \mathbb{R}^d$, but if one wants to consider the market with some additional restriction on strategies, then $\mathcal{P}$ will not consist of all such processes.

Since for $t = 0$ we must know what $\mathcal{F}_{-1}$ means, we assume that $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. For the simplicity of the considerations we will also assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

One can see that for any $\omega \in \Omega$ and for any $t \in \{0, \ldots, T\}$, $\mathbf{\Theta}_t(\omega) \in [0, +\infty)^2 \times \mathbb{R}^d$ is a portfolio, and we assume that this portfolio is held from time $t-1$ up to time t. This justifies our assumption that the process $\{\mathbf{\Theta}_t\}_{t=0,\ldots,T}$ is predictable.

Let us note that for any given strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ on the market $\mathcal{M} = (S, \mathcal{P}, \varphi)$, the value of the strategy at time t can be calculated in two ways, namely as the sum

$$\Theta_t^+ B_t^+ - \Theta_t^- B_t^- + \Theta_t^1 S_t^1 + \ldots + \Theta_t^d S_t^d$$

or as the sum

$$\Theta_{t+1}^+ B_t^+ - \Theta_{t+1}^- B_t^- + \Theta_{t+1}^1 S_t^1 + \ldots + \Theta_{t+1}^d S_t^d.$$

This is because the portfolio $\mathbf{\Theta}_t$ is held during the time interval $(t-1, t)$, while the portfolio $\mathbf{\Theta}_{t+1}$ is held during the time interval $(t, t+1)$. Thus, for any $t \in \{0, \dots, T-1\}$, we consider two possible different values of the strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ at time t, called the value before transaction and the value after transaction.

Definition 2.4

Assume that we are given a strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ defined on the market $\mathcal{M} = (S, \mathcal{P}, \varphi)$. Then, the *value* of the strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ at time t *before transaction*, we define as

$$V_{t-}^{\mathbf{\Theta}} := \Theta_t^+ B_t^+ - \Theta_t^- B_t^- + \Theta_t^1 S_t^1 + \ldots + \Theta_t^d S_t^d$$

and the *value* of the strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ at time t *after transaction*, we define as

$$V_{t+}^{\mathbf{\Theta}} := \Theta_{t+1}^+ B_t^+ - \Theta_{t+1}^- B_t^- + \Theta_{t+1}^1 S_t^1 + \ldots + \Theta_{t+1}^d S_t^d.$$

For the terminal date T, we can only consider the value before transaction

$$V_{T-}^{\mathbf{\Theta}} := \Theta_T^+ B_T^+ - \Theta_T^- B_T^- + \Theta_T^1 S_T^1 + \ldots + \Theta_T^d S_T^d.$$

One can notice that, for a given strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$, the values $V_{t-}^{\mathbf{\Theta}}$ and $V_{t+}^{\mathbf{\Theta}}$ can be different. The inequality $V_{t+}^{\mathbf{\Theta}}(\omega) > V_{t-}^{\mathbf{\Theta}}(\omega)$, for some $\omega \in \Omega$, means that we add an amount of money to the system/strategy, while the inequality $V_{t+}^{\mathbf{\Theta}}(\omega) < V_{t-}^{\mathbf{\Theta}}(\omega)$, for some $\omega \in \Omega$, can mean that we subtract an amount of money from the system/strategy.

Definition 2.5

Let us assume that the market $\mathcal{M} = (S, \mathcal{P}, \varphi)$ is given. The process $\varphi = \{(\varphi_t^1, \dots, \varphi_t^d)\}_{t=0,1,\dots,T}$ is called the *transaction cost process*, and is assumed that for each $i \in \{1, \dots, d\}$, each $\omega \in \Omega$, each $t \in \{0, 1, \dots, T\}$ and each $x \in \mathbb{R}$, the value of the function $\varphi_t^i(\omega)\colon \mathbb{R} \to [0, +\infty)$ at x, denoted by $\varphi_t^i(\omega).x$, is equal to the transaction cost of buying x shares of i-th stock at time t if $x > 0$, and is equal to the transaction cost of selling $|x|$ shares of i-th stock at time t if $x < 0$.

For a given strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$ defined on the market $\mathcal{M} = (S, \mathcal{P}, \varphi)$, the *total transaction cost* at time $t \in \{0, 1, \dots, T\}$ is defined as

$$C_t^{\Theta} = \sum_{i=1}^{d} \varphi_t^i.(\Theta_{t+1}^i - \Theta_t^i) =: \sum_{i=1}^{d} \varphi_t^i.\Delta\Theta_t^i,$$

where for any function $\zeta\colon \Omega \to \mathbb{R}$, by $\varphi_t^i.\zeta$ we denote the function $\omega \mapsto \varphi_t^i(\omega).\zeta(\omega)$. For the terminal date T, when (by assumption) all shares are sold, the total transaction cost is defined as

$$C_T^{\Theta} = \sum_{i=1}^{d} \varphi_T^i.(-\Theta_T^i).$$

The following condition means that the market $\mathcal{M}$ is assumed to be without transaction costs

$$\forall\, i \in \{1,\dots,d\},\ t \in \{0,1,\dots,T\},\ \omega \in \Omega \quad \varphi_t^i(\omega) \equiv 0.$$

In case the market $\mathcal{M}$ is without transaction costs, we will write $\mathcal{M} = (S,\mathcal{P})$ instead of $\mathcal{M} = (S,\mathcal{P},\varphi)$.

On the other hand, if for each $i \in \{1,\dots,d\}$, each $t \in \{0,1,\dots,T\}$ and each $\omega \in \Omega$, the functions $\varphi_t^i(\omega)|_{[0,+\infty)}$ and $\varphi_t^i(\omega)|_{(-\infty,0])}$ are linear, we say that $\mathcal{M}$ is a market with proportional transaction costs. Usually, it is assumed that the functions $\varphi_t^i(\omega)$ do not depend on t and ω.

Now we can define the value of the strategy at terminal time T, after transactions.

Definition 2.6
For a given strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\dots,T}$ defined on the market $\mathcal{M}$, we set (by definition)

$$V_{T+}^{\boldsymbol{\Theta}} = V_{T-}^{\boldsymbol{\Theta}} - C_T^{\boldsymbol{\Theta}}.$$

The value $V_{T+}^{\boldsymbol{\Theta}}$ will be called *terminal value* of the strategy $\boldsymbol{\Theta}$ and also denoted by $V_T^{\boldsymbol{\Theta}}$.

Definition 2.7
The strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\dots,T}$ (defined on the market $\mathcal{M} = (S,\mathcal{P},\varphi)$) is called a *self-financing* strategy if

$$V_{t+}^{\boldsymbol{\Theta}} = V_{t-}^{\boldsymbol{\Theta}} - C_t^{\boldsymbol{\Theta}} \qquad \text{for all } t \in \{0,\dots,T\}. \tag{1}$$

If the market $\mathcal{M}$ is without transaction costs, the above condition simplifies as follows

$$V_{t+}^{\boldsymbol{\Theta}} = V_{t-}^{\boldsymbol{\Theta}} \qquad \text{for all } t \in \{0,\dots,T\}. \tag{2}$$

Let us notice that the condition (1) or (2) must be checked only for $t \in \{0,\dots,T-1\}$, because for $t = T$ the condition is valid by the definition of V_{T+}^{Θ}.

Thus for a self-financing strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\dots,T}$ defined on the market without transaction costs, the common value $V_{t+}^{\boldsymbol{\Theta}} = V_{t-}^{\boldsymbol{\Theta}}$, for $t \in \{0,\dots,T\}$, is called the value of the strategy at time t and denoted by $V_t^{\boldsymbol{\Theta}}$. In particular, $V_0^{\boldsymbol{\Theta}}$ is called the *initial value* of the strategy.

In a general situation, by the initial value of the strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\dots,T}$, we mean $V_{0-}^{\boldsymbol{\Theta}}$, which can also be denoted by $V_0^{\boldsymbol{\Theta}}$.

Definition 2.8
A self-financing strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\dots,T}$ is called an *arbitrage opportunity* or *arbitrage strategy* if $V_0^{\boldsymbol{\Theta}} = 0$ and the terminal value of the strategy satisfies

$$P(V_T^{\boldsymbol{\Theta}} \geq 0) = 1 \qquad \text{and} \qquad P(V_T^{\boldsymbol{\Theta}} > 0) > 0.$$

In the sequel we will assume that the market $\mathcal{M}$ is without transaction costs. The following lemma is an easy but useful observation.

Lemma 2.9
If there is a trading strategy $\{\mathbf{\Theta}_t\}_{t=0,\dots,T}$, defined on the market $\mathcal{M}=(S,\mathcal{P})$, such that $V_{0-}^{\mathbf{\Theta}}=0$, $V_{t-}^{\mathbf{\Theta}}\geq V_{t+}^{\mathbf{\Theta}}$ for $t\in\{0,\dots,T\}$, and $P(V_{T+}^{\mathbf{\Theta}}\geq 0)=1$, $P(V_{T+}^{\mathbf{\Theta}}>0)>0$, then there is an arbitrage opportunity on the market $\mathcal{M}$.

Proof. Let's create a strategy $\{\mathbf{\Phi}_t\}_{t=0,\dots,T}=\{(\Phi_t^+,\Phi_t^-,\Phi_t^1,\dots,\Phi_t^d)\}_{t=0,\dots,T}$ in the following way

$$\mathbf{\Phi}_0=(\Theta_0^+,\Theta_0^-,\Theta_0^1,\dots,\Theta_0^d)$$

and

$$\mathbf{\Phi}_{t+1}=(\Phi_{t+1}^+,\Theta_{t+1}^-,\Theta_{t+1}^1,\dots,\Theta_{t+1}^d)$$

for $t=0,\dots,T-1$, where

$$\mathbf{\Phi}_{t+1}^+=\Phi_t^+-\Theta_t^++\frac{V_{t-}^{\mathbf{\Theta}}-V_{t+}^{\mathbf{\Theta}}}{B_t^+}+\Theta_{t+1}^+.$$

We have proved that the process $\{\mathbf{\Phi}_t\}_{t=0,\dots,T}$ is indeed a strategy and then that it is an arbitrage opportunity on the market $\mathcal{M}$.

To see that $\{\mathbf{\Phi}_t\}_{t=0,\dots,T}\in\mathcal{P}$, we must check that $\Phi_t^+\geq 0$ for all $t\in\{0,\dots,T\}$, so first we verify this by using induction that $\Phi_t^+\geq\Theta_t^+$ for any $t\in\{0,\dots,T\}$. For $t=0$, we have $\Phi_0^+=\Theta_0^+$. Let $t_0\in\{0,\dots,T-1\}$ be given and assume $\Phi_{t_0}^+\geq\Theta_{t_0}^+$. Then, because $\Phi_{t_0}^+-\Theta_{t_0}^+\geq 0$ and $V_{t_0-}^{\mathbf{\Theta}}\geq V_{t_0+}^{\mathbf{\Theta}}$, we have $\Phi_{t_0+1}^+=\Phi_{t_0}^+-\Theta_{t_0}^++\frac{V_{t_0-}^{\mathbf{\Theta}}-V_{t_0+}^{\mathbf{\Theta}}}{B_{t_0}^+}+\Theta_{t_0+1}^+\geq\Theta_{t_0+1}^+$ and the proof of the induction step is completed.

Next we check that the strategy $\{\mathbf{\Phi}_t\}_{t=0,\dots,T}$ is self-financing. Note that

$$V_{t-}^{\mathbf{\Phi}}=\Phi_t^+B_t^+-\Theta_t^-B_t^-+\Theta_t^1S_t^1+\dots+\Theta_t^dS_t^d=\Phi_t^+B_t^++V_{t-}^{\mathbf{\Theta}}-\Theta_t^+B_t^+\geq V_{t-}^{\mathbf{\Theta}}. \quad (3)$$

Therefore,

$$\begin{aligned}
V_{t+}^{\mathbf{\Phi}} &= \Phi_{t+1}^+B_t^+-\Theta_{t+1}^-B_t^-+\Theta_{t+1}^1S_t^1+\dots+\Theta_{t+1}^dS_t^d\\
&= \Big(\Phi_t^+-\Theta_t^++\frac{V_{t-}^{\mathbf{\Theta}}-V_{t+}^{\mathbf{\Theta}}}{B_t^+}+\Theta_{t+1}^+\Big)B_t^+-\Theta_{t+1}^-B_t^-+\Theta_{t+1}^1S_t^1\\
&\quad+\dots+\Theta_{t+1}^dS_t^d\\
&= \Phi_t^+B_t^+-\Theta_t^+B_t^++V_{t-}^{\mathbf{\Theta}}-V_{t+}^{\mathbf{\Theta}}+\Theta_{t+1}^+B_t^+-\Theta_{t+1}^-B_t^-+\Theta_{t+1}^1S_t^1\\
&\quad+\dots+\Theta_{t+1}^dS_t^d\\
&= \Phi_t^+B_t^+-\Theta_t^+B_t^++V_{t-}^{\mathbf{\Theta}}\\
&= V_{t-}^{\mathbf{\Phi}}.
\end{aligned}$$

Finally, since $V_0^{\mathbf{\Phi}}=V_{0-}^{\mathbf{\Theta}}=0$ and (3) $V_T^{\mathbf{\Phi}}=V_{T-}^{\mathbf{\Phi}}\geq V_{T-}^{\mathbf{\Theta}}=V_{T+}^{\mathbf{\Theta}}$, we have $P(V_0^{\mathbf{\Phi}}=0)=1$, $P(V_T^{\mathbf{\Phi}}\geq 0)\geq P(V_T^{\mathbf{\Theta}}\geq 0)=1$ and $P(V_T^{\mathbf{\Phi}}>0)\geq P(V_{T+}^{\mathbf{\Theta}}>0)>0$. Thus we have proved that the strategy $\{\mathbf{\Phi}_t\}_{t=0,\dots,T}$ is an arbitrage opportunity on the market $\mathcal{M}$.

Theorem 2.10
For any predictable process $\{(\Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$ and any number $v_0\in\mathbb{R}$ there is exactly one predictable process $\{(\Theta_t^+,\Theta_t^-)\}_{t=0,\dots,T}$ with values in $[0,+\infty)^2$ such that

(i) the process $\{(\Theta_t^+,\Theta_t^-,\Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$ is a self-financing strategy with the initial value v_0,

(ii) for any $t\in\{0,\dots,T\}$ we have $\Theta_t^+\cdot\Theta_t^-=0$ (in other words $\{\Theta_t^+=0\}\cup\{\Theta_t^-=0\}=\Omega$).

Proof. First, we prove the existence of a predictable process $\{(\Theta_t^+,\Theta_t^-)\}_{t=0,\dots,T}$. Denote $X_{t-}=\Theta_t^1S_t^1+\ldots+\Theta_t^dS_t^d$ and $X_{t+}=\Theta_{t+1}^1S_t^1+\ldots+\Theta_{t+1}^dS_t^d$. Now, for the strategy $\Theta=\{(\Theta_t^+,\Theta_t^-,\Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$ given by an arbitrary predictable process $\{(\Theta_t^+,\Theta_t^-)\}_{t=0,\dots,T}$ with values in $[0,+\infty)^2$, we can write $V_{t-}^{\boldsymbol\Theta}=\Theta_t^+B_t^+-\Theta_t^-B_t^-+X_{t-}$ and $V_{t+}^{\boldsymbol\Theta}=\Theta_{t+1}^+B_t^+-\Theta_{t+1}^-B_t^-+X_{t+}$.

Let us observe that X_{0-} is a constant number and X_{t+} is $\mathcal{F}_t$-measurable for $t=0,\dots,T-1$. We will construct a process $\{(\Theta_t^+,\Theta_t^-)\}_{t=0,\dots,T}$ inductively with respect to $t\in\{0,\dots,T\}$. For $t=0$, we set

$$(\Theta_0^+,\Theta_0^-):=\begin{cases}(v_0-X_{0-},0) & \text{if } v_0-X_{0-}\ge 0,\\ (0,X_{0-}-v_0) & \text{otherwise,}\end{cases}\tag{4}$$

for $t\in\{0,\dots,T-1\}$, assuming that we have already constructed $\mathcal{F}_{t-1}$-measurable variables Θ_t^+ and Θ_t^-, we put

$$\Theta_{t+1}^+:=\frac{V_{t-}^{\boldsymbol\Theta}-X_{t+}}{B_t^+}\mathbb{1}_{\{V_{t-}^{\Theta}-X_{t+}\ge 0\}}\tag{5}$$

and

$$\Theta_{t+1}^-:=\frac{X_{t+}-V_{t-}^{\boldsymbol\Theta}}{B_t^-}\mathbb{1}_{\{V_{t-}^{\Theta}-X_{t+}<0\}}.\tag{6}$$

Regardless of the fact that in the equations above a process $\{(\Theta_t^+,\Theta_t^-)\}_{t=0,\dots,T}$ is not entirely defined, the value of this strategy at time t before the transaction $V_{t-}^{\boldsymbol\Theta}$ is known because we have assumed that Θ_t^+ and Θ_t^- have already been constructed.

Now we may easily check that the process $\boldsymbol\Theta=\{(\Theta_t^+,\Theta_t^-,\Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$ defined by (4)–(6) is predictable and satisfies conditions of the theorem.

First notice that, by (4), we have

$$V_{0-}^{\boldsymbol\Theta}=\Theta_0^+B_0^+-\Theta_0^-B_0^-+X_{0-}=\Theta_0^+-\Theta_0^-+X_{0-}=v_0.$$

On the other hand, by (5) and (6), for any $t\in\{0,\dots,T-1\}$ we get

$$\begin{aligned}V_{t+}^{\boldsymbol\Theta}&=\Theta_{t+1}^+B_t^+-\Theta_{t+1}^-B_t^-+X_{t+}\\&=(V_{t-}^{\boldsymbol\Theta}-X_{t+})\mathbb{1}_{\{V_{t-}^{\boldsymbol\Theta}-X_{t+}\ge 0\}}-(X_{t+}-V_{t-}^{\boldsymbol\Theta})\mathbb{1}_{\{V_{t-}^{\boldsymbol\Theta}-X_{t+}<0\}}+X_{t+}\\&=V_{t-}^{\boldsymbol\Theta}.\end{aligned}$$

To see that Θ_t^+,Θ_t^- are $\mathcal{F}_{t-1}$-measurable, for $t\in\{0,\dots,T\}$, let us notice, that Θ_0^+,Θ_0^- are constants and that for a given $t\in\{0,\dots,T-1\}$ if Θ_t^+,Θ_t^- are $\mathcal{F}_{t-1}$-measurable, then by (5) and (6) $\Theta_{t+1}^+,\Theta_{t+1}^-$ are $\mathcal{F}_t$-measurable. Thus, we have already checked that the condition (i) is satisfied. The condition (ii) is also satisfied because of the definition of Θ_t^+ and Θ_t^- (see equations (4)–(6).

Now we prove the uniqueness of the process $\{(\Theta_t^+, \Theta_t^-)\}_{t=0,\dots,T}$. Suppose, conversely, that we have two processes $\{(\Theta_t^+, \Theta_t^-)\}_{t=0,\dots,T}$ and $\{(\Phi_t^+, \Phi_t^-)\}_{t=0,\dots,T}$ satisfying conditions (i) and (ii) from the theorem. Since $\Theta_0^+ - \Theta_0^- + X_{0-} = v_0 = \Phi_0^+ - \Phi_0^- + X_{0-}$, $\Theta_0^+, \Theta_0^-, \Phi_0^+, \Phi_0^- \in [0,\infty)$ and $\Theta_0^+ \cdot \Theta_0^- = 0 = \Phi_0^+ \cdot \Phi_0^-$ it follows that $\Theta_0^+ = \Phi_0^+$ and $\Theta_0^- = \Phi_0^-$.

Next, using induction with respect to $t \in \{0,\dots,T\}$, we show that $\Theta_t^+ = \Phi_t^+$ and $\Theta_t^- = \Phi_t^-$ for all $t \in \{0,\dots,T\}$. So, assume that for $t_0 \in \{0,\dots,T-1\}$, we have $\Theta_{t_0}^+ = \Phi_{t_0}^+$ and $\Theta_{t_0}^- = \Phi_{t_0}^-$. Then, $V_{t_0-}^{\mathbf{\Theta}} = \Theta_{t_0}^+ B_{t_0}^+ - \Theta_{t_0}^- B_{t_0}^- + X_{t_0-} = \Phi_{t_0}^+ B_{t_0}^+ - \Phi_{t_0}^- B_{t_0}^- + X_{t_0-} = V_{t_0-}^{\mathbf{\Phi}}$. Since $V_{t_0+}^{\mathbf{\Theta}} = V_{t_0-}^{\mathbf{\Theta}}$ and $V_{t_0+}^{\mathbf{\Phi}} = V_{t_0-}^{\mathbf{\Phi}}$, we obtain $\Theta_{t_0+1}^+ B_{t_0}^+ - \Theta_{t_0+1}^- B_{t_0}^- + X_{t_0+} = V_{t_0+}^{\mathbf{\Theta}} = V_{t_0+}^{\mathbf{\Phi}} = \Phi_{t_0+1}^+ B_{t_0}^+ - \Phi_{t_0+1}^- B_{t_0}^- + X_{t_0+}$, and thus

$$\Theta_{t_0+1}^+ B_{t_0}^+ - \Theta_{t_0+1}^- B_{t_0}^- = \Phi_{t_0+1}^+ B_{t_0}^+ - \Phi_{t_0+1}^- B_{t_0}^-. \tag{7}$$

Since $\Phi_{t_0+1}^+ \cdot \Phi_{t_0+1}^- = 0$, $B_{t_0}^+, B_{t_0}^- > 0$ and $\Theta_{t_0+1}^+, \Theta_{t_0+1}^-, \Phi_{t_0+1}^+, \Phi_{t_0+1}^- \geq 0$, we see by (7), that $\{\Theta_{t_0+1}^- = 0\} \subset \{\Phi_{t_0+1}^- = 0\}$ and $\{\Theta_{t_0+1}^+ = 0\} \subset \{\Phi_{t_0+1}^+ = 0\}$. Through this symmetry, we also have $\{\Phi_{t_0+1}^- = 0\} \subset \{\Theta_{t_0+1}^- = 0\}$ and $\{\Phi_{t_0+1}^+ = 0\} \subset \{\Theta_{t_0+1}^+ = 0\}$, and so $\{\Theta_{t_0+1}^- = 0\} = \{\Phi_{t_0+1}^- = 0\}$ and $\{\Theta_{t_0+1}^+ = 0\} = \{\Phi_{t_0+1}^+ = 0\}$. Now, on the set $\{\Theta_{t_0+1}^- = 0\} = \{\Phi_{t_0+1}^- = 0\}$ equation (7) simplifies to

$$\Theta_{t_0+1}^+ B_{t_0}^+ = \Phi_{t_0+1}^+ B_{t_0}^+. \tag{8}$$

Since $B_{t_0}^+ > 0$, it follows from (8)) that $\Theta_{t_0+1}^+ = \Phi_{t_0+1}^+$ on the set $\{\Theta_{t_0+1}^- = 0\}$. Of course, on the set $\{\Theta_{t_0+1}^+ = 0\} = \{\Phi_{t_0+1}^+ = 0\}$ we also have $\Theta_{t_0+1}^+ = \Phi_{t_0+1}^+$. Thus, we get $\Theta_{t_0+1}^+ = \Phi_{t_0+1}^+$, on the set $\{\Theta_{t_0+1}^+ = 0\} \cup \{\Theta_{t_0+1}^- = 0\} = \Omega$. Similar arguments show that $\Theta_{t_0+1}^- = \Phi_{t_0+1}^-$ on Ω.

Theorem 2.11

Assume that we are given a predictable process $\{(\Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$. *Let* $\{(\Theta_t^+, \Theta_t^-)\}_{t=0,\dots,T}$ *and* $\{(\Phi_t^+, \Phi_t^-)\}_{t=0,\dots,T}$ *be two predictable processes such that both processes*

$$\Theta = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T} \quad \text{and} \quad \Phi = \{(\Phi_t^+, \Phi_t^-, \Theta_t^1,\dots,\Theta_t^d)\}_{t=0,\dots,T}$$

are self-financing strategies. If $V_0^{\mathbf{\Theta}} \geq V_0^{\mathbf{\Phi}}$ *and for all* $t \in \{0,\dots,T-1\}$ *we have* $\Theta_t^+ \cdot \Theta_t^- = 0$*, then for all* $t \in \{0,\dots,T\}$,

$$V_t^{\mathbf{\Theta}} \geq V_t^{\mathbf{\Phi}}. \tag{9}$$

Moreover, if $t_0 \in \{0,\dots,T-1\}$ *is such that*

$$P\Big(\Phi_{t_0+1}^+ > 0, \Phi_{t_0+1}^- > 0, \frac{B_{t_0+1}^+}{B_{t_0}^+} < \frac{B_{t_0+1}^-}{B_{t_0}^-}\Big) > 0, \tag{10}$$

then for all $t \in \{t_0+1,\dots,T\}$,

$$P(V_t^{\mathbf{\Theta}} > V_t^{\mathbf{\Phi}}) > 0. \tag{11}$$

Proof. We prove (9) using induction on t. Let $t = 0$, then (9) is satisfied by the assumption. Assume now that the claim is true for t, where $t \in \{0, \ldots, T-1\}$. We show that it is true for $t+1$.

Since

$$\boldsymbol{\Theta} = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T} \quad \text{and} \quad \boldsymbol{\Phi} = \{(\Phi_t^+, \Phi_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$$

are self-financing strategies, we have

$$V_t^{\boldsymbol{\Theta}} = V_{t+}^{\boldsymbol{\Theta}} = \Theta_{t+1}^+ B_t^+ - \Theta_{t+1}^- B_t^- + \Theta_{t+1}^1 S_t^1 + \ldots + \Theta_{t+1}^d S_t^d$$

and

$$V_t^{\boldsymbol{\Phi}} = V_{t+}^{\boldsymbol{\Phi}} = \Phi_{t+1}^+ B_t^+ - \Phi_{t+1}^- B_t^- + \Theta_{t+1}^1 S_t^1 + \ldots + \Theta_{t+1}^d S_t^d.$$

Inequality $V_t^{\boldsymbol{\Theta}} \geq V_t^{\boldsymbol{\Phi}}$ implies

$$\Theta_{t+1}^+ B_t^+ - \Theta_{t+1}^- B_t^- \geq \Phi_{t+1}^+ B_t^+ - \Phi_{t+1}^- B_t^-. \tag{12}$$

First, consider the situation on the set $\{\Theta_{t+1}^+ = 0\}$. Multiplying (12) by $\frac{B_{t+1}^-}{B_t^-}$ and using $\frac{B_{t+1}^+}{B_t^+} \leq \frac{B_{t+1}^-}{B_t^-}$, we obtain

$$-\Theta_{t+1}^- B_{t+1}^- \geq \Phi_{t+1}^+ B_t^+ \cdot \frac{B_{t+1}^-}{B_t^-} - \Phi_{t+1}^- B_{t+1}^- \geq \Phi_{t+1}^+ B_{t+1}^+ - \Phi_{t+1}^- B_{t+1}^-.$$

Therefore, on the set $\{\Theta_{t+1}^+ = 0\}$ we get

$$\begin{aligned} V_{t+1}^{\boldsymbol{\Theta}} = V_{(t+1)-}^{\boldsymbol{\Theta}} &= -\Theta_{t+1}^- B_{t+1}^- + \Theta_{t+1}^1 S_{t+1}^1 + \ldots + \Theta_{t+1}^d S_{t+1}^d \\ &\geq \Phi_{t+1}^+ B_{t+1}^+ - \Phi_{t+1}^- B_{t+1}^- + \Theta_{t+1}^1 S_{t+1}^1 + \ldots + \Theta_{t+1}^d S_{t+1}^d = V_{(t+1)-}^{\boldsymbol{\Phi}} \\ &= V_{t+1}^{\boldsymbol{\Phi}}. \end{aligned}$$

Consider now the situation on the set $\{\Theta_{t+1}^- = 0\}$. Through multiplying (12) by $\frac{B_{t+1}^+}{B_t^+}$ and using the inequality $\frac{B_{t+1}^+}{B_t^+} \leq \frac{B_{t+1}^-}{B_t^-}$, we obtain

$$\Theta_{t+1}^+ B_{t+1}^+ \geq \Phi_{t+1}^+ B_{t+1}^+ - \Phi_{t+1}^- B_t^- \cdot \frac{B_{t+1}^+}{B_t^+} \geq \Phi_{t+1}^+ B_{t+1}^+ - \Phi_{t+1}^- B_{t+1}^-.$$

Hence, also on the set $\{\Theta_{t+1}^- = 0\}$, we have

$$\begin{aligned} V_{t+1}^{\boldsymbol{\Theta}} = V_{(t+1)-}^{\boldsymbol{\Theta}} &= \Theta_{t+1}^+ B_{t+1}^+ + \Theta_{t+1}^1 S_{t+1}^1 + \ldots + \Theta_{t+1}^d S_{t+1}^d \\ &\geq \Phi_{t+1}^+ B_{t+1}^+ - \Phi_{t+1}^- B_{t+1}^- + \Theta_{t+1}^1 S_{t+1}^1 + \ldots + \Theta_{t+1}^d S_{t+1}^d = V_{(t+1)-}^{\boldsymbol{\Phi}} \\ &= V_{t+1}^{\boldsymbol{\Phi}}, \end{aligned}$$

which completes the proof of (9), because $\{\Theta_{t+1}^+ = 0\} \cup \{\Theta_{t+1}^- = 0\} = \Omega$.

Next we prove (11) by using induction on t. Let us take any

$$\omega \in \Big\{\Phi_{t_0+1}^+ > 0,\, \Phi_{t_0+1}^- > 0,\, \frac{B_{t_0+1}^+}{B_{t_0}^+} < \frac{B_{t_0+1}^-}{B_{t_0}^-}\Big\}. \tag{13}$$

In the sequel, we will write $v_t^{\mathbf{\Theta}}$ for $V_t^{\mathbf{\Theta}}(\omega)$, $v_t^{\mathbf{\Phi}}$ for $V_t^{\mathbf{\Phi}}(\omega)$, θ_t^+, θ_t^-, θ_t^i for, respectively, $\Theta_t^+(\omega)$, $\Theta_t^-(\omega)$, $\Theta_t^i(\omega)$ and ϕ_t^+, ϕ_t^-, ϕ_t^i for, respectively, $\Phi_t^+(\omega)$, $\Phi_t^-(\omega)$, $\Phi_t^i(\omega)$. We will use similar notation for $B_t^+(\omega)$, $B_t^-(\omega)$ and $S_t^i(\omega)$ (there is no possibility of confusion with the cost process, because we consider the market without transaction costs).

First we show that $V_{t_0+1}^{\mathbf{\Theta}}(\omega) > V_{t_0+1}^{\mathbf{\Phi}}(\omega)$. Since Θ is a self-financing strategy, we have

$$v_{t_0}^{\mathbf{\Theta}} = V_{t_0+}^{\mathbf{\Theta}}(\omega) = \theta_{t_0+1}^+ b_{t_0}^+ - \theta_{t_0+1}^- b_{t_0}^- + \theta_{t_0+1}^1 s_{t_0}^1 + \ldots + \theta_{t_0+1}^d s_{t_0}^d.$$

Let us set

$$\begin{aligned} x_{t_0} &:= \theta_{t_0+1}^1 s_{t_0}^1 + \ldots + \theta_{t_0+1}^d s_{t_0}^d, \\ x_{t_0+1} &:= \theta_{t_0+1}^1 s_{t_0+1}^1 + \ldots + \theta_{t_0+1}^d s_{t_0+1}^d, \end{aligned}$$

additionally $v_{t_0}^{\mathbf{\Theta}} - x_{t_0} = \theta_{t_0+1}^+ b_{t_0}^+ - \theta_{t_0+1}^- b_{t_0}^-$. Consider the situation when $v_{t_0}^{\mathbf{\Theta}} - x_{t_0} \geq 0$. Then $\theta_{t_0+1}^- = 0$ and $v_{t_0}^{\mathbf{\Theta}} - x_{t_0} = \theta_{t_0+1}^+ b_{t_0}^+$, so

$$\theta_{t_0+1}^+ = \frac{v_{t_0}^{\mathbf{\Theta}} - x_{t_0}}{b_{t_0}^+}. \tag{14}$$

Note that

$$\begin{aligned} v_{t_0}^{\mathbf{\Phi}} = v_{t_0+}^{\mathbf{\Phi}} &= \phi_{t_0+1}^+ b_{t_0}^+ - \phi_{t_0+1}^- b_{t_0}^- + \theta_{t_0+1}^1 s_{t_0}^1 + \ldots + \theta_{t_0+1}^d s_{t_0}^d \\ &= \phi_{t_0+1}^+ b_{t_0}^+ - \phi_{t_0+1}^- b_{t_0}^- + x_{t_0}, \end{aligned} \tag{15}$$

therefore

$$\phi_{t_0+1}^+ = \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0} + \phi_{t_0+1}^- b_{t_0}^-}{b_{t_0}^+}. \tag{16}$$

One can make the following obvious observation that, because of (13) and the fact that $\phi_{t_0+1}^- > 0$ we have

$$\phi_{t_0+1}^- \frac{b_{t_0}^-}{b_{t_0}^+} b_{t_0+1}^+ - \phi_{t_0+1}^- b_{t_0+1}^- < 0. \tag{17}$$

Now, using (9) and (14)–(17), we obtain

$$\begin{aligned} v_{t_0+1}^{\mathbf{\Theta}} &= \theta_{t_0+1}^+ b_{t_0+1}^+ - \theta_{t_0+1}^- b_{t_0+1}^- + \theta_{t_0+1}^1 s_{t_0+1}^1 + \ldots + \theta_{t_0+1}^d s_{t_0+1}^d \\ &= \theta_{t_0+1}^+ b_{t_0+1}^+ + x_{t_0+1} = \frac{v_{t_0}^{\mathbf{\Theta}} - x_{t_0}}{b_{t_0}^+} b_{t_0+1}^+ + x_{t_0+1} \geq \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0}}{b_{t_0}^+} b_{t_0+1}^+ + x_{t_0+1} \\ &> \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0}}{b_{t_0}^+} b_{t_0+1}^+ + \phi_{t_0+1}^- \frac{b_{t_0}^-}{b_{t_0}^+} b_{t_0+1}^+ - \phi_{t_0+1}^- b_{t_0+1}^- + x_{t_0+1} \\ &= (v_{t_0}^{\mathbf{\Phi}} - x_{t_0} + \phi_{t_0+1}^- b_{t_0}^-) \frac{b_{t_0+1}^+}{b_{t_0}^+} - \phi_{t_0+1}^- b_{t_0+1}^- + x_{t_0+1} \\ &= \phi_{t_0+1}^+ b_{t_0+1}^+ - \phi_{t_0+1}^- b_{t_0+1}^- + x_{t_0+1} \\ &= v_{t_0+1}^{\mathbf{\Phi}}. \end{aligned}$$

Further, if $v_{t_0}^{\mathbf{\Theta}} - x_{t_0} < 0$, then $\theta_{t_0+1}^{+} = 0$ and $v_{t_0}^{\mathbf{\Theta}} - x_{t_0} = -\theta_{t_0+1}^{-} b_{t_0}^{-}$, and thus

$$-\theta_{t_0+1}^{-} = \frac{v_{t_0}^{\mathbf{\Theta}} - x_{t_0}}{b_{t_0}^{-}}.$$

As in earlier examples, using (9), (16) and (17), we have

$$\begin{aligned}
v_{t_0+1}^{\mathbf{\Theta}} &= -\theta_{t_0+1}^{-} b_{t_0+1}^{-} + x_{t_0+1} = \frac{v_{t_0}^{\mathbf{\Theta}} - x_{t_0}}{b_{t_0}^{-}} b_{t_0+1}^{-} + x_{t_0+1} \\
&\geq \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0}}{b_{t_0}^{-}} b_{t_0+1}^{-} + x_{t_0+1} > \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0}}{b_{t_0}^{+}} b_{t_0+1}^{+} + x_{t_0+1} \\
&> \frac{v_{t_0}^{\mathbf{\Phi}} - x_{t_0}}{b_{t_0}^{+}} b_{t_0+1}^{+} + \phi_{t_0+1}^{-} \frac{b_{t_0}^{-}}{b_{t_0}^{+}} b_{t_0+1}^{+} - \phi_{t_0+1}^{-} b_{t_0+1}^{-} + x_{t_0+1} \\
&= \phi_{t_0+1}^{+} b_{t_0+1}^{+} - \phi_{t_0+1}^{-} b_{t_0+1}^{-} + x_{t_0+1} \\
&= v_{t_0+1}^{\mathbf{\Phi}}.
\end{aligned}$$

Since the afore-mentioned arguments are valid for arbitrary $\omega \in \left\{\Phi_{t_0+1}^{+} > 0, \Phi_{t_0+1}^{-} > 0, \frac{B_{t_0+1}^{+}}{B_{t_0}^{+}} < \frac{B_{t_0+1}^{-}}{B_{t_0}^{-}}\right\}$, we see that

$$\left\{\Phi_{t_0+1}^{+} > 0, \Phi_{t_0+1}^{-} > 0, \frac{B_{t_0+1}^{+}}{B_{t_0}^{+}} < \frac{B_{t_0+1}^{-}}{B_{t_0}^{-}}\right\} \subset \{V_{t_0+1}^{\mathbf{\Theta}} > V_{t_0+1}^{\mathbf{\Phi}}\}.$$

Assume now that $P(V_t^{\mathbf{\Theta}} > V_t^{\mathbf{\Phi}}) > 0$ for t, where $t \in \{t_0 + 1, \ldots, T - 1\}$. We show that $\{V_t^{\mathbf{\Theta}} > V_t^{\mathbf{\Phi}}\} \subset \{V_{t+1}^{\mathbf{\Theta}} > V_{t+1}^{\mathbf{\Phi}}\}$. So, let us fix $\omega \in \{V_t^{\mathbf{\Theta}} > V_t^{\mathbf{\Phi}}\}$ and use the notation which we used in the proof that $P(V_{t_0+1}^{\mathbf{\Theta}} > V_{t_0+1}^{\mathbf{\Phi}}) > 0$. Thus, we have

$$v_t^{\mathbf{\Theta}} = v_{t+}^{\mathbf{\Theta}} = \theta_{t+1}^{+} b_t^{+} - \theta_{t+1}^{-} b_t^{-} + x_t.$$

Consider the situation when $v_t^{\mathbf{\Theta}} - x_t \geq 0$. Then $\theta_{t+1}^{-} = 0$ and $\theta_{t+1}^{+} = \frac{v_t^{\mathbf{\Theta}} - x_t}{b_t^{+}}$. On the other hand, we have

$$v_t^{\mathbf{\Phi}} = \phi_{t+1}^{+} b_t^{+} - \phi_{t+1}^{-} b_t^{-} + x_t,$$

so $\phi_{t+1}^{+} = \frac{v_t^{\mathbf{\Phi}} - x_t + \phi_{t+1}^{-} b_t^{-}}{b_t^{+}}$. Since $\phi_{t+1}^{+} \geq 0$, we see that $\phi_{t+1}^{-} \geq \frac{x_t - v_t^{\mathbf{\Phi}}}{b_t^{-}}$. Since $\frac{b_{t+1}^{+}}{b_t^{+}} \leq \frac{b_{t+1}^{-}}{b_t^{-}}$ and $\phi_{t+1}^{-} \geq 0$, we get

$$\phi_{t+1}^{-} \frac{b_t^{-}}{b_t^{+}} b_{t+1}^{+} - \phi_{t+1}^{-} b_{t+1}^{-} \leq 0.$$

Therefore, we have

$$v_{t+1}^{\mathbf{\Theta}} = \theta_{t+1}^{+} b_{t+1}^{+} + x_{t+1} = \frac{v_t^{\mathbf{\Theta}} - x_t}{b_t^{+}} b_{t+1}^{+} + x_{t+1} > \frac{v_t^{\mathbf{\Phi}} - x_t}{b_t^{+}} b_{t+1}^{+} + x_{t+1}$$

$$\geq \frac{v_t^{\mathbf{\Phi}} - x_t}{b_t^+} b_{t+1}^+ + \phi_{t+1}^- \frac{b_t^-}{b_t^+} b_{t+1}^+ - \phi_{t+1}^- b_{t+1}^- + x_{t+1}$$
$$= \phi_{t+1}^+ b_{t+1}^+ - \phi_{t+1}^- b_{t+1}^- + x_{t+1} = v_{t+1}^{\mathbf{\Phi}}.$$

If $v_t^{\mathbf{\Theta}} - x_t < 0$, then $\theta_{t+1}^+ = 0$ and $-\theta_{t+1}^- = \frac{v_t^{\mathbf{\Theta}} - x_t}{b_t^-}$. As in previous cases, we obtain

$$v_{t+1}^{\mathbf{\Theta}} = -\theta_{t+1}^- b_{t+1}^- + x_{t+1} = \frac{v_t^{\mathbf{\Theta}} - x_t}{b_t^-} b_{t+1}^- + x_{t+1} > \frac{v_t^{\mathbf{\Phi}} - x_t}{b_t^-} b_{t+1}^- + x_{t+1}$$
$$\geq \frac{v_t^{\mathbf{\Phi}} - x_t}{b_t^+} b_{t+1}^+ + x_{t+1} \geq \frac{v_t^{\mathbf{\Phi}} - x_t}{b_t^+} b_{t+1}^+ + x_{t+1} + \phi_{t+1}^- \frac{b_t^-}{b_t^+} b_{t+1}^+ - \phi_{t+1}^- b_{t+1}^-$$
$$= v_{t+1}^{\mathbf{\Phi}}.$$

Thus, we have proved that $\{V_t^{\mathbf{\Theta}} > V_t^{\mathbf{\Phi}}\} \subset \{V_{t+1}^{\mathbf{\Theta}} > V_{t+1}^{\mathbf{\Phi}}\}$ for all $t \in \{t_0 + 1, \ldots, T - 1\}$. This completes the proof of the theorem.

As a consequence of these theorems we obtain the following fact.

Corollary 2.12
If there is an arbitrage strategy on the market $\mathcal{M} = (S, \mathcal{P})$, then there is, also, an arbitrage strategy satisfying the condition (ii) of Theorem 2.10.

Proof. Indeed, if $\mathbf{\Theta} = \{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$ is any arbitrage strategy on the market $\mathcal{M} = (S, \mathcal{P})$, then by Theorem 2.10 there is a predictable process $\{(\Phi_t^+, \Phi_t^-)\}_{t=0,\ldots,T}$ such that the process $\mathbf{\Phi} = \{(\Phi_t^+, \Phi_t^-, \Theta_t^1, \ldots, \Theta_t^d)\}_{t=0,\ldots,T}$ is a self-financing strategy with the initial value 0. Then, by Theorem 2.11, we have

$$P(V_T^{\mathbf{\Phi}} \geq 0) \geq P(V_T^{\mathbf{\Theta}} \geq 0) = 1$$

and

$$P(V_T^{\mathbf{\Phi}} > 0) \geq P(V_T^{\mathbf{\Theta}} > 0) > 0.$$

3. Martingale property

Since we have two processes $\{B_t^+\}_{t=0,\ldots,T}$ and $\{B_t^-\}_{t=0,\ldots,T}$ that can be considered as the processes of the value of money in time, it is not clear how to define discounted price processes of risky assets. The same reason makes it unclear how to define the notion of a martingale measure. To avoid this difficulty we propose the following concept of a martingale pair.

Definition 3.1
Let us assume that we are given the market $\mathcal{M} = (S, \mathcal{P})$ defined on the probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t=0,\ldots,T}$.

If there exist a predictable process $\{B_t\}_{t=0,\ldots,T}$ and a probability measure P^* on $(\Omega, \mathcal{F}_T)$ such that

(i) P^* is equivalent to P,

(ii) $B_0 = 1$ and for all $t \in \{0, \ldots, T-1\}$ we have

$$\frac{B^+_{t+1}}{B^+_t} \le \frac{B_{t+1}}{B_t} \le \frac{B^-_{t+1}}{B^-_t},$$

(iii) the process $\{(\frac{S^1_t}{B_t}, \ldots, \frac{S^d_t}{B_t})\}_{t=0,\ldots,T}$ is P^*-martingale,

then we say that the pair $(\{B_t\}_{t=0,\ldots,T}, P^*)$ is a *martingale pair* for the market $\mathcal{M} = (S, \mathcal{P})$.

Let us assume that we are given a predictable process $\{B_t\}_{t=0,\ldots,T}$ satisfying condition (ii) of Definition 3.1. Then, we consider the model $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$, where the process $\tilde{S}$ of prices is defined as $\{(B_t, S^1_t, \ldots, S^d_t)\}_{t=0,1,\ldots,T}$, the set $\tilde{\mathcal{P}}$ consists of all predictable processes $\{(\Theta^0_t, \Theta^1_t, \ldots, \Theta^d_t)\}_{t=0,1,\ldots,T}$ with values in $\mathbb{R}^{d+1}$.

The values of the strategy $\boldsymbol{\Theta} = \{(\Theta^0_t, \Theta^1_t, \ldots, \Theta^d_t)\}_{t=0,1,\ldots,T}$ in the model $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$ are defined by

$$V^{\boldsymbol{\Theta}}_{t-} = \Theta^0_t B_t + \Theta^1_t S^1_t + \ldots + \Theta^d_t S^d_t \qquad \text{for } t = 0, 1, \ldots, T$$

and

$$V^{\boldsymbol{\Theta}}_{t+} = \Theta^0_{t+1} B_t + \Theta^1_{t+1} S^1_t + \ldots + \Theta^d_{t+1} S^d_t \qquad \text{for } t = 0, 1, \ldots, T.$$

Similarly to the proof of Lemma 2.9, one can prove the following

LEMMA 3.2
If there is a trading strategy $\{\boldsymbol{\Theta}_t\}_{t=0,\ldots,T}$, defined on the market $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$, such that $V^{\boldsymbol{\Theta}}_{0-} = 0$, $V^{\boldsymbol{\Theta}}_{t-} \ge V^{\boldsymbol{\Theta}}_{t+}$ for $t \in \{0, \ldots, T\}$, and $P(V^{\boldsymbol{\Theta}}_{T+} \ge 0) = 1$, $P(V^{\boldsymbol{\Theta}}_{T+} > 0) > 0$, then there is an arbitrage opportunity on the market $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$.

Using the above lemma, we can prove the following

LEMMA 3.3
If there is an arbitrage opportunity on the market $\mathcal{M}$, then there is, also, an arbitrage opportunity on the market $\widetilde{\mathcal{M}}$.

Proof. Let us assume that $\{\boldsymbol{\Theta}_t\}_{t=0,\ldots,T}$, defined on the market $\mathcal{M} = (S, \mathcal{P})$ is an arbitrage opportunity. Let us define a strategy $\{\boldsymbol{\Psi}_t\}_{t=0,\ldots,T}$, defined on the market $\widetilde{\mathcal{M}}$, as follows

$$\Psi^0_0 = \Theta^+_0 B^+_0 - \Theta^-_0 B^-_0 = \Theta^+_0 - \Theta^-_0$$

and

$$\Psi^0_t = \frac{\Theta^+_t B^+_{t-1} - \Theta^-_t B^-_{t-1}}{B_{t-1}} \tag{18}$$

for $t \in \{1, \ldots, T\}$ and

$$\Psi^i_t = \Theta^i_t \tag{19}$$

for $t \in \{0, \ldots, T\}$ and $i \in \{1, \ldots, d\}$. Observe that $V^{\boldsymbol{\Psi}}_{0-} = V^{\boldsymbol{\Theta}}_{0-} = 0$. We will prove that $V^{\boldsymbol{\Psi}}_{t-} \ge V^{\boldsymbol{\Psi}}_{t+}$, for $t \in \{0, \ldots, T\}$. Denote $X_{t-} = \Theta^1_t S^1_t + \ldots + \Theta^d_t S^d_t$ and

$X_{t+} = \Theta^1_{t+1} S^1_t + \ldots + \Theta^d_{t+1} S^d_t$. Since $\{\mathbf{\Theta}_t\}_{t=0,\ldots,T}$ is a self-financing strategy, we have $\Theta^+_t B^+_t - \Theta^-_t B^-_t + X_{t-} = \Theta^+_{t+1} B^+_t - \Theta^-_{t+1} B^-_t + X_{t+}$ and so

$$X_{t-} - X_{t+} = (\Theta^+_{t+1} - \Theta^+_t) B^+_t - (\Theta^-_{t+1} - \Theta^-_t) B^-_t. \tag{20}$$

Observe that, by (18)–(20) and the condition (ii) of Definition 3.1, we get

$$\begin{aligned}
V^{\mathbf{\Psi}}_{t-} - V^{\mathbf{\Psi}}_{t+} &= \Psi^0_t B_t + X_{t-} - \Psi^0_{t+1} B_t - X_{t+} \\
&= \frac{\Theta^+_t B^+_{t-1} - \Theta^-_t B^-_{t-1}}{B_{t-1}} B_t - \frac{\Theta^+_{t+1} B^+_t - \Theta^-_{t+1} B^-_t}{B_t} B_t \\
&\quad + (\Theta^+_{t+1} - \Theta^+_t) B^+_t - (\Theta^-_{t+1} - \Theta^-_t) B^-_t \\
&= \Theta^+_t \Big(\frac{B^+_{t-1}}{B_{t-1}} B_t - B^+_t \Big) - \Theta^-_t \Big(\frac{B^-_{t-1}}{B_{t-1}} B_t - B^-_t \Big) \\
&\geq 0.
\end{aligned}$$

Also, by (18)–(20) and the condition (ii) of Definition 3.1, we obtain

$$\begin{aligned}
V^{\mathbf{\Psi}}_T = V^{\mathbf{\Psi}}_{T-} &= \frac{\Theta^+_T B^+_{T-1} - \Theta^-_T B^-_{T-1}}{B_{T-1}} B_T + X_{T-} \\
&= \Theta^+_T B^+_{T-1} \frac{B_T}{B_{T-1}} - \Theta^-_T B^-_{T-1} \frac{B_T}{B_{T-1}} + X_{T-} \\
&\geq \Theta^+_T B^+_T - \Theta^-_T B^-_T + X_{T-} = V^{\mathbf{\Theta}}_{T-} \\
&= V^{\mathbf{\Theta}}_T.
\end{aligned}$$

By the above inequality, we have

$$P(V^{\mathbf{\Psi}}_T \geq 0) \geq P(V^{\mathbf{\Theta}}_T \geq 0) = 1$$

and

$$P(V^{\mathbf{\Psi}}_T > 0) \geq P(V^{\mathbf{\Theta}}_T > 0) = 1$$

and so by Lemma 3.2, there is an arbitrage opportunity on the market $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$.

Example 3.4

Let us consider a one-period model $\mathcal{M}$ of a financial market with one risky asset S_t and two processes $B^-_t = (1,1)^t$ and $B^+_t = (1,02)^t$. The stock price at time $t = 1$ can take two different values $S_1(\omega_1) = 108$, $S_1(\omega_2) = 104$ and $S_0 = 100$. Next, we consider the model $\widetilde{\mathcal{M}} = (\tilde{S}, \tilde{\mathcal{P}})$, where the process $\tilde{S}$ of prices is defined as $\{(B_t, S_t)\}_{t=0,1,\ldots,T}$, the set $\tilde{\mathcal{P}}$ consists of all predictable processes $\{(\Theta^0_t, \Theta_t)\}_{t=0,1,\ldots,T}$ and $B_t = (1,02)^t$.

Let us consider a strategy Θ on the market $\widetilde{\mathcal{M}}$ defined $(\Theta^0_0, \Theta^1_0) = (0,0)$ and $(\Theta^0_1, \Theta^1_1) = (-100, 1)$. Note that Θ is a self-financing strategy, because $\Theta^0_0 B_0 + \Theta^1_0 S_0 = \Theta^0_1 B_0 + \Theta^1_1 S_0$ and an arbitrage strategy, because $V^{\Theta}_0 = 0$ and $V^{\Theta}_1 = \Theta^0_1 B_1 + \Theta^1_1 S_1 = -100 + S_1 > 0$.

We will show that there is not an arbitrage strategy on the market $\mathcal{M}$. Suppose that φ is an arbitrage strategy on the market $\mathcal{M}$, then φ satisfies $\varphi_1^+ \geq 0$, $\varphi_1^- \geq 0$,

$$\varphi_0^+ - \varphi_0^- + \varphi_0^1 S_0 = 0, \qquad \varphi_1^+ - \varphi_1^- + \varphi_1^1 S_0 = 0 \tag{21}$$

and

$$1,02\varphi_1^+ - 1,1\varphi_1^- + \varphi_1^1 S_1 > 0. \tag{22}$$

Using (21)–(22) we have $(\varphi_1^- - 100\varphi_1^1)1,02 - 1,1\varphi_1^- + \varphi_1^1 S_1 > 0$, which is equivalent to $\varphi_1^- < \frac{(S_1-100)\varphi_1^1}{0,08}$. If $\varphi_1^1 \leq 0$ we obtain a contradiction with $\varphi_1^- \geq 0$, so we need $\varphi_1^1 > 0$. Then $\varphi_1^+ = \varphi_1^- - 100\varphi_1^1 < \frac{(S_1-100)\varphi_1^1}{0,08} - 100\varphi_1^1 = \frac{(S_1-108)\varphi_1^1}{0,08} \leq 0$, which contradicts $\varphi_1^+ \geq 0$. We have proved that there is no $(\varphi_t^+, \varphi_t^-)$ satisfying conditions (21)–(22) and $\varphi_1^+ \geq 0$, $\varphi_1^- \geq 0$, so the market $\mathcal{M}$ is arbitrage free.

Now, we are in a position to prove the first main result.

Theorem 3.5

If there exists a martingale pair for the market $\mathcal{M} = (S, \mathcal{P})$, then there is no arbitrage opportunity on the market $\mathcal{M} = (S, \mathcal{P})$.

Proof. Let us assume that there exists a martingale pair $(\{B_t\}_{t=0,\ldots,T}, P^*)$ for the market $\mathcal{M} = (S, \mathcal{P})$. Then the process $\{(\frac{S_t^1}{B_t}, \ldots, \frac{S_t^d}{B_t})\}_{t=0,\ldots,T}$ is P^*-martingale. This means that P^* is a martingale measure in the model $\widetilde{\mathcal{M}}$. Using The First Fundamental Theorem of Asset Pricing we show that $\widetilde{\mathcal{M}}$ is arbitrage-free. It follows from Lemma 3.3 that there is not any arbitrage opportunity on the market $\mathcal{M}$.

We also have the following result.

Theorem 3.6

If $(\{B_t\}_{t=0,\ldots,T}, P^*)$ is a martingale pair for the market $\mathcal{M} = (S, \mathcal{P})$, then for any $h \in L^2(\Omega, \mathcal{F}_T, P)$ letting

$$C_t = E_{P^*}(B_T^{-1} B_t h \,|\, \mathcal{F}_t)$$

the extended model $\bar{\mathcal{M}} = (\bar{S}, \bar{\mathcal{P}})$ is arbitrage free, where

$$\bar{S} = \{(B_t^+, B_t^-, S_t^1, \ldots, S_t^d, C_t)\}_{t=0,1,\ldots,T}$$

and $\bar{\mathcal{P}}$ is the set of all predictable processes $\{(\Theta_t^+, \Theta_t^-, \Theta_t^1, \ldots, \Theta_t^d, \Theta_t^{d+1})\}_{t=0,1,\ldots,T}$ with values in $[0, +\infty)^2 \times \mathbb{R}^{d+1}$.

Proof. Observe that the process $\{(\frac{S_t^1}{B_t}, \ldots, \frac{S_t^d}{B_t})\}_{t=0,\ldots,T}$ is P^*-martingale and $\frac{C_t}{B_t} = E_{P^*}(B_T^{-1} h \,|\, \mathcal{F}_t)$ is also P^*-martingale, so $\{(\frac{S_t^1}{B_t}, \ldots, \frac{S_t^d}{B_t}, \frac{C_t}{B_t})\}_{t=0,\ldots,T}$ is P^*-martingale. Hence $(\{B_t\}_{t=0,\ldots,T}, P^*)$ is a martingale pair for the market $\bar{\mathcal{M}} = (\bar{S}, \bar{\mathcal{P}})$. Thus, we can use Theorem 3.5.

4. Some special cases

In this section we will examine some special cases in which the implication of Theorem 3.5 can be replaced by equivalence. We will start with the easiest; a one-period two-state model of a financial market with one risky asset S_t and two different deterministic interest rates for loans (r_l) and deposits (r_d).

Example 4.1
Let us assume that the probability space $(\Omega, \mathcal{F}, P)$ is given as follows: $\Omega = \{u, d\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(u), P(d) > 0$ with $P(u) + P(d) = 1$. We also consider filtration $\{\mathcal{F}\}_{t=0,1}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$, and assume that the process $\{(B_t^+, B_t^-, S_t)\}_{t=0,1}$ is given by $B_0^+ = B_0^- \equiv 1$, $S_0 \in (0, +\infty)$, $B_1^+ \equiv 1 + r_d$, $B_1^- \equiv 1 + r_l$, where positive numbers r_d and r_l satisfy the obvious relation $r_d < r_l$, and $S_1(u) = S_1^u$, $S_1(d) = S_1^d$ with $0 < S_1^d < S_1^u$. Of course r_d and r_l denote, respectively, the interest rates under which the bank account and the bank loans are subjected.

One can easily check that there is an arbitrage opportunity if $S_0(1 + r_d) \geq S_1^u$ or $S_1^d \geq S_0(1 + r_l)$ (see Definition 2.8) In other words, the necessary conditions for the considered market to be arbitrage free are

$$S_0(1 + r_d) < S_1^u \qquad \text{and} \qquad S_1^d < S_0(1 + r_l).$$

The following easy lemma will be used to prove the existence of a martingale pair for the arbitrage free models considered in this section.

Lemma 4.2
Let us consider four positive numbers d, u, r_d, r_l such that $d < u$, $r_d < r_l$, $1 + r_d < u$ and $d < 1 + r_l$. Then, we have the following inequality

$$\max\{d, 1 + r_d\} < \min\{u, 1 + r_l\}.$$

Proof. There are two cases: $d \geq 1 + r_d$ or $d < 1 + r_d$. In the first case, we have $\max\{d, 1 + r_d\} = d$. Since, by assumptions, $d < u$ and $d < 1 + r_l$, it follows that $\max\{d, 1 + r_d\} = d < \min\{u, 1 + r_l\}$. In the second case, it is true that $\max\{d, 1 + r_d\} = 1 + r_d$ and so $\max\{d, 1 + r_d\} = 1 + r_d < \min\{u, 1 + r_l\}$, because per the assumptions we have $1 + r_d < 1 + r_l$ and $1 + r_d < u$.

Now we will provide the necessary and sufficient conditions for the market of Example 4.1 to be arbitrage free. In other words, in the context of Example 4.1, we reverse the implication of Theorem 3.5.

Theorem 4.3
Let us consider the model of the financial market described in Example 4.1. Then, the following conditions are equivalent

(a) *the model is arbitrage free,*

(b) $S_0(1 + r_d) < S_1^u$ *and* $S_1^d < S_0(1 + r_l)$,

(c) *the model permits a martingale pair.*

Proof. Since the implication (*a*)⇒(*b*) was already mentioned above in Example 4.1 (the necessary conditions) and the implication (*c*)⇒(*a*) is a consequence of Theorem 3.5, thus we only need to show the implication (*b*)⇒(*c*).

Now, assume that the condition (*b*) is satisfied. Then, by Lemma 4.2 for $d = \frac{S_1(d)}{S_0}$ and $u = \frac{S_1(u)}{S_0}$, there exists a positive number r such that

$$\max\Big\{\frac{S_1(d)}{S_0}, 1 + r_d\Big\} < 1 + r < \min\Big\{\frac{S_1(u)}{S_0}, 1 + r_l\Big\}. \tag{23}$$

Let us consider the process $\{B_t\}_{t=0,1}$ given by $B_0 \equiv 1$, $B_1 \equiv 1+r$ and the function $Q\colon \mathcal{F} \to \mathbb{R}$ given by

$$Q(u) = \frac{S_0(1+r) - S_1(d)}{S_1(u) - S_1(d)} \qquad \text{and} \qquad Q(d) = \frac{S_1(u) - S_0(1+r)}{S_1(u) - S_1(d)}.$$

Since, by (23), $S_1(d) < S_0(1 + r) < S_1(u)$, one can see that $Q(u), Q(d) > 0$. We also see, by the definition of Q, that $Q(u) + Q(d) = 1$. Thus, the function Q is a probability measure, and moreover this probability measure is equivalent to P (because we have $Q(u), Q(d), P(u), P(d) > 0$). From (23), we also obtain

$$\frac{B_1^+}{B_0^+} = 1 + r_d < 1 + r = \frac{B_1}{B_0} = 1 + r < 1 + r_l = \frac{B_1^-}{B_0^-}.$$

One can, also, easily see that $E_Q(\frac{S_1}{B_1}) = \frac{S_0}{B_0}$, where E_Q denotes the mean value with respect to probability measure Q. But, this means that the process $\{\frac{S_t}{B_t}\}_{t=0,1}$ is a Q-martingale.

So we are checked that the pair $(\{B_t\}_{t=0,1}, Q)$ is a martingale pair for the considered market and thus the proof of implication (*b*)⇒(*c*) is finished.

The next special case that we will examine is the following one-period multi-state model.

Example 4.4

Now, we assume that the probability space $(\Omega, \mathcal{F}, P)$ is given as follows: $\Omega = \{\omega_1, \ldots, \omega_n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(\omega_1), \ldots, P(\omega_n) > 0$ with $P(\omega_1) + \ldots + P(\omega_n) = 1$ for some $n \geq 2$. We also consider filtration $\{\mathcal{F}\}_{t=0,1}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$. Assume that the process $\{(B_t^+, B_t^-, S_t)\}_{t=0,1}$ is given by $B_0^+ = B_0^- \equiv 1$, $S_0 \in (0, +\infty)$, $B_1^+ \equiv 1 + r_d$, $B_1^- \equiv 1 + r_l$, where positive numbers r_d and r_l satisfy $r_d < r_l$ and $S_1(\omega_1), \ldots, S_1(\omega_n)$ are positive numbers. Without loss of generality, we can assume that $S_1(\omega_1) < S_1(\omega_2) < \ldots < S_1(\omega_n)$.

One can easily check that there is an arbitrage opportunity if $S_0(1 + r_d) \geq S_1(\omega_n)$ or $S_1(\omega_1) \geq S_0(1 + r_l)$. In other words, the necessary conditions for the considered market to be arbitrage free are

$$S_0(1 + r_d) < S_1(\omega_n) \qquad \text{and} \qquad S_1(\omega_1) < S_0(1 + r_l).$$

Now, we prove the following generalization of Theorem 4.3.

Theorem 4.5
Let us consider the model of the financial market described in Example 4.4. Then, the following conditions are equivalent

(a) *the model is arbitrage free,*

(b) $S_0(1+r_d) < S_1(\omega_n)$ *and* $S_1(\omega_1) < S_0(1+r_l)$,

(c) *the model permits a martingale pair.*

Proof. As in the proof of Theorem 4.3, we only need to show the implication $(b)\Rightarrow(c)$. Thus, assume that the condition (b) is satisfied. By Lemma 4.2 for $d=\frac{S_1(\omega_1)}{S_0}$ and $u=\frac{S_1(\omega_n)}{S_0}$, there exists a positive number r such that

$$\max\Big\{\frac{S_1(\omega_1)}{S_0}, 1+r_d\Big\} < 1+r < \min\Big\{\frac{S_1(\omega_n)}{S_0}, 1+r_l\Big\}.$$

Moreover, we can choose r such that $1+r\notin\{\frac{S_1(\omega_1)}{S_0},\ldots,\frac{S_1(\omega_n)}{S_0}\}$. Let $k\in\{1,\ldots,n-1\}$ be such that

$$\frac{S_1(\omega_k)}{S_0} < 1+r < \frac{S_1(\omega_{k+1})}{S_0}.$$

Without loss of generality, we can assume that $k\le n-k$ (if the inequality $k>n-k$ holds, the argument is similar). Now, we can choose a partition $I_1,\ldots,I_k$ of the set $\{k+1,\ldots,n\}$ with $I_l\neq\emptyset$ for $l=1,\ldots,k$.

Now, let us consider the process $\{B_t\}_{t=0,1}$ given by $B_0\equiv 1$ and $B_1\equiv 1+r$ and the function $Q\colon\mathcal{F}\to\mathbb{R}$ given by

$$Q(\omega_l)=\frac{\sum_{i\in I_l}|S_1(\omega_i)-S_0(1+r)|}{\sum_{i=1}^{k}\#I_i|S_1(\omega_i)-S_0(1+r)|+\sum_{i=k+1}^{n}|S_1(\omega_i)-S_0(1+r)|}$$

for $l=1,\ldots,k$ and

$$Q(\omega_l)=\frac{|S_1(\omega_j)-S_0(1+r)|}{\sum_{i=1}^{k}\#I_i|S_1(\omega_i)-S_0(1+r)|+\sum_{i=k+1}^{n}|S_1(\omega_i)-S_0(1+r)|}$$

for $l\in I_j$. From the definition of Q it is obvious that $Q(\omega_1),\ldots,Q(\omega_n)>0$. One can, also, check that $Q(\omega_1)+\ldots+Q(\omega_n)=1$. Thus, the function Q is a probability measure that is equivalent to P. The same reasons, as in the proof of Theorem 4.3 give $\frac{B_1^+}{B_0^+}<\frac{B_1}{B_0}<\frac{B_1^-}{B_0^-}$. To see that $E_Q(\frac{S_1}{B_1})=S_0=\frac{S_0}{B_0}$, we make the following calculations

$$\begin{aligned}E_Q\Big(\frac{S_1}{B_1}\Big)&=\sum_{i=1}^{n}\frac{S_1(\omega_i)}{1+r}Q(\omega_i)=\sum_{i=1}^{n}\frac{S_0(1+r)+[S_1(\omega_i)-S_0(1+r)]}{1+r}Q(\omega_i)\\&=S_0+\frac{1}{1+r}\sum_{i=1}^{k}\Big\{[S_1(\omega_i)-S_0(1+r)]Q(\omega_i)\\&\quad+\sum_{j\in I_i}[S_1(\omega_j)-S_0(1+r)]Q(\omega_j)\Big\}\end{aligned}$$

and notice that, for all $i \in \{1,\dots,k\}$, we have

$$[S_1(\omega_i) - S_0(1+r)]Q(\omega_i) + \sum_{j\in I_i}[S_1(\omega_j) - S_0(1+r)]Q(\omega_j) = 0.$$

So we have checked that the pair $(\{B_t\}_{t=0,1}, Q)$ is a martingale pair for the considered market and have also finished the proof of implication $(b)\Rightarrow(c)$.

Before we present, in the next section, a general result for finite models, we will examine the next model which is similar to the model of Cox-Ross-Rubinstein. We will call it a CRR-type model.

Example 4.6
This model can be realized on the probability space $(\Omega, \mathcal{F}, P)$ defined as follows: $\Omega = \{\omega_1,\dots,\omega_n\}$ with $n = 2^T$, where T is the time horizon, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(\omega_1),\dots,P(\omega_n) > 0$ with $P(\omega_1)+\ldots+P(\omega_n) = 1$.

We also consider filtration $\{\mathcal{F}\}_{t=0,\dots,T}$ with $\mathcal{F}_0 = \{\emptyset,\Omega\}$, $\mathcal{F}_1 = \sigma(\{A^1_u, A^1_d\})$, where $A^1_d = \{\omega_1,\dots,\omega_{\frac{n}{2}}\}$, $A^1_u = \{\omega_{\frac{n}{2}+1},\dots,\omega_n\}$, $\mathcal{F}_2 = \sigma(\{A^2_{uu}, A^2_{ud}, A^2_{du}, A^2_{dd}\})$, where $A^2_{dd} = \{\omega_1,\dots,\omega_{\frac{n}{4}}\}$, $A^2_{du} = \{\omega_{\frac{n}{4}+1},\dots,\omega_{\frac{n}{2}}\}$, $A^2_{ud} = \{\omega_{\frac{n}{2}+1},\dots,\omega_{3\cdot\frac{n}{4}}\}$ and $A^2_{uu} = \{\omega_{3\cdot\frac{n}{4}+1},\dots,\omega_n\}$, and so on. To be precise, for $k \in \{1,\dots,T\}$, we define $\mathcal{F}_k$ as the σ-field generated by the partition of the set Ω in to the 2^k subsets $A^k_{\varepsilon_1\ldots\varepsilon_k}$ with $\varepsilon_1,\dots,\varepsilon_k \in \{u,d\}$, where $A^k_{\varepsilon_1\ldots\varepsilon_k} = \{\omega_{\varphi_k(\varepsilon_1\ldots\varepsilon_k)\cdot\frac{n}{2^k}+1},\dots,\omega_{(\varphi_k(\varepsilon_1\ldots\varepsilon_k)+1)\cdot\frac{n}{2^k}}\}$ and $\varphi_k(\varepsilon_1\ldots\varepsilon_k) = \varepsilon_1\ldots\varepsilon_{k\,(2)}$, assuming the value of the k-digit binary sequence $\varepsilon_1\ldots\varepsilon_k$, where we assign value 1 to the 'digit' u and value 0 to the 'digit' d.

To define the process $\{(B^+_t, B^-_t, S_t)\}_{t=0,\dots,T}$, we assume that u and d denote, depending on the context, symbols acting as short-cuts of *u*p and *d*own (like in the above-mentioned definition of filtration), and positive numbers (like below in the definition of the process $\{S_t\}_{t=0,\dots,T}$). Of course, if u and d are considered as positive numbers, we assume that $d < u$.

Now, we can define $\{(B^+_t, B^-_t, S_t)\}_{t=0,\dots,T}$. First of all, we put $B^+_t \equiv (1+r_d)^t$ and $B^-_t \equiv (1+r_l)^t$ for $t = 0,1,\dots,T$ and

$$S_k = \sum_{\varepsilon,\dots,\varepsilon_k\in\{u,d\}} \varepsilon_1\cdots\varepsilon_k S_0\cdot\mathbf{1}_{A^k_{\varepsilon_1\ldots\varepsilon_k}},$$

where $\mathbf{1}_{A^k_{\varepsilon_1\ldots\varepsilon_k}}$ denotes the characteristic function of the set $A^k_{\varepsilon_1\ldots\varepsilon_k}$. One can easily verify that there is an arbitrage opportunity if $1+r_d \geq u$ or $d \geq 1+r_l$. In other words, the necessary conditions for the considered market to be arbitrage free are

$$1+r_d < u \qquad \text{and} \qquad d < 1+r_l.$$

The requirements for the model in Example 4.6 to be arbitrage free are the following.

Theorem 4.7
Let us consider the model of the financial market described in Example 4.6. Then, the following conditions are equivalent

(a) *the model is arbitrage free,*

(*b*) $1 + r_d < u$ *and* $d < 1 + r_l$,

(*c*) *the model permits a martingale pair.*

Proof. As in the proofs of Theorems 4.3 and 4.5, we only need to show the implication (*b*)⇒(*c*). So choose a positive number r such that

$$\max\{d, 1 + r_d\} < 1 + r < \min\{u, 1 + r_l\}.$$

Using this number r, we define

$$p = \frac{(1 + r) - d}{u - d} \qquad \text{and} \qquad q = 1 - p = \frac{u - (1 + r)}{u - d}.$$

By the definition of the filtration $\{\mathcal{F}_t\}_{t=0,\dots,T}$, we have for any $k \in \{1, \dots, T - 1\}$ and for any $\varepsilon_1, \dots, \varepsilon_k \in \{u, d\}$ that $A^k_{\varepsilon_1\dots\varepsilon_k} A^{k+1}_{\varepsilon_1\dots\varepsilon_k d} \cup A^{k+1}_{\varepsilon_1\dots\varepsilon_k u}$. We also have $\Omega = A^1_d \cup A^1_u$. Thus, the probability measure $Q\colon \mathcal{F} \to \mathbb{R}$, we can define such that $Q(A^1_u) = p$, $Q(A^1_d) = q$ and for any $k \in \{1, \dots, T - 1\}$ and any $\varepsilon_1, \dots, \varepsilon_k \in \{u, d\}$,

$$Q\big(A^{k+1}_{\varepsilon_1\dots\varepsilon_k u} \,|\, A^k_{\varepsilon_1\dots\varepsilon_k}\big) = p \qquad \text{and} \qquad Q\big(A^{k+1}_{\varepsilon_1\dots\varepsilon_k d} \,|\, A^k_{\varepsilon_1\dots\varepsilon_k}\big) = q.$$

By this definition of Q, as one can check, we have for any $k \in \{1, \dots, T - 1\}$ and any $\varepsilon_1, \dots, \varepsilon_k \in \{u, d\}$, such that

$$E_Q\Big(\frac{S_{k+1}}{B_{k+1}} \,\Big|\, A^k_{\varepsilon_1\dots\varepsilon_k}\Big) = \frac{\varepsilon_1 \cdots \varepsilon_k u S_0}{(1 + r)^{k+1}} \cdot p + \frac{\varepsilon_1 \cdots \varepsilon_k d S_0}{(1 + r)^{k+1}} \cdot q \frac{\varepsilon_1 \cdots \varepsilon_k S_0}{(1 + r)^k} = \frac{S_k}{B_k}\Big|_{A^k_{\varepsilon_1\dots\varepsilon_k}},$$

which means that $E_Q(\frac{S_{k+1}}{B_{k+1}} \,|\, \mathcal{F}_k) = \frac{S_k}{B_k}$ for $k = 1, \dots, T - 1$. Of course, we also have $E_Q(\frac{S_1}{B_1} \,|\, \mathcal{F}_0) = E_Q(\frac{S_1}{B_1}) = \frac{S_0}{B_0}$.

Since, also, $\frac{B^+_{t+1}}{B^+_t} = 1 + r_d < 1 + r = \frac{B_{t+1}}{B_t} = 1 + r < 1 + r_l = \frac{B^-_{t+1}}{B^-_t}$ for any $t \in \{0, \dots, T - 1\}$, it follows that the pair $(\{\mathcal{F}_t\}_{t=0,\dots,T}, Q)$ is a martingale pair for the model.

5. The first fundamental-type theorem for finite models with two different interest rates

Example 5.1
Let us consider a model of a financial market with one risky asset S_t. Assume that $\{B^+_t\}_{t=0,1,\dots,T}$ and $\{B^-_t\}_{t=0,1,\dots,T}$ are predictable stochastic processes, with values of $(0, +\infty)$, such that the process $\{B^+_t\}_{t=0,1,\dots,T}$ is a deposit process and the process $\{B^-_t\}_{t=0,1,\dots,T}$ is a loan process (see Definition 2.1). This model can be realized on the finite probability space $(\Omega, \mathcal{F}, P)$ defined as follows: $\Omega = \{\omega_1, \dots, \omega_n\}$, $T \in \mathbb{N}_+$ is the time horizon, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(\omega_1), \dots, P(\omega_n) > 0$ with $P(\omega_1) + \ldots + P(\omega_n) = 1$.

We also define filtration $\{\mathcal{F}_t\}_{t=0,\dots,T}$ on the measurable space $(\Omega, \mathcal{F})$ as follows $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and the filtration $\{\mathcal{F}_t\}_{t=0,\dots,T}$ is described by the sequence of partitions

$$\mathcal{A}^{(t)} = \{A^{(t)}_i \,|\, i = 1, \dots, r_t\} \qquad \text{for all } t \in \{0, \dots, T\}$$

with $\mathcal{A}^{(0)} = \{\Omega\}$ (and $r_0 = 1$), such that for all $t \in \{0,\dots,T-1\}$ there is a partition $\{I_1^{(t)},\dots,I_{r_t}^{(t)}\}$ of the set $\{1,\dots,r_{t+1}\}$ such that for all $i \in \{1,\dots,r_t\}$, we have $A_i^{(t)} = \bigcup_{j\in I_i^{(t)}} A_j^{(t+1)}$. We also assume that $\mathcal{F}_T = \mathcal{F}$.

Each set of the partition $\mathcal{A}^{(t)}$ represents one of the possible states of the world at time t, and the number r_t can be interpreted as the number of states in which the world can arrive at the moment t. Let us fix $t \in \{0,\dots,T-1\}$ and $i \in \{1,\dots,r_t\}$. Since the functions B_t^+, B_t^-, S_t are constant on the sets of the form $A_i^{(t)}$, by $B_t^+(A_i^{(t)})$ and so on we will denote this constant value. Note that $B_{t+1}^+(A_j^{(t+1)}) = B_{t+1}^+(A_i^{(t)})$ and $B_{t+1}^-(A_j^{(t+1)}) = B_{t+1}^-(A_i^{(t)})$ for all $j \in I_i^{(t)}$.

One can check that the necessary conditions for the considered market to be arbitrage free are

$$\frac{B_{t+1}^+(A_i^{(t)})}{B_t^+(A_i^{(t)})} < \frac{\max_{j\in I_i^{(t)}}\{S_{t+1}(A_j^{(t+1)})\}}{S_t(A_i^{(t)})}$$

and

$$\frac{B_{t+1}^-(A_i^{(t)})}{B_t^-(A_i^{(t)})} > \frac{\min_{j\in I_i^{(t)}}\{S_{t+1}(A_j^{(t+1)})\}}{S_t(A_i^{(t)})},$$

for all $t \in \{0,\dots,T-1\}$ and all $i \in \{1,\dots,r_t\}$.

The necessary and sufficient conditions for the model of Example 5.1 to be arbitrage free are the following.

Theorem 5.2

Let us consider the model of financial market described in Example 5.1. Then, the following conditions are equivalent:

(a) *the model is arbitrage free ,*

(b) $\frac{B_{t+1}^+(A_i^{(t)})}{B_t^+(A_i^{(t)})} < \frac{\max_{j\in I_i^{(t)}}\{S_{t+1}(A_j^{(t+1)})\}}{S_t(A_i^{(t)})}$ *and* $\frac{B_{t+1}^-(A_i^{(t)})}{B_t^-(A_i^{(t)})} > \frac{\min_{j\in I_i^{(t)}}\{S_{t+1}(A_j^{(t+1)})\}}{S_t(A_i^{(t)})}$ *for all* $t \in \{0,\dots,T-1\}$ *and all* $i \in \{1,\dots,r_t\}$.

(c) *the model permits a martingale pair.*

Proof. We only need to show the implication $(b)\Rightarrow(c)$. For $t=1$ we choose $r_1^{(1)}$ and $k_1^{(1)}$ as in Theorem 4.5, such that

$$\max\left\{\frac{\min_s S_1(A_s^{(1)})}{S_0}, \frac{B_1^+}{B_0^+}\right\} < 1 + r_1^{(1)} < \min\left\{\frac{\max_s S_1(A_s^{(1)})}{S_0}, \frac{B_1^-}{B_0^-}\right\}.$$

Consider the process $\{B_t\}_{t=0,1}$, and the function Q on $\mathcal{F}_1$ defined analogously as in Theorem 4.5. Let us fix $t \in \{2,\dots,T-1\}$ and $i \in \{1,\dots,r_t\}$. By analogy we choose $r_i^{(t)}$ such that

$$\max\left\{\frac{\min_{j\in I_i^{(t)}} S_t(A_j^{(t)})}{S_{t-1}(A_i^{(t-1)})}, \frac{B_t^+(A_i^{(t-1)})}{B_{t-1}^+(A_i^{(t-1)})}\right\} < 1 + r_i^{(t)}$$

$$< \min\left\{\frac{\max_{j\in I_i^{(t)}} S_t(A_j^{(t)})}{S_{t-1}(A_i^{(t-1)})}, \frac{B_t^-(A_i^{(t-1)})}{B_{t-1}^-(A_i^{(t-1)})}\right\}.$$

Let the process $\{B_s\}_{s=0,\dots,t}$ be given by $B_0 \equiv 1$ and $B_s(A_j^{(s-1)}) = B_{s-1}(A_j^{(s-1)})(1+r_j^{(s)})$ for $s \in \{1,\dots,t\}$ and $j \in I_i^{(s)}$.

Now, we consider the conditional probability $Q(A_j^{(t+1)} \mid A_i^{(t)})$ for all $t \in \{1,\dots,T-1\}$, all $i \in \{1,\dots r_t\}$ and all $j \in I_t^{(i)}$, and define it analogously as in the proof of Theorem 4.5. The definition of Q on $\mathcal{F}_1$ gives the definition of $Q\colon \mathcal{F}_T \to \mathcal{R}$.

By this definition of Q (similar to the proof in Theorem 4.5) we get for any $t = 1,\dots,T-1$ and for any $i \in \{1,\dots,r_t\}$,

$$E_Q\Big(\frac{S_{t+1}}{B_{t+1}}\Big| A_i^t\Big) = \sum_{j\in I_i^{(t)}} \frac{S_{t+1}(A_j^{(t+1)})}{B_{t+1}(A_j^{(t+1)})}\cdot Q(A_j^{(t+1)} \mid A_i^{(t)}) = \frac{S_t}{B_t}\Big|_{A_i^{(t)}},$$

which means that $E_Q(\frac{S_t}{B_t} \mid \mathcal{F}_{t-1}) = \frac{S_{t-1}}{B_{t-1}}$ for $t = 2,\dots,T$. Of course, we also have $E_Q(\frac{S_1}{B_1} \mid \mathcal{F}_0) = E_Q(\frac{S_1}{B_1}) = \frac{S_0}{B_0}$.

Since, also, $\frac{B_{t+1}^+}{B_t^+} < \frac{B_{t+1}}{B_t} < \frac{B_{t+1}^-}{B_t^-}$ for any $t \in \{0,\dots,T-1\}$, it follows that the pair $(\{\mathcal{F}_t\}_{t=0,\dots,T}, Q)$ is a martingale pair for the model.

We will show the application of Theorem 5.2, considering the phenomenon of different access to arbitration in a certain sense in the same market for two different investors.

Example 5.3

Let us assume that the probability space $(\Omega, \mathcal{F}, P)$ is given as follows $\Omega = \{u, d\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(u), P(d) > 0$ with $P(u) + P(d) = 1$. Let $\{\mathcal{F}\}_{t=0,1}$ be filtration such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$, and the process $\{(B_t^+, B_t^-, B_t, S_t)\}_{t=0,1}$ be given by $B_0^+ = B_0^- = B_0 \equiv 1$, $S_0 \in (0,+\infty)$, $B_1^+ \equiv 1 + r_d$, $B_1^- \equiv 1 + r_l$ and $B_1 = B_1^+$, where positive numbers r_d and r_l, because of their interpretation, satisfy the obvious relation $r_d < r_l$. Finally let $S_1(u) > S_1(d) > 0$ be real numbers.

Of course r_d and r_l denote, respectively, the interest rates under which the bank account and the bank loans are subjected. Furthermore, we assume that $1 + r_d < \frac{S_1(d)}{S_0} < 1 + r_l < \frac{S_1(u)}{S_0}$. We also make an assumption that a small player can not take any position in B_t while a big player can take any position in B_t (including a short position).

From the small player's point of view the considered model is indifferent from the model $\{(B_t^+, B_t^-, S_t)\}_{t=0,1}$ without any constraints. This model by Theorem 4.3, is arbitrage free and so the small player does not have an arbitrage opportunity in the model $\{(B_t^+, B_t^-, B_t, S_t)\}_{t=0,1}$.

From the big player's point of view the situation is completely different because of the possibility of taking a position in B_t, especially a short position, means that the big player can borrow money at the same interest rate as he can make a deposit. Since $1 + r_d < \frac{S_1(d)}{S_0} < \frac{S_1(u)}{S_0}$, the big player has an arbitrage opportunity in the model.

The Example 5.3 shows that a big player who has the same interest rate for deposits and loans is in a prime position compared to a small player who has two different interest rates. Moreover, we can create a market model in which both players have two different interest rates, but the big player is still in a better position. We consider that situation in the next example.

Example 5.4
Let us assume that the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_t\}_{t=0,1}$ is given in the previous example. Now, consider the following process

$$\{(B^+_{t,s}, B^-_{t,s}, B^+_{t,b}, B^-_{t,b}, S_t)\}_{t=0,1} \tag{24}$$

with $B^+_{0,s} = B^-_{0,s} = B^+_{0,b} = B^-_{0,b} \equiv 1$, $B^+_{1,s} \equiv 1+r_{d,s}$, $B^-_{1,s} \equiv 1+r_{l,s}$, $B^+_{1,b} \equiv 1+r_{d,b}$, $B^-_{1,b} \equiv 1+r_{l,b}$, where positive numbers $r_{d,s}$, $r_{l,s}$, $r_{d,b}$, $r_{l,b}$ satisfy the following relation $r_{d,s} \leq r_{d,b} \leq r_{l,b} \leq r_{l,s}$, $S_0 \in (0, +\infty)$ and let $S_1(u) > S_1(d) > 0$ be real numbers. We also assume that the small player cannot take any position in $B^+_{t,b}$ and $B^-_{t,b}$. While the big player can take long position not only in $B^+_{t,s}$ and $B^-_{t,s}$ (as the small player can) but also in $B^+_{t,b}$ and $B^-_{t,b}$ (both kinds of players can take long and short position in S_t). From the small player's point of view the considered model gives exactly the same possibilities as the model $\{(B^+_{t,s}, B^-_{t,s}, S_t)\}_{t=0,1}$ gives.

While from the big player's point of view the considered model gives even more possibilities from the model $\{(B^+_{t,b}, B^-_{t,b}, S_t)\}_{t=0,1}$. Now, suppose that one of the following is satisfied

1. $1 + r_{d,s} \leq 1 + r_{d,b} \leq 1 + r_{l,b} \leq \frac{S_1(d)}{S_0} < 1 + r_{l,s} < \frac{S_1(u)}{S_0}$,
2. $\frac{S_1(d)}{S_0} < 1 + r_{d,s} < \frac{S_1(u)}{S_0} \leq 1 + r_{d,b} \leq 1 + r_{l,b} < 1 + r_{l,s}$.

Now, using Theorem 4.3 for the model $\{(B^+_{t,s}, B^-_{t,s}, S_t)\}_{t=0,1}$ we obtain, that the small player does not have an arbitrage opportunity in the model (24). By Theorem 4.3 using the model $\{(B^+_{t,b}, B^-_{t,b}, S_t)\}_{t=0,1}$ and the above discussion, we conclude that a big player does have an arbitrage opportunity in the model.

Remark 5.5
Note that we can generalize Example 5.4 and consider the market model, as in Example 5.1, with two players with different deposit processes and loan processes. With properly selected processes the small player doesn't have an arbitrage opportunity, because of Theorem 5.2, in contrast to the big player, who has an arbitrage opportunity. The examples show that different investors, who have different access to deposits and loans, have different positions related to arbitrage.

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Marek Karaś
AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30
30-059 Kraków
Poland
E-mail: mkaras@mat.agh.edu.pl

Anna Serwatka
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University in Kraków
ul. Łojasiewicza 6
30-348 Kraków
Poland
E-mail: Anna.Serwatka@im.uj.edu.pl

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