DE GRUYTER OPEN

Ann. Univ. Paedagog. Crac. Stud. Math. 16 (2017), 73-76
DOI: 10.1515/aupcsm-2017-0006

FOLIA 206

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVI (2017)

Raghavendra G. Kulkarni

Intersect a quartic to extract its roots

Communicated by Justyna Szpond

Abstract. In this note we present a new method for determining the roots of a quartic polynomial, wherein the curve of the given quartic polynomial is intersected by the curve of a quadratic polynomial (which has two unknown coefficients) at its root point; so the root satisfies both the quartic and the quadratic equations. Elimination of the root term from the two equations leads to an expression in the two unknowns of quadratic polynomial. In addition, we introduce another expression in one unknown, which leads to determination of the two unknowns and subsequently the roots of quartic polynomial.

1. Introduction

This note presents a new method to determine the roots of a quartic polynomial. In this method, the curve of the given quartic polynomial is intersected by the curve of a quadratic polynomial (which has two unknown coefficients) at its root point, say $(r, 0)$. Hence the root, r, satisfies both the quartic and the quadratic equations. Elimination of r from the two equations leads to an expression in the two unknowns of quadratic polynomial.

In addition, we introduce another expression of our choice in one unknown, to enable us to determine the two unknowns from the two expressions; and subsequently the roots of quartic polynomial are determined. The method is explained in detail as below.

AMS (2010) Subject Classification: 12E05, 12E12.

Keywords and phrases: Intersection of curves, quartic polynomial, quadratic polynomial, roots, resolvent cubic equation.

2. Intersection of polynomials

Without loss of any generality, we consider here a depressed quartic polynomial (which has no x^3 term),

$$p(x) = x^4 + ax^2 + bx + c, \tag{1}$$

where the coefficients, a, b and c, are real and the roots of $p(x)$ are in the complex field. Our aim is to determine these roots using the method of intersection proposed here. The other methods of obtaining roots of quartic polynomials are mentioned in [1, 2]. Let us consider a quadratic polynomial,

$$q(x) = x^2 - dx - f, \tag{2}$$

where the coefficients, d and f, are unknowns ($d, f \in \mathbb{C}$). We stipulate that the curve of the quadratic polynomial (2) intersects the curve of the given quartic polynomial (1) at its root, r. Therefore the root must satisfy the two equations, $p(r) = 0$ and $q(r) = 0$, which means:

$$r^4 + ar^2 + br + c = 0, \tag{3}$$

$$r^2 - dr - f = 0. \tag{4}$$

Eliminating r^4, and then subsequently eliminating r^2 from (3) using (4), and after some algebraic manipulations we get an expression for the root, r, in terms of d and f,

$$r = -(d^2 f + f^2 + af + c)/(d^3 + 2df + ad + b). \tag{5}$$

Eliminating r from (4) using (5) yields an expression in the two unknowns, d and f,

$$\begin{aligned} f^4 + 2af^3 + (ad^2 - 3bd + a^2 + 2c)f^2 - (bd^3 - 4cd^2 + abd + b^2 - 2ac)f \\ + cd^4 + acd^2 + bcd + c^2 = 0. \end{aligned} \tag{6}$$

We need one more equation, which can facilitate the determination of two unknowns. To obtain such an equation, first we rearrange (6) as,

$$\begin{aligned} [f + (a/2)]^4 + [ad^2 - 3bd + 2c - (a^2/2)]f^2 \\ - [bd^3 - 4cd^2 + abd + b^2 + (a^3/2) - 2ac]f \\ + cd^4 + acd^2 + bcd + c^2 - (a^4/16) = 0. \end{aligned}$$

Making a substitution, $g = f + (a/2)$, in the above expression yields,

$$g^4 + hg^2 + jg + k = 0, \tag{7}$$

where h, j and k are given by:

$$\begin{aligned} h &= ad^2 - 3bd + 2c - (a^2/2), \\ j &= -[bd^3 + (a^2 - 4c)d^2 - 2abd + b^2], \end{aligned}$$

$$k = cd^4 + (ab/2)d^3 + [(a^3/4) - ac]d^2 + [bc - (a^2b/4)]d$$
$$+ c^2 + (ab^2/2) - (a^2c/2) + (a^4/16).$$

Notice that setting $j = 0$ gives us the desired equation, which is a cubic equation in d,

$$d^3 + [(a^2 - 4c)/b]d^2 - 2ad + b = 0, \tag{8}$$

known as resolvent cubic equation. Solving (8) we determine three values of d. Also note that $j = 0$ makes (7) a quadratic equation in g^2,

$$g^4 + hg^2 + k = 0. \tag{9}$$

Solving (9), we determine g^2 as,

$$g^2 = (-h \pm \sqrt{h^2 - 4k})/2,$$

leading to determination of four values of g,

$$g = \pm\sqrt{(-h \pm \sqrt{h^2 - 4k})/2}. \tag{10}$$

Further using the relation, $f = g - (a/2)$, four values of f are obtained from (10). Notice that for each value of d, we get four values of f. One can use any one of three values of d and corresponding four values of f to determine the four roots of given quartic polynomial (1) from the expression (5). However, if for a certain set of $\{d, f\}$, the denominator of (5) goes to zero, then such set has to be avoided and the other sets can be used to determine the roots of the quartic polynomial.

3. Numerical example

We solve one numerical example with the proposed method. Let the curve of the quartic polynomial, $p(x) = x^4 + 2x^2 + 4x + 2$, be intersected by the curve of a quadratic polynomial, $q(x) = x^2 - dx - f$. The resolvent cubic equation (8) is obtained as, $d^3 - d^2 - 4d + 4 = 0$, and its three roots are obtained as, $d = 1, 2$, and -2. Choosing $d = 1$, the quartic equation (9) is obtained as: $g^4 - 8g^2 + 25 = 0$. Now, from (10) we determine four values of g as:

$$2.121320343559 + 0.7071067811865i, \quad -2.121320343559 - 0.7071067811865i,$$
$$2.121320343559 - 0.7071067811865i, \quad -2.121320343559 + 0.7071067811865i.$$

Using the relation, $f = g - (a/2)$, the corresponding four values of f are obtained as:

$$1.121320343559 + 0.7071067811865i, \quad -3.121320343559 - 0.7071067811865i,$$
$$1.121320343559 - 0.7071067811865i, \quad -3.121320343559 + 0.7071067811865i.$$

We use (5) to determine the four roots of given quartic polynomial as:

$$-0.7071067811865 - 0.2928932188134i, \quad 0.7071067811865 - 1.707106781186i,$$
$$-0.7071067811865 + 0.2928932188134i, \quad 0.7071067811865 + 1.707106781186i.$$

Use of other values of d to find the roots of quartic polynomial is left as an exercise to the readers.

4. Conclusions

We have presented a new method to obtain the roots of a quartic polynomial, wherein the curve of a quadratic polynomial with two unknown coefficients intersects the curve of the quartic at a common root point $(r, 0)$. So the root r satisfies both the quadratic and the quartic equations. Elimination of r from these equations leads to an expression in the two unknowns. Introduction of a cubic equation (known as resolvent cubic equation) in one of the unknowns leads to the determination of both unknowns, and further it results in the extraction of the roots of quartic polynomial.

Acknowledgment

The author is grateful to the anonymous referee, whose valuable comments resulted in improved manuscript. The author thanks the management of PES University, Bengaluru, for supporting this work.

References

[1] Dickson, Leonard E. *First course in the theory of equations.* New York: J. Wiley & sons, inc., 1922. Cited on 74.

[2] Kulkarni, Raghavendra G. "Shifting the origin to solve quartic equations." *The Mathematical Gazette* 97, no. 539 (2013): 268–270. Cited on 74.

Department of Electronics & Communication Engineering
PES University
100 Feet Ring Road, BSK III Stage Bengaluru - 560085
India
E-mail: dr_rgkulkarni@yahoo.com; raghavendrakulkarni@pes.edu

Received: September 25, 2017; final version: November 4, 2017;
available online: November 28, 2017.