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## Translation equation and the Jordan non-measurable continuous functions

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**Abstract.** A connection between the continuous translation equation and the Jordan non-measurable continuous functions is given.

It is well known that a continuous function is Lebesgue measurable. It is not true for the Jordan measurability (in short: measurability). We give an example of a non-measurable continuous function by the solution of the translation equation.

### 1. Continuous solutions of translation equation

Every continuous solution of the translation equation

$$F(F(x, t), s) = F(x, t + s), \quad (1)$$

where  $F: I \times \mathbb{R} \rightarrow I$  and  $I$  is a non-degenerated interval, is of the form

$$F(x, t) = \begin{cases} h_n^{-1}[h_n(g(x)) + t], & \text{for } g(x) \in I_n, t \in \mathbb{R}, \\ g(x), & \text{for } g(x) \in g(I) \setminus \bigcup I_n, t \in \mathbb{R}, \end{cases} \quad (2)$$

where  $g: I \rightarrow I$  is a continuous idempotent ( $g \circ g = g$ ),  $I_n \subset g(I)$  for  $n \in N_1 \subset \mathbb{N}$  are open and disjoint intervals and  $h_n: I_n \rightarrow \mathbb{R}$  are homeomorphisms.

Indeed, it is proved in the book [3] that every continuous solution  $F_1$  of the translation equation for which  $F_1(x, 0) = x$  is of the form (1) with  $g(x) = x$ . Let  $F$  be a continuous solution of the translation equation. The function  $F_1 = F|_{F(I, \mathbb{R}) \times \mathbb{R}}$

is a continuous solution of the translation equation for which  $F_1(x, 0) = x$ , since if  $x = F(x_1, t_1)$  for some  $(x_1, t_1) \in I \times \mathbb{R}$ , then

$$F_1(x, 0) = F_1(F(x_1, t_1), 0) = F(F(x_1, t_1), 0) = F(x_1, t_1) = x.$$

Moreover,  $F(x, t) = F(F(x, t), 0) = F_1(F(x, 0), t)$  and  $F(x, 0)$  is a continuous idempotent.

## 2. Main considerations

### DEFINITION

A function  $f: I_1 \rightarrow I_2$ , where  $I_1, I_2$  are the intervals in  $\mathbb{R}$ , is said to be measurable if the set  $\{x \in I_1 : f(x) > a\}$  is measurable for every  $a \in \mathbb{R}$ .

Let  $I \subset \mathbb{R}$  be a non-degenerated interval,  $g: I \rightarrow I$  a continuous idempotent such that  $g(I)$  is a non-degenerated bounded interval. Let  $C$  be a set of the Smith-Volterra-Cantor type in  $g(I)$ , i.e. let  $C$  be a non-measurable set obtained in  $g(I)$  as a modification of the construction of the Cantor set in which  $\frac{1}{4}$  is taken in place of  $\frac{1}{3}$  ([1] p.191) (the Cantor set is here not good since it is of Jordan measure zero as a closed set of Lebesgue measure zero). Let  $I_n$  be the components of the open set  $g(I) \setminus C$ . Let  $F$  be the function given by the formula (2) with these intervals  $I_n$  and arbitrary homeomorphisms  $h_n: I_n \rightarrow \mathbb{R}$ . Fix an arbitrary  $t_0 \neq 0$ . We will prove that the functions  $f(x) = F(x, t_0) - g(x)$  and  $-f$  are continuous and at least one of these functions is non-measurable.

Indeed, they are continuous since  $F$  and  $g$  are continuous functions. We have

- 1)  $f(x) = 0$  for  $g(x) \in C$ ,
- 2)  $f(x) \neq 0$  for  $g(x) \in I_n, n \in N_1 \subset \mathbb{N}$ , otherwise we would have  $g(x) = F(x, t_0) = h_n^{-1}[h_n(g(x)) + t_0]$  and  $h_n(g(x)) = h_n(g(x)) + t_0$ , a contradiction.

Thus,

$$\begin{aligned} \bigcup I_n &= \{x \in \bigcup I_n : f(x) > 0\} \cup \{x \in \bigcup I_n : f(x) < 0\} \\ &= g(I) \cap \{x \in I : f(x) > 0\} \cup g(I) \cap \{x \in I : f(x) < 0\}. \end{aligned}$$

The set  $\bigcup I_n$  is non-measurable since  $\bigcup I_n = g(I) \setminus C$ , thus at least one of the sets  $\{x \in I : f(x) > 0\}$  and  $\{x \in I : f(x) < 0\} = \{x \in I : -f(x) > 0\}$  is non-measurable. The proof is completed.

The type of monotonicity of homeomorphisms  $h_n$  decides partly which function:  $f$  or  $-f$ , is not measurable., e.g. if  $t_0 > 0$  and every  $h_n$  is increasing, then

$$F(x, t_0) = h_n^{-1}[h_n(g(x)) + t_0] > h_n^{-1}[h_n(g(x))] = g(x)$$

for  $g(x) \in I_n$ . Thus we have  $f(x) > 0$  for  $g(x) \in \bigcup I_n$ , hence the function  $f$  is non-measurable. This type of monotonicity of  $h_n$  may be of course different for different  $n$ .

Let  $I$  be the bounded interval and  $g(x) = x$  in (2). In this case the function  $F(x, 0) = x$  is evidently measurable. Moreover, for every  $t_0 \neq 0$ , the function  $F(\cdot, t_0): I \rightarrow I$  is measurable too: for every real number  $a$  the set  $\{x \in I : F(x, t_0) > a\}$  is an interval, as  $F(\cdot, t_0)$  is onto, continuous and increasing.

CONCLUSION

*The difference of measurable functions (even continuous) may be non-measurable.*

It is known that this situation is impossible for the Lebesgue measurable functions.

There exists a continuous solution  $F$  of (1) such that for all  $t \in \mathbb{R}$ , functions  $F(\cdot, t)$  are non-measurable.

Indeed, we put  $h(x) = d(x, C) + 1$  for  $x \in [0, 1)$  and  $h(x) = x$  for  $x \in [1, 2]$ , where  $C$  is the above set on the interval  $[0, 1]$  and  $d(x, C)$  is the distance between  $x$  and  $C$ . This function  $h$  is

- 1) continuous since the function  $d(x, C)$  is continuous ([2] p.103),
- 2) non-measurable since the set  $\{x \in [0, 2] : h(x) > 1\} = (I \setminus C) \cup (1, 2]$  is non-measurable,
- 3) an idempotent function since it is the identity function on the range of the function  $h$  ( $h([0, 2]) = [1, 2]$ ).

Thus the function  $F(x, t) = h(x)$  for  $(x, t) \in [0, 2] \times \mathbb{R}$  is the solution of (1) and  $F(\cdot, t) : [0, 2] \rightarrow [0, 2]$  is a continuous, non-measurable function for every  $t \in \mathbb{R}$ .

**3. Remark**

PROPOSITION

*There exists a solution  $F$  of (1) for which  $F(\cdot, 0)$  is measurable and  $F(\cdot, 1)$  is non-measurable.*

*Proof.* Let  $g_1 : (0, 1] \cap \mathbb{Q} \rightarrow (-\infty, 0] \cap \mathbb{Q}$ ,  $g_2 : (0, 1] \setminus \mathbb{Q} \rightarrow (0, +\infty) \setminus \mathbb{Q}$ ,  $g_3 : (1, 3) \cap \mathbb{Q} \rightarrow (0, +\infty) \cap \mathbb{Q}$  and  $g_4 : (1, 3) \setminus \mathbb{Q} \rightarrow (-\infty, 0] \setminus \mathbb{Q}$  be bijections such that  $g_4((1, 2) \setminus \mathbb{Q}) \subset (-1, 0]$ . The function  $g = g_1 \cup g_2 \cup g_3 \cup g_4$  is a bijection from  $(0, 3)$  onto  $\mathbb{R}$ . This implies that the function  $F(x, t) = g^{-1}[g(x) + t]$  is a solution of (1). The function  $F(x, 0) = x$  is evidently measurable. We prove that the function  $F(\cdot, 1)$  is non-measurable by proving that the set  $S = \{x \in (0, 3) : F(x, 1) > 1\}$  is non-measurable. We have

- i)  $(1, 2) \cap \mathbb{Q} \subset S$  since if  $x \in (1, 2) \cap \mathbb{Q}$ , then  $g(x) \in (0, +\infty) \cap \mathbb{Q}$ , thus  $g(x) + 1 \in (1, +\infty) \cap \mathbb{Q}$  and this yields that  $F(x, 1) = g^{-1}[g(x) + 1] \in (1, 3) \cap \mathbb{Q}$ ,
- ii)  $[(1, 2) \setminus \mathbb{Q}] \cap S = \emptyset$ . Indeed, suppose to the contrary that there exists an  $x_0 \in (1, 2) \setminus \mathbb{Q}$  such that  $F(x_0, 1) > 1$ . We obtain  $g(x_0) = g_4(x_0) \in (-\infty, 0] \setminus \mathbb{Q}$ , thus  $g(x_0)$  and  $g(x_0) + 1$  are irrational numbers. Moreover,  $g(x_0) \in g_4((1, 2) \setminus \mathbb{Q}) \subset (-1, 0]$  hence  $g(x_0) + 1 \in (0, 1]$  and since  $g(x_0) + 1$  is an irrational number, we have  $g(x_0) + 1 \in (0, 1] \setminus \mathbb{Q} \subset (0, +\infty) \setminus \mathbb{Q}$ . From here  $F(x_0, 1) = g^{-1}[g(x_0) + 1] = g_2^{-1}[g(x_0) + 1] \in (0, 1] \setminus \mathbb{Q}$ . We obtain a contradiction since  $F(x_0, 1) > 1$ .

By i) and ii), the set  $S$  is non-measurable.

The function  $F$  from the above proof is evidently discontinuous since, e.g. the set  $F((0, 1], 1)$  is not an interval.

#### QUESTION

Does there exist a continuous solution of (1) which has the property as in the Proposition?

Such a solution, if it exists, must be of the form (2) with  $N_1 \neq \emptyset$  and the function  $g$  which is not the identity function (see section 2).

## References

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