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Report of Meeting

17th International Conference on Functional Equations and Inequalities, Będlewo, Poland, July 9–15, 2017

The **17th International Conference on Functional Equations and Inequalities** (17th ICFEI), dedicated to the memory of Professor Dobiesław Brydak, was held at Będlewo (Poland) in the Mathematical Research and Conference Center (MRCC), on July 9–15, 2017. It was organized by the Department of Mathematics of the Pedagogical University of Cracow.

The Scientific Committee of the 17th ICFEI consisted of Professors: Nicole Brillouët-Belluot (France), Dobiesław Brydak (Poland) – honorary chairman, Janusz Brzdęk (Poland) – chairman, Jacek Chmieliński (Poland), Krzysztof Ciepliński (Poland), Roman Ger (Poland), Zsolt Páles (Hungary), Dorian Popa (Romania), Ekaterina Shulman (Poland), Henrik Stetkær (Denmark), László Székelyhidi (Hungary), Marek Cezary Zdun (Poland).

The Organizing Committee consisted of Janusz Brzdęk (chairman), Jacek Chmieliński (vice-chairman), Zbigniew Leśniak (vice-chairman), Eliza Jabłońska (scientific secretary), Paweł Solarz (technical support), Beata Deręgowska, Paweł Pasteczka, Paweł Wójcik.

48 participants came from 15 countries: Austria (3 participants), Denmark (1), Egypt (1), France (1), Germany (1), Hungary (5), India (1), Iran (3), Japan (1), Morocco (1), Poland (24), Portugal (1), Romania (2), United Kingdom (1) and United States (2).

The conference was opened on Monday, July 10, by Professor Janusz Brzdęk, the Chairman of the Scientific and Organizing Committees, who welcomed participants on behalf of the Organizing Committee. The opening address was given by Professor Jacek Chmieliński, the Head of the Department of Mathematics of Pedagogical University of Cracow. The opening ceremony was completed by a talk of Professor Marek Czerni presenting the life and scientific achievements of Professor Dobiesław Brydak, the creator of ICFEI, who passed away on March 21, 2017.

During 20 scientific sessions, 40 talks were presented; five of them were longer plenary lectures delivered by Professors Jacek Chmieliński, Zbigniew Leśniak, Adam Ostaszewski, Dorian Popa and Ioan Raşa. The talks were devoted mainly to functional equations and inequalities, iteration theory and their applications in various branches of mathematics, as well as some related topics. In particular, the presented talks concerned classical functional equations such as: Cauchy, Jensen, d'Alembert, Gołąb-Schinzel, Baxter, quadratic, exponential, as well as inequalities (e.g., Hlawka's or Kedlaya's type). Moreover, properties of orthogonally additive functions, involutions, convex functions or multivalued mappings were discussed. The problem of Hyers-Ulam stability of some functional equations was also discussed. Finally, answers to some problems from previous meetings were given (e.g., of Butler, Derfel, Baron & Ger, Raşa), as well as, during special sessions, some new open problems and remarks were presented.

Some social events accompanied the conference: a picnic on Tuesday night, a banquet on Thursday and the piano recital performed by Professors Marek Czerni and László Székelyhidi on Wednesday evening. On Wednesday afternoon participants visited Poznań, walking the old city streets and visiting the historical museum.

The Scientific Committee, on its meeting during the conference, accepted the resignation of Professor Krzysztof Ciepliński from the membership in the Committee. Moreover, the Committee entrusted the chairmanship of the Organizing Committee for the next conference to Professor Jacek Chmieliński.

The conference was closed on Saturday, July 15, by Professor Janusz Brzdęk. The subsequent 18th ICFEI was announced to be organized in the year 2019.

1. Abstracts of Talks

Marcin Adam *Alienation of the quadratic, exponential and d'Alembert equations*

Let $(S,+)$ be a commutative semigroup, $\sigma\colon S\to S$ be an endomorphism with $\sigma^2=id$ and let K be a field of characteristic different from 2. Inspired by results obtained in [1] and [2], we study the solutions $f,g,h\colon S\to K$ of Pexider type functional equations

$$f(x+y)+f(x+\sigma y)+g(x+y)=2f(x)+2f(y)+g(x)g(y),\quad x,y\in S, \tag{1}$$

$$\begin{aligned} f(x+y)+f(x+\sigma y)+h(x+y)+h(x+\sigma y)\\ =2f(x)+2f(y)+2h(x)h(y),\quad x,y\in S, \end{aligned} \tag{2}$$

resulting from summing up the generalized version of the quadratic functional equation with the exponential Cauchy equation and the generalized version of the d'Alembert equation side by side, respectively. We show that under some additional assumptions, equations (1) and (2) force f, g, h to solve the quadratic, exponential and d'Alembert functional equations, respectively.

References

[1] P. Sinopoulos, *Functional equations on semigroups*, Aequationes Math. 59 (2000), 255–261.

[2] B. Sobek, *Alienation of the Jensen, Cauchy and d'Alembert equations*, Ann. Math. Sil. 30 (2016), 181–191.

Javid Ali *Stablility and data dependence results for Zamfirescu multivalued mappings*

Approximating fixed points of a nonlinear operator is one the most widely used techniques for solving differential/integral equations. In view of their concrete applications, it is of great interest to know whether these methods are numerically stable or not. In this presentation, we discuss some new stability and data dependence results for the class of multi-valued Zamfirescu operators. Our results generalize and improve several existing results in literature. It is worth mentioning here that our results are new even for single valued mappings.

Anna Bahyrycz *On the stability of functional equation by Baak, Boo and Rassias* (joint work with **Harald Fripertinger** and **Jens Schwaiger**)

We consider the functional equation of the following form

$$rf\Big(\frac{1}{r}\sum_{j=1}^{d} x_j\Big) + \sum_{S\in\binom{\underline{d}}{\ell}} rf\Big(\frac{1}{r}\Big(\sum_{j\notin S} x_j - \sum_{j\in S} x_j\Big)\Big) = \left(\binom{d-1}{\ell} - \binom{d-1}{\ell-1} + 1\right)\sum_{j=1}^{d} f(x_j)$$

in the class of functions f mapping a normed space X into Banach space Y (both over the field $\mathbb{K}$ of characteristic 0), $r \in \mathbb{R}\setminus\{0\}$ is given, ℓ, d are fixed integers satisfying the inequality $1 < \ell < d/2$, and $\binom{\underline{d}}{\ell}$ denotes the set of all ℓ-subsets of $\underline{d} = \{1,\dots,d\}$.

In [1] the authors determined all odd solutions $f\colon X \to Y$ for vector spaces X, Y over $\mathbb{R}$ and $r \in \mathbb{Q}\setminus\{0\}$. In [3] Oubbi considered the same equation but for arbitrary real $r \neq 0$. Generalizing similar results from [1] he additionally investigates certain stability questions for the equation above, but as for that equation itself for odd approximate solutions only.

At the 54th International Symposium on the Functional Equation H. Fripertinger determined the general solution of this equation (the results were obtained jointly with J. Schwaiger).

In our talk we present many stability results for the above equation. They based on the fundamental and very general results in [2], where a priory no additional assumption (oddness) is assumed. The results come from a joint work with H. Fripertinger and J. Schwaiger.

References

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[2] A. Bahyrycz, J. Olko, *On stability of the general linear equation*, Aequationes Math. 89(6) (2015), 1461–1474.

[3] L. Oubbi, *On Ulam stability of a functional equation in Banach modules*, Can. Math. Bull. 60(1) (2017), 173–183.

Karol Baron *On the set of orthogonally additive functions with orthogonally additive second iterate*

Let E be a real inner product space of dimension at least 2. We show that both the set of all orthogonally additive functions mapping E into E having orthogonally additive second iterate and its complement are dense in the space of all orthogonally additive functions from E into E with the Tychonoff topology.

Nicole Brillouët–Belluot *On a generalization of the Baxter functional equation*

We determine all continuous solutions $f\colon \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(af(x)^k y + bf(y)^\ell x + cxy) = f(x)f(y)$$

with $a, b, c \in \mathbb{R}$ and $k, \ell \in \mathbb{N} \cup \{0\}$.

This functional equation generalizes the Baxter functional equation

$$f(f(x)y + f(y)x - xy) = f(x)f(y)$$

and some generalizations of the functional equation of Gołąb-Schinzel

$$f(f(x)^k y + f(y)^\ell x) = f(x)f(y).$$

Janusz Brzdęk *A fixed point theorem and Ulam stability in generalized dq-metric spaces*

Let Y be a nonempty set, $\mathbb{R}_0^+$ denote the set of nonnegative reals, and $\mu\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$. Let $\rho\colon Y \times Y \to \mathbb{R}_0^+$ be a dq μ-metric (dislocated quasi μ-metric), i.e. let the following two conditions be fulfilled:

(a) if $\rho(x, y) = 0$ and $\rho(y, x) = 0$, then $y = x$,

(b) $\rho(x, z) \le \mu(\rho(x, y), \rho(y, z))$ for $x, y, z \in Y$.

A fixed point theorem for some spaces of functions (with values in Y) will be presented, under the assumptions that Y is ρ-complete and $\mu\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous (with regard to the usual topologies in $\mathbb{R}_0^+$ and $\mathbb{R}_0^+ \times \mathbb{R}_0^+$) and nondecreasing with respect to each variable (i.e. $\mu(a, b) \le \mu(a, c)$ and $\mu(b, a) \le \mu(c, a)$ for every $a, b, c \in \mathbb{R}_0^+$ with $b \le c$). The theorem has been motivated by the notion of Ulam stability and is a natural generalization and extension of the classical Banach Contraction Principle and some other more recent results.

Jacek Chmieliński *On a pexiderization of the orthogonality equation and the orthogonality preserving property*

We consider problems connected with preservation of the inner product or the orthogonality relation by a pair of mappings. Namely, we study the properties:

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle$$

and

$$x \perp y \implies f(x) \perp g(y)$$

for all x, y from the joint domain of f and g.

As an introduction, the case of a single mapping will be shortly reviewed.

References

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Jacek Chudziak *A characterization of probability distortion functions of the Goldstein-Einhorn type*

Probability distortion functions play an important role in various models of decision making under risk. In a literature one can find some classes of such functions. In particular, Goldstein and Einhorn [1] introduced the following class

$$g_{a,\gamma}(p) = \frac{ap^\gamma}{ap^\gamma + (1-p)^\gamma} \qquad \text{for } p \in [0,1],$$

where $a, \gamma > 0$. In the talk we present a characterization of the Goldstein-Einhorn type probability distortion functions.

References

[1] W.M. Goldstein, H.J. Einhorn, *Expression theory and the preference reversal phenomenon*, Psychological Review 94 (1987), 236–254.

Bruce Ebanks *Linked additive functions*

We discuss some old and new results about functional equations of the form

$$\sum_{k=1}^{n} x^{m_k} f_k(x^{j_k}) = 0$$

for nonnegative integers m_k, positive integers j_k and additive functions f_k mapping an integral domain into itself. If there is no "duplication" of terms (that is, if $(m_k, j_k) \neq (m_p, j_p)$ for $k \neq p$), then each f_k is the sum of a linear function and a derivation of some order. We also update a problem posed by Kannappan and Kurepa in 1970 concerning similar equations of a somewhat more general form.

El-Sayed El-Hady *On the analytical solutions of some functional equations*

During the last few decades a certain structure of functional equations see [1] arises from many interesting applications like e.g. fog computing and wireless networks see [2] e.g.. The general form of such structure is given by

$$A_1(x,y)f(x,y) = A_2(x,y)f(x,0) + A_3(x,y)f(0,y) + A_4(x,y)f(0,0) + A_5(x,y),$$

where $A_i(x,y)$, $i = 1,\dots,5$, are given polynomials in two complex variables x, y. As far as I know there is no exact-from general solution available for such kind of equations. In this talk I will present some investigations of the analytical solutions of such general class of equations using some special cases.

References

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Hojjat Farzadfard *Precinct theory: a useful tool for iteration theory*

Let X be a nonempty set, G be a group and $\phi\colon X \to G$ be a map. A function $f\colon X \to X$ is said to be *in the realm of* ϕ provided that there exists $\alpha \in G$ such that $\phi(f(x)) = \alpha\phi(x)$ for all $x \in X$. We call α the *index of* f *with respect to* ϕ; it is denoted by $ind_\phi(f)$. The set of all functions which are in the realm of ϕ is called the *realm* of ϕ and is denoted by $Realm(\phi)$. The above notions were first coined by the author in [1] and then developed in [2].

Let X be a nonempty set. A subset $\mathcal{F}$ of the set

$$\mathfrak{F}(X) := \{f : \ f \text{ is a function of } X \text{ into itself}\}$$

is called a *semi-precinct* if it is the realm of a map ϕ of X into a group G. If ϕ is surjective, $\mathcal{F}$ is called a *precinct.* In the present work we discuss some recent applications of the above notions in three areas of iteration theory: 1. The Schröder and Abel equations, 2. (regular) iteration groups, 3. regular iterations.

References

[1] H. Farzadfard, B. Khani Robati, *The structure of disjoint groups of continuous functions*, Abstr. Appl. Anal. 2012, Article ID 790758, 14 pp.

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Włodzimierz Fechner *Systems of functional inequalities for mappings between rings*

Assume that X and Y are compact Hausdorff spaces, $C(X)$ and $C(Y)$ are algebras of all real-valued continuous functions on X and Y, respectively, with pointwise algebraic operations and pointwise order and $T\colon C(X) \to C(Y)$ is an arbitrary mapping. During the talk we will discuss a few systems of functional inequalities for T. We are especially interested in systems involving Hlawka's functional inequality. In particular, we will study the following system:

$$\begin{cases} T(x+y)+T(x+z)+T(y+z) \geq T(x+y+z)+T(x)+T(y)+T(z), \\ T(x\cdot y) \geq T(x)\cdot T(y), \end{cases}$$

postulated for all $x, y \in C(X)$.

Żywilla Fechner *A functional equation motivated by some trigonometric identities*
(joint work with **Włodzimierz Fechner**)

We deal with the following functional equation

$$f(xy) + \lambda f(x)f(y) = \phi(x,y),$$

where λ is a complex number and f and ϕ are complex mappings defined on a semigroup. Moreover, we assume an addition formula of trigonometric type for ϕ. Our research is motivated by some earlier results related to a problem posed by S. Butler in 2003. We discuss some possible questions for future research.

References

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László Horváth *Delay differential and Halanay type inequalities*

In the present talk we develop a framework for a Halanay type nonautonomous delay differential inequality with maxima, and establishes necessary and/or sufficient conditions for the global attractivity of the zero solution. The emphasis is put on the rate of convergence based on the theory of the generalized characteristic inequality and equation. The applicability and the sharpness of the results are illustrated by examples.

Eliza Jabłońska *Solution of a problem posed by K. Baron and R. Ger*
(joint work with **Taras Banakh**)

We introduce a new family of 'small' sets which is tightly connected with two well known σ-ideals: of Haar-null sets and of Haar-meager sets. We define a subset

A of a topological group X to be *null-finite* if there exists a null-sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for every $x \in X$ the set $\{n \in \mathbb{N} : \ x_n + x \in A\}$ is finite. Applying null-finite sets we prove that a mid-point convex function $f\colon G \to \mathbb{R}$ defined on an open convex subset G of a metric linear space X is continuous if it is upper bounded on a subset which is not null-finite and whose closure is contained in G. This gives an alternative short proof of a known generalization of Bernstein-Doetsch theorem.

Since Borel null-finite sets are Haar-meager and Haar-null, we conclude that a mid-point convex function $f\colon G \to \mathbb{R}$ defined on an open convex subset G of a complete linear metric space X is continuous if it is upper bounded on a Borel subset $B \subset G$ which is not Haar-null or not Haar-meager in X. The last result resolves an old problem in the theory of functional equations and inequalities posed by K. Baron and R. Ger in 1983.

References

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Tibor Kiss *On Jensen-differences which are quasidifferences*
(joint work with **Zsolt Páles**)

Let $I \subseteq \mathbb{R}$ be a nonempty open subinterval of $\mathbb{R}$. The aim of the talk is to solve the functional equation

$$f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2} = G(g(x)-g(y)), \qquad (x,y \in I),$$

where $g\colon I \to \mathbb{R}$ and $G\colon g(I)-g(I) \to \mathbb{R}$ are differentiable functions, $f\colon I \to \mathbb{R}$ is continuously differentiable and g' does not vanish on I.

References

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[2] T. Kiss, Zs. Páles, *On a functional equation related to two variable weighted quasi-arithmetic means*, 2017 (submitted).

Zbigniew Leśniak *On properties of Brouwer flows and Brouwer homeomorphisms*

We present properties of Brouwer flows, i.e. flows which contain a Brouwer homeomorphism. In particular, we describe a relationship between the equivalence classes of the codivergency relation and the set of regular points.

We also show the corresponding results which concern Brouwer homeomorphisms that are not necessarily embeddable in a flow. These results are obtained under assumptions on existence of invariant lines. Such lines play a similar role as trajectories in the case where a Brouwer homeomorphism is embeddable in a flow.

Gyula Maksa *Results related to the invariance equation $K \circ (M, N) = K$*
(joint work with **Zoltán Daróczy**)

Let $I \subset \mathbb{R}$ (the reals) be an interval of positive length, $K, M, N: I^2 \to I$ be functions. In this talk, we discuss the following problems.

- Suppose that K and M are means in the sense that

$$K(x,y), M(x,y) \in [\min\{x,y\}, \max\{x,y\}] \qquad (x, y \in I),$$

and the invariance equation holds for the functions K, M and N. Find conditions under which the function N will be a mean itself, as well. We present the solution of this problem supposing that K is Matkowski mean, that is,

$$K(x,y) = (f+g)^{-1}(f(x)+g(y)) \qquad (x, y \in I),$$

where $f, g: I \to \mathbb{R}$ are continuous and strictly monotonic functions in the same sense.

- Under given means K and M, what is the solution N of the invariance equation? In special cases, we give the answer to this question, too.

The main motivation of our investigations is the paper [1].

References

[1] P. Kahlig, J. Matkowski, *Invariant means related to classical weighted means*, Publ. Math. Debrecen, 89/3 (2016), 373–387.

Renata Malejki *On the stability of a generalized Fréchet functional equation with respect to hyperplanes in the parameter space*
(joint work with **Janusz Brzdęk** and **Zbigniew Leśniak**)

In this paper we consider the functional equation

$$\begin{aligned} A_1F(x+y+z) + A_2F(x) + A_3F(y) + A_4F(z) \\ = A_5F(x+y) + A_6F(x+z) + A_7F(y+z), \end{aligned} \tag{1}$$

where $A_1, \dots, A_7 \in \mathbb{K}$ are constants belonging to $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ($\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively), in the class of functions $F: X \to Y$, where X is a commutative group and Y is a Banach space over the field $\mathbb{K}$.

We study the stability of a generalization of the Fréchet functional equation. In the proof of the main result we use a fixed point theorem to get an exact solution of the equation close to a given approximate solution. Our aim is to study the problem of stability of equation (1) for all possible values of coefficients A_i for $i \in \{1, \dots, 7\}$.

Kazimierz Nikodem *Functions generating* $(m, M, \Psi)-$*Schur-convex sums* (joint work with **Silvestru Sever Dragomir**)

The notion of $(m, M, \Psi)-$Schur-convexity is introduced and functions generating $(m, M, \Psi)-$Schur-convex sums are investigated. An extension of the Hardy-Littlewood-Pólya majorization theorem is obtained. A counterpart of the result of Ng stating that a function generates $(m, M, \Psi)-$Schur-convex sums if and only if it is $(m, M, \psi)-$Wright-convex is proved and a characterization of $(m, M, \psi)-$Wright-convex functions is given.

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Jolanta Olko *On a system of two functional inclusions*

Given $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$ and polynomials P, Q satisfying some additional conditions, we study the following system of functional inclusions

$$\begin{cases} F(x+a) \subset F(x) + P(x), \\ F(x+b) \subset F(x) + Q(x), \end{cases} \qquad x \in \mathbb{R},$$

where F is a set-valued function defined on the set of reals. Motivated by [1] we obtain a multivalued counterpart to results concerning inequalities related to single-valued polynomials.

References

[1] J. Schwaiger, *Remarks on a paper about functional inequalities for polynomials and Bernoulli numbers*, Aequationes Math. 78 (2009), 177–183.

Masakazu Onitsuka *On the Hyers–Ulam stability of first-order nonhomogeneous linear difference equations*

Consider the first-order nonhomogeneous linear difference equation

$$\Delta_h x(t) - a x(t) = f(t) \tag{1}$$

on $h\mathbb{Z}$, where

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \qquad \text{and} \qquad h\mathbb{Z} = \{hk|\ k \in \mathbb{Z}\}$$

for given $h > 0$; a is a real number with $a \neq -1/h$; $f(t)$ is a real-valued function on $h\mathbb{Z}$. We call h the "*step-size*". Let I be a nonempty open interval of $\mathbb{R}$. We define $T = h\mathbb{Z} \cap I$ and

$$T^* = \begin{cases} T \setminus \{\max T\}, & \text{if the maximum of } T \text{ exists,} \\ T, & \text{otherwise.} \end{cases}$$

Throughout this talk, we assume that T and T^* are nonempty sets of $\mathbb{R}$. Note here that if a function $x(t)$ exists on T, then $\Delta_h x(t)$ exists on T^*. Now we will define the Hyers–Ulam stability for Eq. (1). We say that Eq. (1) has the "*Hyers–Ulam stability*" on T if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi\colon T \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \leq \varepsilon$ for all $t \in T^*$, then there exists a solution $x\colon T \to \mathbb{R}$ of Eq. (1) such that $|\phi(t) - x(t)| \leq K\varepsilon$ for all $t \in T$. We call such K a "*HUS constant*" for Eq. (1) on T. It is easy to see that if $a = 0$ or $a = -2/h$, then Eq. (1) does not have the Hyers–Ulam stability on $h\mathbb{Z}$. In this study, we treat only the case of $a \neq 0$ and $a \neq -2/h$.

Eq. (1) is an approximation of the ordinary differential equation

$$x' - ax = f(t), \tag{2}$$

where a is a non-zero real number; $f(t)$ is a continuous real-valued function on $\mathbb{R}$. It is well-known that Eq. (2) has the Hyers–Ulam stability with a HUS constant $1/|a|$ on $\mathbb{R}$. Furthermore, the solution $x(t)$ of (2) satisfying $|\phi(t) - x(t)| \leq \varepsilon/|a|$ for all $t \in \mathbb{R}$ is the only one (unique), where $\phi(t)$ is a continuously differentiable function satisfying $|\phi'(t) - a\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$. That is, for Eq. (2), the minimums of HUS constants on $\mathbb{R}$ is $1/|a|$. See [2, 4] and the references therein. On the other hand, the following result is obtained by using a result presented by Brzdęk, Popa and Xu [1]. In the case where $a \neq 0, -1, -2$, $f(t) \equiv 0$ and $h = 1$, Eq. (1) has the Hyers–Ulam stability with a HUS constant $1/||a+1|-1|$ on $\mathbb{Z}$. Furthermore, the solution $x(t)$ of (1) satisfying $|\phi(t) - x(t)| \leq \varepsilon/||a+1|-1|$ for all $t \in \mathbb{Z}$ is the only one, where $\phi(t)$ is a function satisfying $|\Delta_1\phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{Z}$. Comparing the above results under the assumption $f(t) \equiv 0$, we see that if $a \neq 0$ and $a > -1$, then the minimums of HUS constants of (1) with $h = 1$ and (2) are the same. However, if $a \neq -2$ and $a < -1$, then they are different. Now, an important question arises: How does the step-size influence the minimum of HUS constants for Eq. (1) on $h\mathbb{Z}$? The main purpose of this study is to answer the question.

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Adam Ostaszewski *Asymptotic group actions: associated functional equations and inequalities*

The theory of regular variation has two classical settings. The first, due to Karamata in 1930, rests on the *Cauchy functional equation* for additive functions. The second, due to Bojanić and Karamata in 1963 and de Haan (1970 on), rests on the *Gołąb-Schinzel functional equation* (or *Goldie functional equation*, depending on context). One can unify these two cases by suitable use of an algebraic approach due to Popa and Javor. This clarifies the interplay between the two formulations above; it also gives a unified proof to the hardest single result in each theory, that on *quantifier weakening*. Key here is a use of functional inequalities, which apparently complements the existing literature.

A less classical but very useful setting is that of *Beurling slow and regular variation*. This artificial-looking but in fact very natural setting originated in Beurling's approach to extending the Wiener Tauberian theory, from convolutions to 'convolution-like' settings. The resulting Wiener-Beurling Tauberian theorem is the natural tool for the Tauberian theorem for the Borel summability method (and for Riesz means, and Beurling moving averages).

As the above use of the term convolution suggests, this hinges on group structure, and group actions; as mention of Tauberian theory suggests, this is analysis, involving limits. We give a topological treatment involving asymptotic group actions.

Lahbib Oubbi *On the Ulam-Hyers stability of a functional equation of C. Baak et al.*

Let X and Y be linear spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $f\colon X \to Y$ be an odd mapping. For any rational number $r \neq 0$, C. Baak, D.H. Boo, and Th.M. Rassias [1] introduced the following functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{i(j)\in\{0,1\} \\ \sum_{j=1}^{d} i(j)=\ell}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{i(j)}x_j}{r}\right) = (C_{d-1}^{\ell} - C_{d-1}^{\ell-1} + 1)\sum_{j=1}^{d} f(x_j),$$

where d and ℓ are positive integers so that $1 < \ell < \frac{d}{2}$, and $C_q^p := \frac{q!}{(q-p)!p!}$, $p, q \in \mathbb{N}$ with $p \leq q$. The authors solved this equation and, whenever Y is a Banach space, they showed that it is Hyers-Ulam stable for $r \neq 2$.

In this talk, we will solve the same equation and show its Hyers-Ulam stability for arbitrary non zero scalar $r \in \mathbb{K}$ with Y only a sequentially complete locally pseudo-convex space. To this aim, we will first give a theorem of Forti-Brzdęk type in uniformizable spaces.

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Zsolt Páles *Generalizations of the Bernstein–Doetsch theorem*
(joint work with **Carlos Gonzáles, Attila Gilányi, Kazimierz Nikodem** and **Gari Roa**)

Given a convex set D of a linear space X and two set-valued maps $A, B\colon (D-D) \to 2^Y$, we consider the Jensen convexity type

$$\frac{F(x)+F(y)}{2} + A(x-y) \subseteq \mathrm{cl}\Big(F\Big(\frac{x+y}{2}\Big) + B(x-y)\Big), \qquad x, y \in D$$

and the Jensen concavity type

$$F\Big(\frac{x+y}{2}\Big) + A(x-y) \subseteq \mathrm{cl}\Big(\frac{F(x)+F(y)}{2} + B(x-y)\Big), \qquad x, y \in D$$

inclusions and derive their convexity/concavity consequences. These results involve the Takagi and Tabor transforms of the set-valued maps A and B and generalize those results that have been obtained for real and vector valued functions.

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Paweł Pasteczka *Weighted Kedlaya inequality*
(joint work with **Zsolt Páles)**

In 2016 we proved that for every symmetric, repetition invariant and Jensen concave mean M the Kedlaya-type inequality

$$A(x_1, M(x_1,x_2), \dots, M(x_1,\dots,x_n)) \le M(x_1, A(x_1,x_2), \dots, A(x_1,\dots,x_n))$$

holds for an arbitrary (x_n) (A stands for the arithmetic mean). We are going to prove the weighted counterpart of this inequality. More precisely, if (x_n) is a vector with corresponding (non-normalized) weights (μ_n) and $M\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \mu_1 & \mu_2 & \dots & \mu_n \end{pmatrix}$

denotes the weighted mean then, under analogous conditions on M, the inequality

$$A\left(\begin{matrix} x_1 & M\begin{pmatrix} x_1 & x_2 \\ \mu_1 & \mu_2 \end{pmatrix} & \dots & M\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \mu_1 & \mu_2 & \dots & \mu_n \end{pmatrix} \\ \mu_1 & \mu_2 & \dots & \mu_n \end{matrix}\right)$$

$$\leq M\left(\begin{matrix} x_1 & A\begin{pmatrix} x_1 & x_2 \\ \mu_1 & \mu_2 \end{pmatrix} & \dots & A\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \mu_1 & \mu_2 & \dots & \mu_n \end{pmatrix} \\ \mu_1 & \mu_2 & \dots & \mu_n \end{matrix}\right)$$

holds for every (x_n) and (μ_n) such that the sequence $(\frac{\mu_k}{\mu_1+\ldots+\mu_k})$ is decreasing.

Dorian Popa *Ulam stability of linear operators*

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and X a linear space over the field $\mathbb{K}$. A function $\rho_X\colon X \to [0,+\infty]$ is called a gauge if

(i) $\rho_X(\lambda x) = |\lambda| \cdot \rho_X(x)$ for all $x \in X$, $\lambda \in \mathbb{K}$, $\lambda \neq 0$;

(ii) $\rho_X(x) = 0$ if and only if $x = 0$.

Let A, B be linear spaces over $\mathbb{K}$, ρ_A, ρ_B gauges on A and B and $L\colon A \to B$ a linear operator. We say that L is Ulam stable with constant $K \geq 0$ if for every $x \in A$ such that $\rho_B(Lx) \leq 1$ there exists $z \in \operatorname{Ker} L$ with $\rho_A(z - x) \leq K$.

We obtain a characterization of Ulam stability for a linear operator L and a representation of the best Ulam constant for L. As applications we give some results on the stability of linear operators acting on normed spaces and of some differential and partial differential operators.

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Teresa Rajba *Muirhead inequality for convex orders and a problem of I. Raşa* (joint work with **Andrzej Komisarski)**

We present ([2]) a new, very short proof of a conjecture by I. Raşa (25-years-old problem), which is an inequality involving basic Bernstein polynomials and convex functions.

PROBLEM.
Prove or disprove that

$$\sum_{i,j=0}^{n} \big(b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\big) f\Big(\frac{i+j}{2n}\Big) \geq 0 \qquad (1)$$

for each convex function $f \in \mathbb{C}([0,1])$ and for all $x, y \in [0,1]$.

It was affirmed positively very recently by J. Mrowiec, T. Rajba and S. Wąsowicz [3] by the use of stochastic convex orderings, as well as by U. Abel [1] who simplified their proof using elementary methods.

Inequality (1) is equivalent to the following stochastic convex ordering relation

$$F_{X+Y} \leq_{cx} \tfrac{1}{2}(F_{X_1+X_2} + F_{Y_1+Y_2}), \tag{2}$$

where (X, Y), (X_1, X_2) and (Y_1, Y_2) are the pairs of independent binomially distributed random variables such that $X, X_1, X_2 \sim B(n, x)$ and $Y, Y_1, Y_2 \sim B(n, y)$.

Using the usual stochastic order relation $\leq_{st}$, we give a useful sufficient condition for the verification of relation (2), which in the case of binomial distributions is equivalent to the I. Raşa inequality (1).

Theorem.
Let X and Y be two independent random variables with finite means, such that $X \leq_{st} Y$ or $Y \leq_{st} X$. Let (X_1, X_2) and (Y_1, Y_2) be two pairs of independent random variables such that X, X_1, X_2, and Y, Y_1, Y_2 are identically distributed. Then

$$F_{X+Y} \leq_{cx} \frac{1}{2}(F_{X_1+X_2} + F_{Y_1+Y_2}). \tag{3}$$

The sufficient condition, that appears in the above theorem, is satisfied for random variables X and Y with distributions from some families (among others) of probability distributions: binomial, Poisson, negative binomial, gamma, exponential, beta and Gaussian distributions. Taking these distributions, as an immediate consequence of the above theorem, we can obtain several I. Raşa type inequalities as generalizations of the I. Raşa inequality (1).

Our methods allow us to give some extended versions of stochastic convex order (3) as well as the I. Raşa type inequalities. In particular, we prove the Muirhead type inequality for convex orderings for convolution polynomials of probability distributions.

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Ioan Raşa *Entropies, Heun functions and convexity*

For a probability distribution $(p_0(x), p_1(x), \ldots)$ depending on a real parameter x, the *index of coincidence* is defined by $S(x) = \sum_i p_i^2(x)$. The Rényi entropy and Tsallis entropy are $R(x) = -\log S(x)$, respectively $T(x) = 1 - S(x)$.

We consider a family of probability distributions for which $S(x)$ satisfies a Heun differential equation. Convexity properties of $S(x)$ are investigated; combined with the Heun property, they lead to bounds of the corresponding entropies.

In particular, we show that $S(x)$ associated with the binomial distribution is log-convex; this proves a conjecture formulated in [2]. The proof is based on the Heun property of $S(x)$ (see [2]) and the relation between $S(x)$ and the Legendre polynomials (see [1]).

The complete monotonicity of entropies (including Shannon entropy) is also discussed.

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Maciej Sablik *Characterizing exponential polynomials*

The authors of [1] considered the following equation

$$\sum_{i=1}^{m} f_i(\alpha_i x + \beta_i y) = \sum_{k=1}^{n} u_k(y)v_k(x) + \sum_{s=1}^{N} P_s(x)w_s(y)e^{\lambda_s(y)x}$$

and proved that under suitable assumptions, every $f_i\colon \mathbb{R}^d \to \mathbb{C}$ is an exponential polynomial. Using methods from [2] we show the following.

Theorem.
If

$$\sum_{r=0}^{M}\sum_{j=0}^{r}\left(\sum_{i=1}^{m_r} f_{rji}(\alpha_{rji}x + \beta_{rji}y)\right)x^{r-j}y^j = \sum_{k=1}^{n} u_k(y)v_k(x) + \sum_{s=1}^{N} P_s(x)w_s(y)e^{\lambda_s(y)x}$$

where $\alpha_{rji}, \beta_{rji} \in \mathbb{R}\setminus\{0\}$ are given scalars, and $f_{rji}, u_k, v_k, w_s, \lambda_s$ map $\mathbb{R}$ into $\mathbb{C}$ while P_s are polynomials, then f_{rji} are exponential polynomials.

Further generalizations, in particular a group-theoretical case, will be presented.

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Farzane Sadeghi Boome *Two orthogonalities based on norm inequalities*
(joint work with **Farzad Dadipour)**

In this talk, we present two orthogonalities in a normed linear space which are based on angular distance inequalities. We also show that one of these orthogonalities is equivalent with the Singer orthogonality and compare these two orthogonalities with each other. Finally some related geometric properties of normed linear spaces are discussed and a characterization of inner product spaces is obtained.

Acknowledgement The second author is supported by the National Elite Foundation through a grant to Professor Mehdi Radjabalipour.

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Cristina Serpa *On systems of iterative functional equations*
(joint work with **Jorge Buescu**)

We formulate a general theoretical framework for systems of iterative functional equations between general spaces. We give general necessary conditions for the existence of solutions such as compatibility conditions (essential hypotheses to ensure problems are well-defined). For topological spaces we characterize continuity of solutions; for metric spaces we find sufficient conditions for existence and uniqueness. Conjugacy equations arise from the problem of identifying dynamical systems from the topological point of view. We show that even in the simplest cases, e.g. piecewise affine maps, solutions of functional equations arising from conjugacy problems may have exotic properties. We provide a general construction for finding solutions, including an explicit formula showing how, in certain cases, a solution can be constructively determined. We show the relevance of this for the representation of real numbers.

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Ekaterina Shulman *On Montel's type statements for mappings on groups*

Let G be a topological group, $f\colon G \to \mathbb{C}$ a continuous function. We call f a *polynomial of degree* $\le n$ if

$$\Delta_{h_{n+1}} \cdots \Delta_{h_2} \Delta_{h_1} f = 0 \qquad \text{for any } h_1, \dots, h_{n+1} \in G. \tag{1}$$

Furthermore, we call f a *semipolynomial of degree* $\le n$ if

$$\Delta_h^{n+1} f = 0 \qquad \text{for any } h \in G. \tag{2}$$

We study the following question: *whether it is sufficient to check the conditions* (1) *and* (2) *only for increments* h_i *from a topologically generating subset* $E \subset G$*?*

For polynomials the question is solved affirmatively.

For semipolynomials, we prove that the answer is positive if

1. E consists of compact elements,
2. G is commutative and E is finite.

In the second case we show that both conditions are essential.

Peter Stadler *Curve shortening by short rulers*

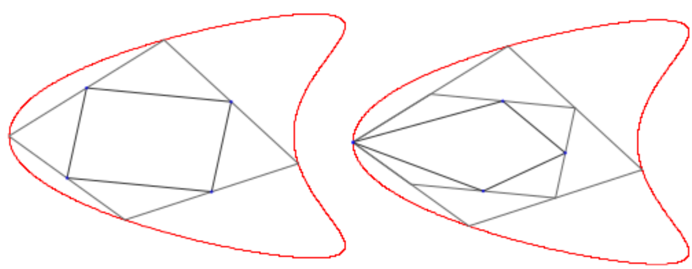
We look at homomorphisms $h\colon (\mathbb{R}, +) \to (G, \circ)$ on a Lie group G

$$h(s+t) = h(s) \circ h(t), \qquad h(0) = e, \qquad h(1) = g.$$

The restriction of h to the interval $[0, 1]$ is a geodesic.

On Riemannian manifolds geodesics are locally shortest lines. The problem is to construct long geodesics. But any curve connecting starting point and end point can be shortened by using a ruler which allows to construct short geodesics.

If starting point and end point coincide, the shortened curve is closed, too.



We look at two possibilities: If we skip the starting point (left picture), we get the Birkhoff curve shortening as described in [1]. Else, we get the short ruler method as proposed by [2]. We investigate the differences we get as we iterate the two methods.

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Henrik Stetkær *The kernel of the second order Cauchy difference*

Let S be a semigroup, not necessarily commutative. The Cauchy difference of $f\colon S\to\mathbb{C}$ is the function $(Cf)(x,y) := f(xy)-f(x)-f(y)$ for $x,y\in S$, and the second order Cauchy difference is the iterated function $(C^2f)(x,y,z) = C\{Cf(x,\cdot)\}(y,z)$ for $x,y,z\in S$.

That $C^2f=0$ is equivalent to Whitehead's functional equation

$$f(xyz)+f(x)+f(y)+f(z)=f(xy)+f(yz)+f(xz),\quad x,y,z\in S.$$

We relate various functional equations by showing that the functions $f\colon S\to\mathbb{C}$ satisfying $C^2f=0$ are those of the form $f(x)=J(x)+B(x,x)$, $x\in S$, where $J\colon S\to\mathbb{C}$ is a solution of (a version of) Jensen's functional equation and $B\colon S\times S\to\mathbb{C}$ is bi-additive.

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Mariusz Sudzik *On a problem of Derfel*

The talk concerns the functional equation

$$f(x)=\frac{1}{2}f(x-1)+\frac{1}{2}f(-2x). \tag{1}$$

Last year, during *21st European Conference on Iteration Theory* held in Innsbruck, Professor Gregory Derfel posed the following:

Problem.
Is there a non-constant bounded continuous function $f\colon\mathbb{R}\to\mathbb{R}$ satisfying (1)?

I will show a partial solution to the Problem. The talk is divided into three parts. In the first part I will prove that every bounded continuous solution of equation (1) which attains its extreme value is constant. It is the main result. Its proof is quite elementary.

In the second part of my presentation I will discuss the properties of solutions of the equation (1) which do not attaining their extreme values. In this case the Problem is still open. I will show, for example, the following fact

LEMMA.
Let $a, b \in \mathbb{R}$ *and let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a bounded continuous solution of equation* (1). *If*

$$\inf\{f(t): t \in \mathbb{R}\} < f(x) < \sup\{f(t): t \in \mathbb{R}\} \quad \textit{for every } x \in \mathbb{R},$$

then

(i) $\inf\{f(x): x \in \mathbb{R}\} = \inf\{f(x): x < a\} = \inf\{f(x): x > b\}$,

(ii) $\sup\{f(x): x \in \mathbb{R}\} = \sup\{f(x): x < a\} = \sup\{f(x): x > b\}$.

The last part of my talk is connected with solutions of equation (1) in a smaller class of functions. For example, we can see that there are no non-trivial, bounded, continuous, with bounded variation solutions of equation (1).

László Székelyhidi *Exponential polynomials on affine groups*

Exponential polynomials are the basic building blocks of spectral synthesis. Recently it has turned out that the fundamental theorem of L. Schwartz about spectral synthesis on the reals can be generalized to several dimensions via some reasonable modification of the original setting. As affine groups play a basic role in this generalization it seems reasonable to study the class of exponential polynomials on these objects.

Peter Volkmann *Comparison theorems for functional, differential, and integral equations*
(joint work with **Gerd Herzog)**

An abstract result will be given, which can be used to establish comparison theorems as indicated in the title.

In particular, this result can be applied to the papers from 2012 (http://doi.org/10.1007/978-3-0348-0249-9$_2$1) and 2016 (http://doi.org/10.5445/IR/1000061837) in the same way as a theorem from 2002 (http://doi.org/10.5445/IR/492002) could be used to derive simpler comparison theorems.

Alfred Witkowski *On involutions preserving convexity*
(joint work with **Janusz Matkowski)**

It is known that if $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a convex function, then so is $g(x) = xf(1/x)$ This means that the operator $F(f)(x) = xf(1/x)$ is a convexity preserving involution on the space of all real functions on $\mathbb{R}_+$.

We give a classification of the linear operators of the form $T(f)(x) = H(x, f \circ h(x))$ sharing the above properties.

2. Problem and Remarks

1. Problem.

Find the general solution of the functional equation

$$C_1(x,y)F(x) + C_2(x,y)G(y) = 0, \qquad (*)$$

$$\forall x, y \in T := \{(x, y) : \; xy = (a_1 + a_2 y + a_3 xy)(b_1 + b_2 x + b_3 xy)\} \subset \mathbb{C}^2,$$

such that:

1. $C_i(x, y)$, $i = 1, 2$ are given functions, in particular rational analytic inside the unit disk,
2. $F, G \colon D \to \mathbb{C}$, where D is the unit disk,
3. The set T is a torus in $\mathbb{C}^2$,
4. $a_i, b_i \in (0, 1)$, $i = 1, 2, 3$,
5. equation $(*)$ arose recently from a network model.

El-Sayed El-Hady

2. Problem. Assume that the restriction of a continuous function $f \colon \mathbb{R}^2 \to \mathbb{R}$ to any straight line is an exponential polynomial. Is it true that f is an exponential polynomial in two variables?

The question is a simple version of a problem posed by J.M. Almira [1, Open Problem 2].

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Ekaterina Shulman

3. Remark. During the 16$^{\text{th}}$ International Conference on Functional Equations and Inequalities a talk was given concerning the stability of the equation appearing in the title (see [2]). My question about the general solution of the equation itself was answered later by Janusz Brzdęk; see [1, p.196].

Contrary to some assertions in the literature the general solution is not an arbitrary quadratic function, but of the form $x \to a(x^2)$ with additive a.

Let $n \in \mathbb{N}$ be ≥ 2. Replacing $\sqrt{x^2 + y^2}$ by $\sqrt[n]{x^n + y^n}$ in the equation it can be shown that the general solution is given by $f(x) = a(x^n)$ with a additive. This is a generalized homogeneous polynomial of degree n. But not every homogeneous polynomial of degree n is of this form:
Let $b \colon \mathbb{R} \to \mathbb{R}$ be additive. Then g with $g(x) := b(x)^n$ is a homogenous polynomial od degree n which is not of the form $x \to a(x^n)$ with additive a unless b is continuous.

Proof. Suppose $a(x^n) = b(x)^n$ for all x. Then $a((\lambda x + \mu y)^n) = (b(\lambda x + \mu y))^n$ for all real x, y and all rational numbers λ, μ. Thus

$$\sum_{l=0}^{n} \binom{n}{l} \lambda^l \mu^{n-l} a(x^l y^{n-l}) = \sum_{l=0}^{n} \binom{n}{l} \lambda^l \mu^{n-l} b(x)^l b(y)^{n-l}.$$

With $y = 1$ and $l = 2$ we get $a(x^2) = b(x)^2 b(1)^{n-2}$. This implies that a is continuous, $a(x) = cx$ for all x and some c. But then $|b(x)| = \sqrt[n]{|c|}\,|x|$, which implies that b also continuous.

The situation is different for $\mathbb{C}$, since there are many discontinuous automorphisms of $\mathbb{C}$.

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Jens Schwaiger

3. List of Participants

1. **ADAM Marcin**, Silesian University of Technology, Gliwice, Poland,
   email: marcin.adam@polsl.pl
2. **ALI Javid**, Aligarh Muslim University, Aligarh, India,
   email: javid@amu.ac.in
3. **BAHYRYCZ Anna**, AGH University of Science and Technology, Kraków, Poland,
   email: bahyrycz@agh.edu.pl
4. **BARON Karol**, University of Silesia, Katowice, Poland,
   email: baron@us.edu.pl
5. **BRILLOUËT-BELLUOT Nicole**, Ecole Centrale de Nantes, Nantes, France,
   email: nicole.belluot@wanadoo.fr
6. **BRZDĘK Janusz**, Pedagogical University of Cracow, Kraków, Poland,
   email: jbrzdek@up.krakow.pl
7. **CHMIELIŃSKI Jacek**, Pedagogical University of Cracow, Kraków, Poland,
   email: jacek@up.krakow.pl
8. **CHUDZIAK Jacek**, University of Rzeszów, Rzeszów, Poland,
   email: chudziak@ur.edu.pl
9. **CZERNI Marek**, Pedagogical University of Cracow, Kraków, Poland,
   email: mczerni@ap.krakow.pl
10. **DADIPOUR Farzad**, Graduate University of Advanced Technology, Kerman, Iran,
    email: dadipoor@yahoo.com
11. **DERĘGOWSKA Beata**, Pedagogical University of Cracow, Kraków, Poland,
    email: bderegowska@up.krakow.pl
12. **EBANKS Bruce**, University of Louisville, Louisville, United States,
    email: ebanks1950@gmail.com
13. **EL-HADY El-Sayed**, Suez Canal University, Ismailia, Egypt,
    email: elsayed_elhady@ci.suez.edu.eg
14. **FARZADFARD Hojjat**, Shiraz Branch, Islamic Azad University, Shiraz, Iran,
    email: hojjat.farzadfard@gmail.com

15. **FECHNER Włodzimierz**, Łódź University of Technology, Łódź, Poland,
email: wlodzimierz.fechner@p.lodz.pl
16. **FECHNER Żywilla**, University of Silesia, Katowice, Poland,
email: zfechner@math.us.edu.pl
17. **FÖRG-ROB Wolfgang**, University of Innsbruck, Innsbruck, Austria,
email: wolfgang.foerg-rob@uibk.ac.at
18. **HORVÁTH László**, University of Pannonia, Veszprém,Hungary,
email: lhorvath@almos.uni-pannon.hu
19. **JABŁOŃSKA Eliza**, Rzeszów University of Technology, Rzeszów, Poland,
email: elizapie@prz.edu.pl
20. **KISS Tibor**, University of Debrecen, Debrecen, Hungary,
email: kiss.tibor@science.unideb.hu
21. **KOMINEK Zygfryd**, University of Silesia, Katowice, Poland,
email: zkominek@math.us.edu.pl
22. **LEŚNIAK Zbigniew**, Pedagogical University of Cracow, Kraków, Poland,
email: zlesniak@up.krakow.pl
23. **MAKAGON Andrzej**, Hampton University, Hampton, United States,
email: andrzej.makagon@hamptonu.edu
24. **MAKSA Gyula**, University of Debrecen, Debrecen, Hungary,
email: maksa@science.unideb.hu
25. **MALEJKI Renata**, Pedagogical University of Cracow, Kraków, Poland,
email: smalejki@o2.pl
26. **NIKODEM Kazimierz**, University of Bielsko-Biała, Bielsko-Biała, Poland,
email: knikodem@ath.bielsko.pl
27. **OLKO Jolanta**, AGH University of Science and Technology, Kraków, Poland,
email: olko@agh.edu.pl
28. **ONITSUKA Masakazu**, Okayama University of Science, Okayama, Japan,
email: onitsuka@xmath.ous.ac.jp
29. **OSTASZEWSKI Adam**, London School of Economics, London, United Kingdom,
email: a.j.ostaszewski@lse.ac.uk
30. **OUBBI Lahbib**, Mohammed V University in Rabat, Rabat, Morocco,
email: oubbi@daad-alumni.de
31. **PÁLES Zsolt**, University of Debrecen, Debrecen, Hungary,
email: pales@science.unideb.hu
32. **PASTECZKA Paweł**, Pedagogical University of Cracow, Kraków, Poland,
email: ppasteczka@up.krakow.pl
33. **POPA Dorian**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
email: Popa.Dorian@math.utcluj.ro
34. **RAJBA Teresa**, University of Bielsko-Biała, Bielsko-Biała, Poland,
email: trajba@ath.bielsko.pl
35. **RAŞA Ioan**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
email: Ioan.Rasa@math.utcluj.ro
36. **SABLIK Maciej**, University of Silesia, Katowice, Poland,
email: maciej.sablik@us.edu.pl
37. **SADEGHI BOOME Farzane**, Shahid Bahonar University of Kerman, Kerman, Iran,
email: sadeghi.farzane@gmail.com
38. **SCHWAIGER Jens**, University of Graz, Graz, Austria,
email: jens.schwaiger@uni-graz.at

39. **SERPA Cristina**, Centro de Matemática Aplicações Fundamentais e Investigação Operacional, CMAF-CIO, Lisbon, Portugal,
email: cristinaserpa@hotmail.com
40. **SHULMAN Ekaterina**, University of Silesia, Katowice, Poland,
email: shulmanka@gmail.com
41. **SOLARZ Paweł**, Pedagogical University of Cracow, Kraków, Poland,
email: psolarz@up.krakow.pl
42. **STADLER Peter**, University of Innsbruck, Innsbruck, Austria,
email: peter.stadler@student.uibk.ac.at
43. **STETKAER Henrik**, Aarhus University, Aarhus, Denmark,
email: stetkaer@imf.au.dk
44. **SUDZIK Mariusz**, University of Zielona Góra, Zielona Góra, Poland,
email: m.sudzik@wmie.uz.zgora.pl
45. **SZÉKELYHIDI László**, University of Debrecen, Debrecen, Hungary,
email: lszekelyhidi@gmail.com
46. **TABOR Józef**, University of Rzeszów, Rzeszów, Poland,
email: tabor@univ.rzeszow.pl
47. **VOLKMANN Peter**, KIT, Karlsruhe, Germany,
email: ju-volkmann@t-online.de
48. **WITKOWSKI Alfred**, UTP University of Science and Technology, Bydgoszcz, Poland,
email: alfred.witkowski@utp.edu.pl

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