FOLIA 84

Annales Universitatis Paedagogicae Cracoviensis

Studia Mathematica IX (2010)

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# Unstable (Stable) system of stable (unstable) functional equations

**Abstract.** In this note the answer to a question by Z. Moszner from the paper [1] about connections of stability of separate equations and the system of them, is given.

Let $\mathcal{K}$ denote the class of real functions $\mathcal{K} = \{f: \mathbb{R} \to \mathbb{R}\}$. In this note, the following functional equation (or the system of equations) is considered

$$L(f) = R(f) \quad (\text{or } L_1(f) = R_1(f),\ L_2(f) = R_2(f)), \tag{1}$$

where $f \in \mathcal{K}$, $L, L_1, L_2, R, R_1, R_2: \mathcal{K} \to \mathcal{K}$.

Definition

We say that the equation (or the system of equations) (1) is *stable in the class* $\mathcal{K}$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function $g \in \mathcal{K}$ satisfying

$$|L(g(x)) - R(g(x))| \leq \delta$$
$$(\text{or } |L_1(g(x)) - R_1(g(x))| \leq \delta,\ |L_2(g(x)) - R_2(g(x))| \leq \delta)$$

for $x \in \mathbb{R}$, there exists a solution $f \in \mathcal{K}$ of equation (or the system of equations) (1) such that

$$|f(x) - g(x)| \leq \varepsilon \qquad \text{for } x \in \mathbb{R}.$$

It is observed in [1] that a system of two stable functional equations may be unstable, e.g. the system

$$|f(x)| = f(x) \quad \text{and} \quad |f(x)| = -f(x)$$

in the class $\{f: \mathbb{R} \to \mathbb{R} \setminus \{0\}\}$. Similarly, a system of functional equations may be stable even though the equations of this system are unstable, e.g. the system

$$f(f(x)) = x \quad \text{and} \quad f(f(x)) = 1$$

in the class $\mathcal{K}$. These systems have no solutions.

AMS (2000) Subject Classification: 39B82.

Volumes I-VII appeared as *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica*.

PROBLEM
*Is this situation possible only if the system has no solutions?*

This is Problem 8 in [1]. The answer to this problem is negative.

We consider the following system of functional equations in $\mathcal{K}$:

$$(|f(x) - 1| - f(x) + 1) \cdot f(x) = 0 \tag{2}$$

and

$$E(f(x)) = 0, \tag{3}$$

here $E(u)$ denotes the integer part of $u$.

PROPOSITION 1
*The equations* (2) *and* (3) *– considered in the class* $\mathcal{K}$ *– are stable separately, but not as a system. However there is an obvious solution* $f_0 \equiv 0$.

*Proof.* One can observe easily that the sets of solutions of the equations (2) and (3) – in the class $\mathcal{K}$ – are as follows:

$$\mathcal{S}ol(2) = \{f \in \mathcal{K} : \ f(\mathbb{R}) \subset [1, +\infty) \cup \{0\}\}$$

and

$$\mathcal{S}ol(3) = \{f \in \mathcal{K} : \ f(\mathbb{R}) \subset [0, 1)\}.$$

Therefore

$$\mathcal{S}ol(2) \cap \mathcal{S}ol(3) = \{f_0 \equiv 0\}.$$

To prove the stability of equation (2), take $\varepsilon > 0$ and put $\delta := \min\{\frac{3}{2}, \frac{\varepsilon}{2}\}$. Set

$$L(u) = (|u - 1| - u + 1) \cdot u \qquad \text{for } u \in \mathbb{R}.$$

Let $g \in \mathcal{K}$ and

$$|L(g(x))| \leq \delta \qquad \text{for } x \in \mathbb{R}.$$

Since

$$g(x) < -\frac{1}{2} \Longrightarrow |L(g(x))| = 2(1 - g(x)) \cdot |g(x)| > 1 - g(x) > \frac{3}{2},$$

we conclude that

$$g(\mathbb{R}) \subset \Big[-\frac{1}{2}, +\infty\Big).$$

We also have

$$\Big(g(x) \in \Big[-\frac{1}{2}, \frac{1}{2}\Big) \wedge |L(g(x))| = 2(1 - g(x)) \cdot |g(x)| \leq \delta\Big) \Longrightarrow |g(x)| \leq \delta \leq \frac{\varepsilon}{2} \tag{4}$$

since $2(1 - g(x)) > 1$ in this case. Moreover,

$$\Big(g(x) \in \Big[\frac{1}{2}, 1\Big) \wedge L(g(x)) = 2(1 - g(x)) \cdot g(x) \leq \delta\Big) \Longrightarrow 1 - g(x) \leq \delta \leq \frac{\varepsilon}{2} \tag{5}$$

since $2g(x) \geq 1$ in this case.

Define

$$f(x) := \begin{cases} 0, & \text{if } -\frac{1}{2} \le g(x) < \frac{1}{2}, \\ 1+\delta, & \text{if } \frac{1}{2} \le g(x) < 1, \\ g(x), & \text{if } g(x) \ge 1. \end{cases}$$

Evidently $f \in \mathcal{S}ol(2)$. Moreover,

$$|f(x) - g(x)| = \begin{cases} |g(x)|, & \text{if } -\frac{1}{2} \le g(x) < \frac{1}{2}, \\ 1 - g(x) + \delta, & \text{if } \frac{1}{2} \le g(x) < 1, \\ 0, & \text{if } g(x) \ge 1. \end{cases}$$

By (4) and (5) we have

$$|f(x) - g(x)| \le \varepsilon \qquad \text{for } x \in \mathbb{R}.$$

The proof of stability of equation (2) is finished.

To prove the stability of equation (3), take $\varepsilon > 0$ and put $\delta := \min\{\frac{1}{2}, \varepsilon\}$. Let $g \in \mathcal{K}$ and

$$|E(g(x))| \le \delta \qquad \text{for } x \in \mathbb{R}.$$

We have

$$g(x) \in [0,1) \qquad \text{for } x \in \mathbb{R},$$

so $g$ is a solution of equation (3). This ends the proof of stability of equation (3).

To prove that the system of equations (2) and (3) is not stable, take $g_n(x) = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We have

$$E(g_n(x)) = 0 \quad \text{and} \quad L(g_n(x)) = 2 \cdot \frac{n-1}{n^2} \qquad \text{for } n \in \mathbb{N},\ x \in \mathbb{R}.$$

Let $\varepsilon := \frac{1}{2}$. For every $\delta > 0$ there exists an integer $n_0 \ge 3$ such that $L(g_{n_0}(x)) \le \delta$ and obviously $g_{n_0}(x) > \varepsilon$ for $x \in \mathbb{R}$.

Now, let us consider the following functional equations

$$(|f(x) - 1| - f(x) + 1) \cdot |f(x)| + |E(f(x))| = 0 \tag{6}$$

and

$$(|f(x) + 1| + f(x) + 1) \cdot |f(x)| + |E(-f(x))| = 0 \tag{7}$$

for $f \in \mathcal{K}$.

Proposition 2

*The equations (6) and (7) are unstable separately and the system of these equations is stable. The function $f_0 \equiv 0$ is only solution of (6) and of (7) thus of the system of equations (6), (7).*

*Proof.* To prove the non-stability of equation (6) it is sufficient to consider the functions $g_n(x) = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Similarly, to prove the non-stability of equation (7) we can consider $g_n(x) = -1 + \frac{1}{n}$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. To prove the stability of the system of these equations, take $\varepsilon > 0$ and put $\delta := \frac{1}{2}$. Set

$$L_1(u) = (|u-1| - u + 1) \cdot |u| + |E(u)| \qquad \text{for } u \in \mathbb{R}$$

and

$$L_2(u) = (|u+1| + u + 1) \cdot |u| + |E(-u)| \qquad \text{for } u \in \mathbb{R}.$$

Let $g \in \mathcal{K}$ and

$$|L_1(g(x))| \le \delta \quad \text{and} \quad |L_2(g(x))| \le \delta \qquad \text{for } x \in \mathbb{R}.$$

We have

$$(g(x) < 0 \text{ or } g(x) \ge 1) \Longrightarrow L_1(g(x)) \ge |E(g(x))| \ge 1.$$

Moreover,

$$0 < g(x) < 1 \Longrightarrow L_2(g(x)) \ge |E(-g(x))| = 1.$$

From the above $g(x) = f_0(x) \equiv 0$. This ends the proof of stability of the system of equations (6), (7).

Remark 1

We say that equation (or the system of equations) (1) is *superstable in the class* $\mathcal{K}$ (*b-stable in the class* $\mathcal{K}$, respectively), if for every function $g \in \mathcal{K}$ for which $L(g(x)) - R(g(x))$ (or $L_1(g(x)) - R_1(g(x))$ and $L_2(g(x)) - R_2(g(x))$) is bounded, $g$ is bounded or $g$ is a solution of (1) (there exists a solution $f \in \mathcal{K}$ of (1) such that $g(x) - f(x)$ is bounded, respectively).

It is easy to see that the superstability of the equations of the system implies superstability of the system. The converse implication is not true. Indeed, the boundedness of the functions $g(2x)-g(x)$ and $g(2x)+g(x)$ implies the boundedness of the function $g$, so the system

$$f(2x) = f(x) \quad \text{and} \quad f(2x) = -f(x) \tag{8}$$

is superstable. The function

$$g(x) := \begin{cases} k, & \text{if } x \in A := \{x \in \mathbb{R} : \ x = 2^k, \ k \in \mathbb{Z}\}, \\ 0, & \text{if } x \in \mathbb{R} \setminus A, \end{cases}$$

is unbounded and $g$ does not satisfy the equation $f(2x) = f(x)$, but the function $g(2x) - g(x)$ is bounded. Similarly, the function

$$g(x) := \begin{cases} (-1)^k \cdot k, & \text{if } x \in A, \\ 0, & \text{if } x \in \mathbb{R} \setminus A, \end{cases}$$

is unbounded and $g$ is not a solution of the equation $f(2x) = -f(x)$, but the function $g(2x)+g(x)$ is bounded. Therefore the equations in (8) are not superstable separately.

We encounter the same situation for b-stability. Namely, the system (8) is b-stable and the equations of this system are not b-stable. If we consider the system

$$L_1(f(x)) = [f(x)-x]\cdot f(x) = 0 \quad \text{and} \quad L_2(f(x)) = \left[f(x)-\left(x+\frac{1}{x^2+1}\right)\right]\cdot f(x) = 0,$$

then one can observe that this system is not b-stable. Indeed, for $g(x) = x$ the functions $L_1(g(x))$ and $L_2(g(x))$ are bounded and $f(x) = 0$ is the unique solution of the system. The separate equations of this system are b-stable, e.g. for $|(g(x) - x) \cdot g(x)| \leq M$ we have $|g(x) - x| \leq \sqrt{M}$ or $|g(x)| \leq \sqrt{M}$. The function

$$f(x) := \begin{cases} x, & \text{if } |g(x) - x| \leq \sqrt{M}, \\ 0, & \text{if } |g(x) - x| > \sqrt{M} \text{ and } |g(x)| \leq \sqrt{M}, \end{cases}$$

is a solution of the equation $L_1(f(x)) = 0$ and $g(x) - f(x)$ is bounded.

Remark 2
If at least one of the equations of the system is stable (or superstable, or b-stable, respectively) and every solution of this equation is a solution of the second equation of the system, then the system is stable (or superstable, or b-stable, respectively).

Remark 3
The stability of the system $L_1(f) = R_1(f)$ and $L_2(f) = R_2(f)$ is equivalent to the stability of the equation

$$|L_1(f) - R_1(f)| + |L_2(f) - R_2(f)| = 0.$$

## References

[1] Z. Moszner, *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.

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*Received: 25 September 2009; final version: 8 February 2010;*
*available online: 12 March 2010.*