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On a generalization of the Popoviciu equation on groups

Abstract. We determine a general solution of the Popoviciu type functional equation on groups.

1. Introduction

In 1965 T. Popoviciu [5], dealing with some inequality for convex functions, has introduced the functional equation

$$3f\Big(\frac{x+y+z}{3}\Big)+f(x)+f(y)+f(z)=2\Big[f\Big(\frac{x+y}{2}\Big)+f\Big(\frac{x+z}{2}\Big)+f\Big(\frac{y+z}{2}\Big)\Big]. \quad (1)$$

The solution and stability of (1) have been studied by W. Smajdor [6] and T. Trif [7]. Recently, J. Brzdęk [1] has considered stability of (1) on a restricted domain. Solution and stability of the following "quadratic" version of (1),

$$9f\Big(\frac{x+y+z}{3}\Big)+f(x)+f(y)+f(z)=4\Big[f\Big(\frac{x+y}{2}\Big)+f\Big(\frac{x+z}{2}\Big)+f\Big(\frac{y+z}{2}\Big)\Big] \quad (2)$$

have been investigated by Y.W. Lee [3]. The results from [3] have been generalized by the same author in [4], where the functional equation

$$\begin{aligned} m^2 f\Big(\frac{x+y+z}{m}\Big)&+f(x)+f(y)+f(z)\\ &=n^2\Big[f\Big(\frac{x+y}{n}\Big)+f\Big(\frac{x+z}{n}\Big)+f\Big(\frac{y+z}{n}\Big)\Big] \end{aligned} \quad (3)$$

has been considered (m, n are nonzero integers such that $m+1=2n$). The case $m=n=1$ has been studied by P. Kannappan [2]. For some further generalization of (1) we refer to [8]. It is remarkable that the results mentioned above (except for [1] and [6]) concern the case, where unknown function f is acting between two real linear spaces. In the present paper we deal with the functional equation

$$Mf\Big(\frac{x+y+z}{m}\Big)+f(x)+f(y)+f(z)=N\Big[f\Big(\frac{x+y}{n}\Big)+f\Big(\frac{x+z}{n}\Big)+f\Big(\frac{y+z}{n}\Big)\Big] \quad (4)$$

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in a more general setting. Namely, we assume that m, n, M, N are positive integers, $(G, +)$ is a commutative group uniquely divisible by m and n, $(H, +)$ is a commutative group uniquely divisible by 2 and $f\colon G \to H$ is an unknown function. Let us recall that a group $(X, +)$ is said to be *uniquely divisible* by a given positive integer k provided, for every $x \in X$, there exists a unique $y \in X$ such that $x = ky$; such an element will be denoted in a sequel by $\frac{x}{k}$. Furthermore, given arbitrary groups $(X, +)$ and $(Y, +)$, a function $Q\colon X \to Y$ is said to be *quadratic* provided

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \qquad \text{for } x, y \in X$$

and a function $A\colon X \to Y$ is said to be *additive* provided

$$A(x+y) = A(x) + A(y) \qquad \text{for } x, y \in X.$$

2. Results

We begin this section with the following theorem, which is a main result of the paper.

THEOREM 1
Let m, n, M, N be positive integers, $(G, +)$ be a commutative group uniquely divisible by m and n, and $(H, +)$ be a commutative group uniquely divisible by 2. Then a function $f\colon G \to H$ satisfies equation (4) for all $x, y, z \in G$ if and only if there exist a quadratic function $Q\colon G \to H$, an additive function $A\colon G \to H$ and a $B \in H$ such that

$$(M - 3N + 3)B = 0, \tag{5}$$

$$(N - n^2)Q(x) = (M - m^2)Q(x) = 0 \qquad \textit{for } x \in G, \tag{6}$$

$$(Mn + mn - 2mN)A(x) = 0 \qquad \textit{for } x \in G \tag{7}$$

and

$$f(x) = Q(x) + A(x) + B \qquad \textit{for } x \in G. \tag{8}$$

Proof. Assume that f satisfies (4). Then, applying (4) with $x = y = z = 0$, we get

$$(M + 3 - 3N)f(0) = 0. \tag{9}$$

Define the functions $Q\colon G \to H$ and $A\colon G \to H$ by

$$Q(x) := \frac{f(x) + f(-x)}{2} - f(0) \qquad \text{for } x \in G$$

and

$$A(x) := \frac{f(x) - f(-x)}{2} \qquad \text{for } x \in G,$$

respectively. Furthermore, let $B := f(0)$. Then it is clear that $A(0) = Q(0) = 0$,

Q is an even function, A is odd and f is of the form (8). Furthermore, in view of (9), (5) is valid. Note also, that by (4), for every $x, y, z \in G$, we get

$$Mf\Big(\frac{-(x+y+z)}{m}\Big) + f(-x) + f(-y) + f(-z) \\ = N\Big[f\Big(\frac{-(x+y)}{n}\Big) + f\Big(\frac{-(x+z)}{n}\Big) + f\Big(\frac{-(y+z)}{n}\Big)\Big].$$

Therefore, taking into account (9), we obtain that Q and A satisfy (4) for every $x, y, z \in G$. Now, we show that Q is a quadratic function. Since Q is even and satisfies (4), for every $x, y \in G$, we have

$$MQ\Big(\frac{y}{m}\Big) + 2Q(x) + Q(y) = MQ\Big(\frac{x+y-x}{m}\Big) + Q(x) + Q(y) + Q(-x) \\ = N\Big[Q\Big(\frac{x+y}{n}\Big) + Q\Big(\frac{x-y}{n}\Big)\Big].$$

Thus

$$MQ\Big(\frac{y}{m}\Big) + 2Q(x) + Q(y) = N\Big[Q\Big(\frac{x+y}{n}\Big) + Q\Big(\frac{x-y}{n}\Big)\Big] \tag{10}$$

for $x, y \in G$. Taking in (10) $y = 0$, we get $2Q(x) = 2NQ(\frac{x}{n})$ for $x \in G$ whence, as H is uniquely divisible by 2, we have

$$Q(x) = NQ\Big(\frac{x}{n}\Big) \qquad \text{for } x \in G. \tag{11}$$

Moreover, putting in (10) $x = 0$, we obtain

$$MQ\Big(\frac{y}{m}\Big) + Q(y) = 2NQ\Big(\frac{y}{n}\Big) \qquad \text{for } y \in G$$

which, together with (11), gives

$$Q(y) = MQ\Big(\frac{y}{m}\Big) \qquad \text{for } y \in G. \tag{12}$$

Now, from (10)–(12) we deduce that Q is quadratic. Furthermore note that, as Q is quadratic, from (11) and (12) it follows (6).

Next, we consider the function A. As we have already noted, A is odd, vanishes at 0 and satisfies (4), that is, for every $x, y, z \in G$, it holds

$$MA\Big(\frac{x+y+z}{m}\Big) + A(x) + A(y) + A(z) \\ = N\Big[A\Big(\frac{x+y}{n}\Big) + A\Big(\frac{x+z}{n}\Big) + A\Big(\frac{y+z}{n}\Big)\Big]. \tag{13}$$

Applying (13) with $z = 0$, and then with $y = z = 0$, we get

$$MA\Big(\frac{x+y}{m}\Big) + A(x) + A(y) = N\Big[A\Big(\frac{x+y}{n}\Big) + A\Big(\frac{x}{n}\Big) + A\Big(\frac{y}{n}\Big)\Big] \tag{14}$$

for $x, y \in G$ and

$$MA\Big(\frac{x}{m}\Big) + A(x) = 2NA\Big(\frac{x}{n}\Big) \qquad \text{for } x \in G, \tag{15}$$

respectively. By (15), for every $x, y \in G$, we get

$$MA\Big(\frac{x+y}{m}\Big) + A(x+y) = 2NA\Big(\frac{x+y}{n}\Big).$$

Thus, in view of (14), we get

$$A(x+y) - A(x) - A(y) = N\Big[A\Big(\frac{x+y}{n}\Big) - A\Big(\frac{x}{n}\Big) - A\Big(\frac{y}{n}\Big)\Big].$$

On the other hand, using the oddness of A and applying (13), for $x, y \in G$, we obtain

$$\begin{aligned}
&N\Big[A\Big(\frac{x+y}{n}\Big) - A\Big(\frac{x}{n}\Big) - A\Big(\frac{y}{n}\Big)\Big]\\
&\quad = N\Big[A\Big(\frac{x+y}{n}\Big) + A\Big(\frac{y-(x+y)}{n}\Big) + A\Big(\frac{x-(x+y)}{n}\Big)\Big]\\
&\quad = MA\Big(\frac{x+y-(x+y)}{m}\Big) + A(x) + A(y) - A(x+y).
\end{aligned}$$

Consequently,

$$A(x+y) - A(x) - A(y) = A(x) + A(y) - A(x+y) \qquad \text{for } x, y \in G,$$

which means that $2A$ is an additive function. Since H is uniquely divisible by 2, we conclude that A is additive. Finally note that as A is additive, (15) implies (7).

Since the converse is easy to check, the proof is completed.

The next two corollaries generalize to some extend Theorem 2.1 in [7] and Theorem 2.1 in [4], respectively.

Corollary 1

Let m, n be positive integers, $(G, +)$ be a commutative group uniquely divisible by m and n, and $(H, +)$ be a commutative group uniquely divisible by 2. Then a function $f\colon G \to H$ satisfies equation

$$\begin{aligned}
&mf\Big(\frac{x+y+z}{m}\Big) + f(x) + f(y) + f(z)\\
&\quad = n\Big[f\Big(\frac{x+y}{n}\Big) + f\Big(\frac{x+z}{n}\Big) + f\Big(\frac{y+z}{n}\Big)\Big] \qquad \text{for } x, y, z \in G
\end{aligned}$$

if and only if there exist a quadratic function $Q\colon G \to H$, an additive function $A\colon G \to H$ and a $B \in H$ such that $(m - 3n + 3)B = 0$, $Q = 0$ whenever $m \neq 1$ or $n \neq 1$; and f is of the form (8).

Corollary 2

Let m, n be positive integers, $(G, +)$ be a commutative group uniquely divisible by m and n, and $(H, +)$ be a commutative group uniquely divisible by 2. Then a function $f\colon G \to H$ satisfies equation (3) for all $x, y, z \in G$ if and only if there exist a quadratic function $Q\colon G \to H$, an additive function $A\colon G \to H$ and a $B \in H$ such that $(m^2 - 3n^2 + 3)B = 0$, $A = 0$ whenever $m + 1 \neq 2n$; and f is of the form (8).

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