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On asymptotic cyclic contractions

Abstract. We introduce asymptotic cyclic contractions as a generalization of cyclic contractions. The new type of mappings is considered under the recently introduced property UC for pairs of subsets of metric spaces. We show that there may be more than one best proximity point.

1. Introduction

Let $\varphi_n\colon [0,\infty) \to [0,\infty)$ be a sequence of functions uniformly convergent to a continuous function $\varphi\colon [0,\infty) \to [0,\infty)$ for which $\varphi(r) < r$ for $r > 0$. In 2003 Kirk [6] pointed out the following generalization of the well-known Banach's Contraction Principle and the Boyd–Wong fixed point theorem:

Theorem 1.1 ([6], Theorem 2.1)
Suppose (M,d) is a complete metric space and suppose $T\colon M \to M$ is an asymptotic contraction, i.e.,

$$d(T^n(x), T^n(y)) \leq \varphi_n(d(x,y)), \qquad x, y \in M$$

for which the mappings φ_n are also continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $z \in M$, and moreover the Picard sequence $(T^n(x))_{n=1}^{\infty}$ converges to z for each $x \in M$.

Almost simultaneously a similar problem was considered by Jachymski and Jóźwik in [5]. More precisely, they proved that if T is a uniformly continuous mapping, the assumptions on φ_n may be weakened, i.e., functions φ_n need not be continuous and it suffices to assume the limit function φ to be upper semicontinuous and such that $\lim_{r\to\infty}(r - \varphi(r)) = \infty$. Moreover, it is not necessary to suppose the boundedness of some orbit $(T^n(x))$.

More recently, these results were generalized by Suzuki. In [8] Suzuki introduced the so-called asymptotic contractions of Meir–Keeler type. Specifically, Theorem 4 in [8] gives us the following corollary devoted to the Kirk's contractions:

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THEOREM 1.2
Let (X, d) be a complete metric space and let T be a continuous, asymptotic contraction on X. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

At the same time, the Banach's Contraction Principle was generalized for the case of so-called cyclic contractions.

DEFINITION 1.3
Let A and B be nonempty subsets of a metric space X and suppose that a mapping $T: A \cup B \to A \cup B$ is such that $T(A) \subset B$, $T(B) \subset A$ and there exists $k \in (0, 1)$ for which

$$d(T(x), T(y)) \leq (1-k)d(x, y) + kd(A, B) \qquad \text{for all } x \in A,\ y \in B.$$

Then T is called *a cyclic contraction.*

In [9] it has been shown that there exists a unique best proximity point for these kinds of mappings, i.e., a point $z \in A$ for which $d(z, T(z)) = d(A, B)$, under the assumption that the pair (A, B) satisfies property UC. The precise definition may be found in Section 2. Moreover, the sequence of Picard iterations $(T^{2n}(x))$ has been proved to be convergent to z for all $x \in A$. Later on, further developments were considered on this topic by weakening assumptions on T (see [1] and [9]) or on sets A and B (see [2] and [4]). Our main goal in this work is to give an answer whether for asymptotic cyclic contractions of Kirk's type, mappings introduced below for the first time, there is at least one best proximity point $z \in A$ such that $T^{2n}(x) \to z$ for all $x \in A$. We will suppose that the pair (A, B) satisfies the same property UC as in [9].

2. Preliminaries

Let us begin with some notations. Let A and B be two subsets of a metric space (X, d). By $d(a, B)$ and $d(A, B)$ we denote:

$$d(a, B) = \inf\{d(a, b) |\ b \in B\};$$
$$d(A, B) = \inf\{d(a, B) |\ a \in A\}.$$

Now we proceed to some definitions which we will need in the sequel. The first one gives us a precise definition of an asymptotic cyclic contraction.

DEFINITION 2.1
Let A and B be nonempty subsets of a metric space X and suppose that a mapping $T: A \cup B \to A \cup B$ is such that $T(A) \subset B$, $T(B) \subset A$. Moreover, assume that there exists a sequence $(\varphi_n)_{n=1}^{\infty}$ of functions defined on $[d(A, B), \infty)$ and satisfying the following properties:

(i) (φ_n) tends uniformly to an upper semicontinuous function $\varphi: [d(A, B), \infty) \to [d(A, B), \infty)$;

(ii) $\varphi(r) < r$ for each $r > d(A, B)$;

(iii) $d(T^n(x), T^n(y)) \le \varphi_n(d(x,y))$ for all $x \in A$, $y \in B$ and $n \in \mathbb{N}$.

Then T is called *an asymptotic cyclic contraction.*

Next we define the property UC.

DEFINITION 2.2
Let (X,d) be a metric space and let A and B be nonempty subsets of X. A pair (A,B) is said to satisfy *the property UC*, if for each pair $((x_n)_{n=1}^{\infty}, (x'_n)_{n=1}^{\infty})$ of sequences of points of A and a sequence $(y_n)_{n=1}^{\infty}$ of B such that $d(x_n, y_n) \to d(A,B)$ and $d(x'_n, y_n) \to d(A,B)$, the convergence $d(x_n, x'_n) \to 0$ holds.

A natural question here is to find conditions under which a pair of subsets of a Banach or a metric space satisfies the property UC. In the case of a Banach space X and assuming that A is convex, it was proved in [9] that any such pair (A,B) has the property UC if X is, in addition, uniformly convex. If moreover A is supposed to be relatively compact, it suffices that X is uniformly convex in each direction. Now let us consider geodesic metric spaces. In this case (A,B) satisfies the property UC if A is convex and X is a uniformly convex metric space with monotone modulus of convexity (see [2]). For precise definitions the reader may check [3].

3. Main results

Now we are able to proceed to the main result of the paper. More precisely, we will present an existence result of the best proximity point for an asymptotic cyclic contraction $T\colon A\cup B \to A \cup B$. We will also provide some examples showing the significance of the assumptions in our result are needed.

THEOREM 3.1
Let A and B be nonempty subsets of a metric space X such that A is complete and the pair (A,B) satisfies the property UC. Moreover, suppose that $T\colon A\cup B \to A\cup B$ is an asymptotic cyclic contraction that is continuous. Then there is a best proximity point $z \in A$ and $T^{2n}(x) \to z$ for all $x \in A$.

Proof. Let us fix $x \in A$ and $y \in B$. We want to show that

$$d(T^n(x), T^n(y)) \to d(A,B). \tag{3.1}$$

First let us notice that the sequence $(d(T^n(x), T^n(y)))$ is bounded. Indeed, the uniformly convergence of (φ_n) and properties of T imply that

$$d(T^n(x), T^n(y)) \le \varphi_n(d(x,y)) \le \varphi(d(x,y)) + 1$$

for almost all $n \in \mathbb{N}$.

Now let (k_n) be an increasing subsequence of natural numbers chosen in such a way that

$$\lim_{n\to\infty} d(T^{k_n}(x), T^{k_n}(y)) = r > \varphi(r) \ge d(A,B).$$

Since (φ_n) tends uniformly to the upper semicontinuous function φ, one may find $N_1 \in \mathbb{N}$ for which

$$\varphi(d(T^{N_1}(x), T^{N_1}(y))) < \frac{2\varphi(r) + r}{3}$$

and $N_2 \in \mathbb{N}$ such that for $n > N_2$ we obtain

$$d(T^{n+N_1}(x), T^{n+N_1}(y)) \leq \varphi_n(d(T^{N_1}(x), T^{N_1}(y))) < \frac{\varphi(r) + 2r}{3},$$

which leads to

$$\lim_{n\to\infty} d(T^{k_n}(x), T^{k_n}(y)) \leq \frac{\varphi(r) + 2r}{3},$$

a contradiction.

On account of (3.1) and properties of sets A and B, we have

$$d(T^{2n}(x), T^{2n+2}(x)) \to 0. \tag{3.2}$$

Now let $\varepsilon > 0$ and suppose that there are sequences (k_n) and (m_n) for which

$$d(T^{2k_n}(x), T^{2m_n+1}(x)) \geq d(A, B) + \varepsilon. \tag{3.3}$$

According to (3.1)–(3.2) we may assume that m_n is chosen in such a way that

$$d(T^{2k_n}(x), T^{2p+1}(x)) < d(A, B) + \varepsilon, \qquad p \in \{k_n, \ldots, m_n - 1\}. \tag{3.4}$$

Therefore

$$\lim_{n\to\infty} d(T^{2k_n}(x), T^{2m_n+1}(x)) = d(A, B) + \varepsilon \tag{3.5}$$

and moreover,

$$\lim_{n\to\infty} (m_n - k_n) \to \infty. \tag{3.6}$$

Indeed, the boundedness of such a numerical sequence together with (3.1) contradicts (3.5).

In next step we will show that (3.3) leads to a contradiction. This part of the proof was inspired by the one given by Suzuki in [8, Theorem 3]. From (i)–(ii) it follows that there is $\delta > 0$ such that

$$\varphi(r) < \frac{\varphi(d(A, B) + \varepsilon) + d(A, B) + \varepsilon}{2} < d(A, B) + \varepsilon$$

for each $r \in (d(A, B) + \varepsilon - \delta, d(A, B) + \varepsilon)$, which guarantees that

$$\varphi(r) < \max\left\{\frac{\varphi(d(A, B) + \varepsilon) + d(A, B) + \varepsilon}{2}, d(A, B) + \varepsilon - \delta\right\} =: M$$

for $r < d(A, B) + \varepsilon$. On account of (i) one can find $N \in \mathbb{N}$ such that

$$\varphi_{2N}(r) < \frac{M + d(A, B) + \varepsilon}{2} \qquad \text{for } r < d(A, B) + \varepsilon.$$

Combining this with (3.4)–(3.6) and (iii) we get

$$\begin{aligned}
&d(T^{2k_n}(x), T^{2m_n+1}(x)) \\
&\quad \le d(T^{2k_n+2N}(x), T^{2m_n+1}(x)) + \sum_{k=0}^{N-1} d(T^{2k_n+2k}(x), T^{2k_n+2k+2}(x)) \\
&\quad \le \varphi_{2N}(d(T^{2k_n}(x), T^{2m_n-2N+1}(x))) + \sum_{k=0}^{N-1} d(T^{2k_n+2k}(x), T^{2k_n+2k+2}(x)) \\
&\quad < \frac{M + d(A,B) + \varepsilon}{2} + \sum_{k=0}^{N-1} d(T^{2k_n+2k}(x), T^{2k_n+2k+2}(x)) \\
&\quad < d(A,B) + \varepsilon
\end{aligned}$$

for n large enough, which contradicts (3.3).

Thus Lemma 2 in [9] implies the fact that $(T^{2n}(x))$ is a Cauchy sequence and since A is complete, $T^{2n}(x) \to z$. Moreover, since T is continuous we get that z is a best proximity point, i.e., $d(z, T(z)) = d(A, B)$.

To finish the proof we have to get $d(T^{2n}(x_1), T^{2n}(x_2)) \to 0$ for all $x_1, x_2 \in A$, but this follows directly from (3.1) and property UC of the pair of sets (A, B).

Remark 3.2

1. If we compare the above result with the ones given by Suzuki in [8, Theorem 3], it is worth to notice that it does not suffice to assume that there is at least one continuous mapping T^{2n+1}, $n \in \mathbb{N}$. To see it the reader may consider Example 3.3.

2. In contrast to the case of asymptotic contractions (see [5, Theorem 2], [6, Theorem 2.1] and [8, Theorem 3]) or cyclic contractions (compare with [9, Theorem 2]), the best proximity point z obtained in our main theorem does not have to be unique (see Example 3.4). Moreover, z is not necessary a fixed point of T^2.

As it has been mentioned before we are going to show examples supporting to the necessity of assumptions in Theorem 3.1. The first one shows that the continuity of T cannot be weakened by the continuity of some iteration T^{2n+1}.

Example 3.3

Let us define a sequence $(z_n) \subset \mathbb{R}$ as

$$z_0 = 0,\ z_1 = 2,\ z_2 = -1,\ z_3 = 1,\ z_{2n+2} = -\frac{1}{3^{2n-1}},\ z_{2n+3} = 1 + \frac{1}{3^{2n}}, \quad n \in \mathbb{N}.$$

Then defining a mapping T and sets A and B by $T(z_{n-1}) = z_n$, $n \in \mathbb{N}$ and

$$A = \{z_0, z_2, z_4, \ldots\} \quad \text{and} \quad B = \{z_1, z_3, z_5, \ldots\},$$

we obtain that $T\colon A \cup B \to A \cup B$ is an asymptotic cyclic contraction such that T^3 is a continuous mapping. Indeed, since $d(A, B) = 1$ one may take $\varphi(a) = \frac{a+2}{3}$. It is sufficient to consider pairs with one element equal to z_0 or z_3. In any other case it is easy to check that $d(T^{n+2}(u), T^{n+2}(v)) \le \varphi(d(u, v))$, $u \in A$, $v \in B$, $n \in \mathbb{N}$.

1. If $u = z_0$ and $v = z_{2n-1}$, $n \neq 2$, then

$$d(T^{m+3}(u), T^{m+3}(v)) = 1 + \frac{1}{3^m} + \frac{1}{3^{m+2n-1}} \leq \varphi(d(u,v)) + \frac{1}{3^m}, \qquad m \in \mathbb{N}.$$

2. Taking $v = z_3$ and $u = z_{2n}$, we obtain

$$d(T^m(u), T^m(v)) \leq 1 + \frac{1}{3^m} + \frac{1}{3^{m+2n-3}} \leq \varphi(d(u,v)) + \frac{1}{3^{m-1}}, \qquad m \in \mathbb{N}.$$

3. In the same time $u = z_0$ and $v = z_3$, imply

$$d(T^{m+3}(u), T^{m+3}(v)) = 1 + \frac{1}{3^m} + \frac{1}{3^{m+3}} \leq \varphi(d(u,v)) + \frac{2}{3^m}, \qquad m \in \mathbb{N}.$$

Hence $\varphi_m = \varphi + \frac{2}{3^{m-3}}$ for all $m > 3$.

Moreover, the pair (A, B) satisfies property UC and $T^{2n}(z_{2m-2})$ tends to z_0 if $n \to \infty$, but z_0 is not a best proximity point.

Example 3.4

Let us slightly modify the above example. More precisely, let us consider a sequence $(z_n) \subset \mathbb{R}^2$ and two points z' and z'' defined by

$$z_0 = (0, -1),\ z_1 = (1, -1),\ z_{2n} = \Big(-\frac{1}{3^{2n-1}}, 0\Big),\ z_{2n+1} = \Big(1 + \frac{1}{3^{2n}}, 0\Big), \quad n \in \mathbb{N};$$

$$z' = (0, 0),\ z'' = (1, 0).$$

Taking $T(z_{n-1}) = z_n$, $T(z') = z''$, $T(z'') = z'$,

$$A = \{z', z_0, z_2, \ldots\} \quad \text{and} \quad B = \{z'', z_1, z_3, \ldots\},$$

we obtain that $T\colon A \cup B \to A \cup B$ is a continuous asymptotic cyclic contraction with two best proximity points z_0 and z'. Indeed, let us take $\varphi(r) = d(A, B) = 1$ for each $r \geq d(A, B)$. Then fixing $u \in A$ and $v \in B$ we get

$$d(T^n(u), T^n(v)) \leq d(T^n(z_0), T^n(z_1)) \leq 1 + \frac{1}{3^{n-1}} + \frac{1}{3^n} = \varphi(d(u,v)) + \frac{4}{3^n}$$

for all $n \in \mathbb{N}$. It suffices to define a sequence (φ_n) as $\varphi_n(r) = \varphi(r) + 3^{2-n}$ for all $r \geq d(A, B)$ and $n \in \mathbb{N}$.

The last example is devoted to properties of the pair of sets (A, B). It is worth to notice that it is still an open question whether the property UC may be weakened. In case of cyclic contractions it was shown in [2, 4] that the property UC may be taken out by the property WUC and under some additional assumptions on the metric space by the property HW. Under the assumptions considered in [7] the reader may find examples proving the existence of cyclic Meir–Keeler contractions without best proximity points. For precise definitions the reader may check [2, 4, 7].

EXAMPLE 3.5
Let us consider the set of natural numbers $\mathbb{N}$ and the function $d\colon \mathbb{N}\times\mathbb{N}\to[0,\infty)$ defined by

$$d(i,j)=d(j,i)=\begin{cases}0, & i=j,\\ 3-\sum_{k=1}^{j-i-1}\frac{1}{2^k}, & i<j.\end{cases}$$

It is clear that $(\mathbb{N},d)$ is a complete metric space. Indeed, the fact that

$$2<d(i,k)<4=2+2<d(i,j)+d(j,k),\qquad i,j,k\in\{1,2,\ldots\},\ i\neq j\neq k$$

trivially proves the triangle inequality.

Let A be the set of all odd numbers and B the set of all even numbers, and consider the mapping $T\colon(\mathbb{N},d)\to(\mathbb{N},d)$ given by $T(i)=i+1$. Then T is a cyclic contraction on $A\cup B$ so this is also an asymptotic cyclic contraction with no best proximity points.

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