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Studia Mathematica X

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Nonlocal Robin problem in a plane domain with a boundary corner point

Abstract. We investigate the behavior of weak solutions to the nonlocal Robin problem for linear elliptic divergence second order equations in a neighborhood of the boundary corner point. We find an exponent of the solution decreasing rate under the minimal assumptions on the problem coefficients.

1. Introduction

Our article is devoted to the linear elliptic divergence second order equations with the nonlocal Robin boundary condition in a plane bounded domain with a boundary corner point. The nonlocal condition means that the values of the unknown function u on the lateral side of a domain are connected with the values of u inside a domain. This problem appears often in different fields of physics and engineering. For example, nonlocal elliptic boundary value problems have important applications to the theory of diffusion processes, in the theory of turbulence etc. Various problems in this field have been studied by many mathematicians. We refer for the history of this problem and the extensive citation to [4, 11]. Questions of the solvability to nonlocal elliptic value boundary problems were considered by Skubachevskii [11]. In the same place there were obtained a priori estimates of solutions in the Sobolev spaces: both weighted and unweighted. All results in [11] relate to equations with *infinite–differentiable* coefficients. Gurevich [4] considered asymptotics of solutions for nonlocal elliptic problems for equations with *constant* coefficients in plane angles.

The aim of our article is the type $|u(x)| = O(|x|^{\alpha})$ estimate of the weak solution modulus for our problem near an angular boundary point. A principal new feature of our work is the establishing of the weak solution decrease rate exponent under the consideration of *the minimal smoothness* required on the coefficients of the problem. Moreover, we derive global and local estimates for weighted and unweighted Dirichlet integrals applying different methods from those in [4, 11] that allows us to obtain more detailed and exact estimates of these integrals than previously known. We investigate the behavior of weak solutions for the considered

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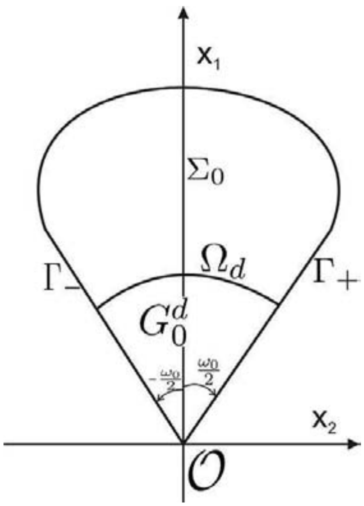
problem in a neighborhood of the boundary corner point by integro–differential inequalities and Kondratiev's ring methods developed in [1]. For this purpose we use *the Friedrichs–Wirtinger type inequality* which is adapted to the our problem. All obtained results are new and distinguishes our work from cited above.

Setting of nonlocal problem. Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G = \overline{\Gamma}_+ \cup \overline{\Gamma}_-$ being a smooth curve everywhere except at the origin $\mathcal{O} \in \partial G$, where near the point $\mathcal{O}$ curves $\Gamma_\pm$ are lateral sides of an angle with the measure $\omega_0 \in [0, 2\pi)$ and the vertex at $\mathcal{O}$. Let $\Sigma_0 = G \cap \{x_2 = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$.

We will use the following notations:

- S^1: the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$;
- (r, ω): the polar coordinates of $x = (x_1, x_2) \in \mathbb{R}^2$ with pole $\mathcal{O}$: $x_1 = r\cos\omega$, $x_2 = r\sin\omega$;
- $\mathcal{C}$: the angle $\{x_1 > r\cos\frac{\omega_0}{2};\ -\infty < x_2 < \infty\}$ with vertex $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral sides of $\mathcal{C}$: $x_1 = r\cos\frac{\omega_0}{2}$, $x_2 = \pm r\sin\frac{\omega_0}{2}$;
- Ω: an arc obtained by intersecting the angle $\mathcal{C}$ with S^1: $\Omega = \mathcal{C} \cap S^1$;
- $G_a^b = \{(r, \omega);\ 0 \le a < r < b;\ \omega \in \Omega\} \cap G$: a ring domain in $\mathbb{R}^2$;
- $\Gamma_{a\pm}^b = \{(r, \omega);\ 0 \le a < r < b;\ \omega = \pm\frac{\omega_0}{2}\} \cap \partial G$: the lateral sides of G_a^b;
- $G_d = G \setminus G_0^d$; $\Gamma_{d\pm} = \Gamma_\pm \setminus \Gamma_{0\pm}^d$, $d > 0$;
- $\Omega_\rho = G_0^d \cap \{|x| = \varrho\}$; $0 < \varrho < d$;
- $\operatorname{meas} G$: the Lebesgue measure of the set G.



We shall consider an elliptic equation with nonlocal boundary condition connecting the values of the unknown function u on the curve Γ_+ with its values of u on the Σ_0:

$$\begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + b^i(x)u_{x_i} + c(x)u = f(x), & x \in G; \\ \mathcal{B}_+[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_+ \dfrac{u(x)}{|x|} + \dfrac{b}{|x|}u(\gamma(x)) = g(x), & x \in \Gamma_+; \\ \mathcal{B}_-[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_- \dfrac{u(x)}{|x|} = h(x), & x \in \Gamma_-; \end{cases} \qquad (L)$$

here:

- $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\overrightarrow{n}, x_i)\frac{\partial}{\partial x_j}$ and $\overrightarrow{n}$ denotes the unit vector outwards with respect to G normal to $\partial G \setminus \mathcal{O}$ (summation over repeated indices from 1 to 2 is understood);
- γ is a diffeomorphism mapping of Γ_+ onto Σ_0; we assume that there exists $d > 0$ such that in the neighborhood Γ^d_{0+} of the point $\mathcal{O}$ the mapping γ is the rotation by the angle $-\frac{\omega_0}{2}$, that is $\gamma(\Gamma^d_{0+}) = \Sigma^d_0$.

Remark 1.1
We observe that

$$u(\gamma(x))|_{\Gamma^d_{0+}} = u(r, 0), \qquad 0 < r < d.$$

In fact, $\gamma(x) = \gamma(x_1, x_2) = \gamma(r\cos\frac{\omega_0}{2}, r\sin\frac{\omega_0}{2}) = (r, 0)$, because of in the neighborhood Γ^d_{0+} of the point $\mathcal{O}$ the mapping γ is the rotation by the angle $-\frac{\omega_0}{2}$.

We use also standard function spaces: $C^k(\overline{G})$ with the norm $|u|_{k,G}$, Lebesgue space $L_p(G)$, $p \geq 1$ with the norm $\|u\|_{p,G}$, the Sobolev space $W^{k,p}(G)$ with the norm $\|u\|_{p,k,(G)} = (\int_G \sum_{|\beta|=0}^k |D^\beta u|^p\,dx)^{\frac{1}{p}}$. We define the weighted Sobolev space: $V^k_{p,\alpha}(G)$ for integer $k \geq 0$ and real α as the closure of $C_0^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{V^k_{p,\alpha}(G)} = \left(\int\limits_G \sum_{|\beta|=0}^k r^{\alpha + p(|\beta| - k)}|D^\beta u|^p\,dx\right)^{\frac{1}{p}}.$$

We write $W^k(G)$ for $W^{k,2}(G)$ and $\overset{\circ}{W}{}^k_\alpha(G)$ for $V^k_{2,\alpha}(G)$.

Let us recall some well known formulae related to polar coordinates (r, ω) centered at the point $\mathcal{O}$:

- $dx = rdrd\omega$,
- $d\Omega_\rho = \rho d\omega$,
- ds denotes the length element on ∂G,
- $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \omega})^2$,

- $\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \omega^2}$.

$C = C(\ldots)$, $c = c(\ldots)$ denote constants depending only on the quantities appearing in parentheses. In what follows, the same letters C, c will (generally) be used to denote different constants depending on the same set of arguments.

Without loss of generality we can assume that there exists $d > 0$ such that G_0^d is *an angle* with the vertex at $\mathcal{O}$ and the measure $\omega_0 \in (0, 2\pi)$, thus

$$\Gamma_{0\pm}^d = \Big\{\Big(r, \pm\frac{\omega_0}{2}\Big)\Big|\ x_1 = \pm\cot\frac{\omega_0}{2}\cdot x_2;\ r \in (0,d)\Big\}.$$

By means of the direct calculation we obtain

Lemma 1.2

$$\cos(\vec{n}, x_1)|_{\Gamma_{0\pm}^d} = -\sin\frac{\omega_0}{2};\quad x_i\cos(\vec{n}, x_i)|_{\Gamma_{0\pm}^d} = 0;\quad x_i\cos(\vec{n}, x_i)|_{\Omega_\varrho} = \varrho.$$

Definition 1.3

A function $u(x)$ is called a *weak* solution of problem (L) provided that $u(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$ and satisfies the integral identity

$$\begin{aligned}
&\int_G \{a^{ij}(x)u_{x_j}\eta_{x_i} - b^i(x)u_{x_i}\eta(x) - c(x)u(x)\eta(x)\}\,dx\\
&\qquad+\int_{\Gamma_+}\Big(\beta_+\frac{u(x)}{r} + \frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds + \beta_-\int_{\Gamma_-}\frac{u(x)}{r}\eta(x)\,ds \qquad (II)\\
&\quad= \int_{\Gamma_+} g(x)\eta(x)\,ds + \int_{\Gamma_-} h(x)\eta(x)\,ds - \int_G f(x)\eta(x)\,dx
\end{aligned}$$

for all functions $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$.

Lemma 1.4

Let $u(x)$ be a weak solution of (L). For any function $\eta(x) \in C^0(\overline{G}) \cap \overset{\circ}{W}{}_0^1(G)$ the equality

$$\begin{aligned}
&\int_{G_0^\varrho} \{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u(x))\eta(x)\}\,dx\\
&\quad= \int_{\Omega_\varrho} a^{ij}(x)u_{x_j}\eta(x)\cos(r, x_i)\,d\Omega_\varrho\\
&\qquad+\int_{\Gamma_{0+}^\varrho}\Big(g(x) - \beta_+\frac{u(x)}{r} - \frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds \qquad (II)_{loc}\\
&\qquad+\int_{\Gamma_{0-}^\varrho}\Big(h(x) - \beta_-\frac{u(x)}{r}\Big)\eta(x)\,ds
\end{aligned}$$

holds for almost every $\varrho \in (0,d)$.

Proof. Let $\chi_\varrho(x)$ be the characteristic function of the set G_0^ϱ. We consider the integral identity (II) replacing $\eta(x)$ by $\eta(x)\chi_\varrho(x)$. As the result we obtain

$$\int\limits_{G_0^\varrho} \{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u(x))\eta(x)\}\,dx$$

$$= -\int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx + \int\limits_{\Gamma_{0+}^\varrho} \Big(g(x) - \beta_+\frac{u(x)}{r} - \frac{b}{r}u(\gamma(x))\Big)\eta(x)\,ds$$

$$+ \int\limits_{\Gamma_{0-}^\varrho} \Big(h(x) - \beta_-\frac{u(x)}{r}\Big)\eta(x)\,ds.$$

Because of formula (7') of subsection 3 §1 chapter 3 in [2]

$$\chi_{x_i} = -\frac{x_i}{r}\delta(\varrho - r),$$

where $\delta(\varrho - r)$ is the Dirac distribution lumped on the circle $r = \varrho$, we get (see Example 4 of subsection 3 §1 chapter 3 [2])

$$-\int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\chi_{x_i}\,dx = \int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\frac{x_i}{r}\delta(\varrho - r)\,dx$$

$$= \int\limits_{\Omega_\varrho} a^{ij}u_{x_j}\eta(x)\cos(r, x_i)\,d\Omega_\varrho.$$

Thus the required statement follows.

We will make the following **assumptions**:

(a) *the condition of the uniform ellipticity:*

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \qquad \forall x \in \overline{G},\ \forall \xi \in \mathbb{R}^2;$$
$$\nu, \mu = const > 0 \qquad and \qquad a^{ij}(0) = \delta_i^j,$$

where δ_i^j *is the Kronecker symbol;*

(b) $a^{ij}(x) \in C^0(\overline{G})$, $b^i(x) \in L_p(G)$, $c(x) \in L_{\frac{p}{2}}(G) \cap L_2(G)$; $\forall p > \widetilde{n}$, $\forall \widetilde{n} > 2$; *for them the inequalities*

$$\Big(\sum_{i,j=1}^{2} |a^{ij}(x) - a^{ij}(y)|^2\Big)^{\frac{1}{2}} \le \mathcal{A}(|x - y|);$$

$$|x|\Big(\sum_{i=1}^{2} |b^i(x)|^2\Big)^{\frac{1}{2}} + |x|^2|c(x)| \le \mathcal{A}(|x|)$$

hold for $x, y \in \overline{G}$, *where* $\mathcal{A}(r)$ *is a monotonically increasing function,* **continuous at 0**, *with* $\mathcal{A}(0) = 0$;

(c) $c(x) \le 0$ *in* G; $b > 0$, $\beta_\pm \ge \beta_0 > 0$ *and* $\beta_+ > \max\{0; \frac{b^2\omega_0}{4} - b; \frac{b^2\omega_0}{4\nu} - b; \frac{4b^2\omega_0}{\nu} - 2b\}$;

(d) $f(x) \in L_{\frac{p}{2}}(G) \cap L_2(G)$, $g(x) \in L_2(\Gamma_+)$, $h(x) \in L_2(\Gamma_-)$ *and there exist numbers* $f_0 \ge 0$, $g_0 \ge 0$, $h_0 \ge 0$, $s > 2 - \frac{2}{p}$ *such that*

$$|f(x)| \le f_0|x|^{s-2}, \qquad |g(x)| \le g_0|x|^{s-1}, \qquad |h(x)| \le h_0|x|^{s-1};$$

(e) $M_0 = \max_{x\in\overline{G}} |u(x)|$ *is known.*

Our main result is the following theorem. Let

$$\lambda = \sqrt{\vartheta\Big(1 + \frac{b}{4\beta_+}\big(2 + \sqrt{4 + 2\omega_0\beta_+}\,\big)\Big)}, \tag{1.1}$$

where ϑ is the smallest positive eigenvalue of problem (EVP) (see Subsection 2.1).

Theorem 1.5

Let u be a weak solution of problem (L), satisfying the assumptions $(a) - (e)$ with $\mathcal{A}(r)$ Dini-continuous at zero. Then there are $d \in (0, \frac{1}{e})$, where e is the Euler number, and a constant $C > 0$ depending only on ν, μ, p, λ, $\| \sum_{i=1}^{2} |b^i(x)|^2 \|_{L_{\frac{p}{2}}(G)}$, ω_0, b, β_+, β_-, M_0, f_0, h_0, g_0, β_0, s, $\operatorname{meas} G$, $\operatorname{meas}\Gamma_+$, $\operatorname{meas}\Gamma_-$ and on the quantity $\int_0^{\frac{1}{e}} \frac{\mathcal{A}(r)}{r}\,dr$ such that for all $x \in G_0^d$

$$|u(x)| \le C \begin{cases} |x|^{\frac{\lambda k}{\sqrt{q}}}, & \text{if } s > \dfrac{\lambda k}{\sqrt{q}}, \\ |x|^{\frac{\lambda k}{\sqrt{q}}} \ln\left(\dfrac{1}{|x|}\right), & \text{if } s = \dfrac{\lambda k}{\sqrt{q}}, \\ |x|^s, & \text{if } s < \dfrac{\lambda k}{\sqrt{q}}, \end{cases} \tag{1.2}$$

where

$$k = 1 + \frac{b}{2\beta_+} - \frac{b\sqrt{1+\omega_0\beta_+}}{2\beta_+} \quad \textit{and} \quad q = 1 + \frac{b}{4\beta_+}\big(2 + \sqrt{4 + 2\omega_0\beta_+}\,\big). \tag{1.3}$$

To prove the main theorem (see Section 6) one ought to derive the following statements:

- the local estimate of the maximum modulus (see Section 3),
- the global estimate of the weighted Dirichlet integral (see Section 4),
- the local estimate of the weighted Dirichlet integral (see Section 5).

2. Preliminaries

2.1. Auxiliary inequalities

In what follows we need some statements and inequalities.

The eigenvalue problem. Let $\Omega = (-\frac{\omega_0}{2}, \frac{\omega_0}{2})$. We consider the following eigenvalue problem:

$$\begin{cases} \psi''(\omega) + \vartheta\psi(\omega) = 0, & \omega \in \Omega, \\ \psi'\Big(\dfrac{\omega_0}{2}\Big) + \beta_+\psi\Big(\dfrac{\omega_0}{2}\Big) = 0, & \\ -\psi'\Big(-\dfrac{\omega_0}{2}\Big) + \beta_-\psi\Big(-\dfrac{\omega_0}{2}\Big) = 0, & \end{cases} \tag{EVP}$$

with $\beta_\pm > 0$, which consist of the determination of all values ϑ (eigenvalues) for which (EVP) has nonzero weak solutions (eigenfunctions).

Definition 2.1

Function ψ is called a *weak solution* of problem (EVP) provided that $\psi \in W^1(\Omega) \cap C^0(\overline{\Omega})$ and satisfies the integral identity

$$\begin{aligned} \int\limits_\Omega (\psi'(\omega)\eta'(\omega) - \vartheta\psi(\omega)\eta(\omega))\, d\omega + \beta_+\psi\Big(\frac{\omega_0}{2}\Big)\eta\Big(\frac{\omega_0}{2}\Big) & \\ + \beta_-\psi\Big(-\frac{\omega_0}{2}\Big)\eta\Big(-\frac{\omega_0}{2}\Big) = 0 & \end{aligned} \tag{2.1}$$

for all $\eta(\omega) \in W^1(\Omega) \cap C^0(\overline{\Omega})$.

Remark 2.2

We observe that $\vartheta = 0$ is not an eigenvalue of (EVP). In fact, setting in (2.1) $\eta = \psi$ and $\vartheta = 0$ we have

$$\int\limits_\Omega |\psi'(\omega)|^2\, d\omega + \beta_+\Big|\psi\Big(\frac{\omega_0}{2}\Big)\Big|^2 + \beta_-\Big|\psi\Big(-\frac{\omega_0}{2}\Big)\Big|^2 = 0 \implies \psi(\omega) \equiv 0,$$

since $\beta_\pm > 0$.

Now, let us introduce the following functionals on $W^1(\Omega) \cap C^0(\Omega)$

$$F[\psi] = \int\limits_\Omega (\psi'(\omega))^2\, d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big),$$

$$G[\psi] = \int\limits_\Omega \psi^2(\omega)\, d\omega,$$

$$H[\psi] = \int\limits_\Omega ((\psi'(\omega))^2 - \vartheta\psi^2(\omega))\, d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big).$$

We introduce also corresponding bilinear forms

$$\mathcal{F}[\psi,\eta] = \int\limits_{\Omega} (\psi'(\omega)\eta'(\omega))\,d\omega + \beta_+\psi\Big(\frac{\omega_0}{2}\Big)\eta\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi\Big(-\frac{\omega_0}{2}\Big)\eta\Big(-\frac{\omega_0}{2}\Big),$$

$$\mathcal{G}[\psi,\eta] = \int\limits_{\Omega} \psi(\omega)\eta(\omega)\,d\omega.$$

We define the set $K = \{\psi \in W^1(\Omega)\cap C^0(\overline{\Omega}) |\, G[\psi] = 1\}$. Since $K \subset W^1(\Omega)\cap C^0(\overline{\Omega})$, $F[\psi]$ is bounded from below for $\psi \in K$. We denote by ϑ the greatest lower bound of $F[\psi]$ for this family:

$$\vartheta := \inf_{\psi\in K} F[\psi].$$

We formulate the following statement:

Theorem 2.3
Let $\Omega \subset S^1$ be an arc. Then there exist $\vartheta > 0$ and a function $\psi \in K$ such that

$$\mathcal{F}[\psi,\eta] - \vartheta\mathcal{G}[\psi,\eta] = 0 \quad \textit{for arbitrary } \eta \in W^1(\Omega)\cap C^0(\overline{\Omega}).$$

In particular $F[\psi] = \vartheta$.

Proof. The proof is similar to Theorem 2.18 [1].

Now from the variational principle we obtain **the Friedrichs–Wirtinger type inequality**:

Theorem 2.4
Let ϑ be the smallest positive eigenvalue of problem (EVP) (it exists according to Theorem 2.3). Let $\Omega \subset S^1$ and assume that $\psi \in W^1(\Omega)\cap C^0(\overline{\Omega})$ satisfies in the weak sense boundary conditions from (EVP). Then

$$\vartheta\int\limits_{\Omega} \psi^2(\omega)\,d\omega \le \int\limits_{\Omega} \Big(\frac{\partial\psi}{\partial\omega}\Big)^2 d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big).$$

Because of (1.1) and the definition of q by (1.3), the Friedrichs–Wirtinger inequality will be written in the following form

$$\int\limits_{\Omega} \psi^2(\omega)\,d\omega \le \frac{q}{\lambda^2}\Bigg\{\int\limits_{\Omega} \Big(\frac{\partial\psi}{\partial\omega}\Big)^2 d\omega + \beta_+\psi^2\Big(\frac{\omega_0}{2}\Big) + \beta_-\psi^2\Big(-\frac{\omega_0}{2}\Big)\Bigg\} \tag{2.2}$$

for all $\psi(\omega) \in W^1(\Omega)\cap C^0(\overline{\Omega})$ satisfying boundary conditions from (EVP) in the weak sense.

We formulate the classical Hardy inequality (see Theorem 330 [5]).

Proposition 2.5
Let $v \in C^0[0,d] \cap W^1(0,d)$, $d>0$ with $v(0)=0$. Then

$$\int_0^d r^{\alpha-3} v^2(r)\,dr \le \frac{4}{(2-\alpha)^2} \int_0^d r^{\alpha-1} \Big(\frac{\partial v}{\partial r}\Big)^2 dr \tag{2.3}$$

for $\alpha<2$, provided that the integral on the right hand side is finite.

Proof. It is the corollary of the classical Hardy inequality (see e.g. §2.1 [1]).

Now we use the Hardy inequality and then we get:

Proposition 2.6 (The Hardy–Friedrichs–Wirtinger inequality)
Let $u \in C^0(\overline{G_0^d}) \cap \overset{\circ}{W}{}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$. Then

$$\begin{aligned} \int_{G_0^d} r^{\alpha-4} u^2(x)\,dx \le \frac{1}{\frac{(2-\alpha)^2}{4} + \frac{\lambda^2}{q}} \cdot \Bigg\{ \int_{G_0^d} r^{\alpha-2} |\nabla u|^2\,dx \\ + \beta_+ \int_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\,ds + \beta_- \int_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\,ds \Bigg\}. \end{aligned} \tag{2.4}$$

Proof. Multiplying inequality (2.2) by $r^{\alpha-3}$ and integrating over $r \in (0,d)$ we obtain

$$\begin{aligned} \int_{G_0^d} r^{\alpha-4} u^2(x)\,dx \le \frac{q}{\lambda^2} \Bigg\{ \int_{G_0^d} r^{\alpha-2} \frac{1}{r^2} \Big(\frac{\partial u}{\partial \omega}\Big)^2 dx \\ + \beta_+ \int_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\,ds + \beta_- \int_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\,ds \Bigg\}. \end{aligned} \tag{2.5}$$

Hence (2.4) follows for $\alpha=2$. Now, let $\alpha<2$. We shall show that $u(0)=0$. In fact, from $u(0)=u(x)-(u(x)-u(0))$ using the Cauchy inequality we have $\frac{1}{2}|u(0)|^2 \le |u(x)|^2 + |u(x)-u(0)|^2$. Multiplying this inequality by $r^{\alpha-4}$, integrating over G_0^d and using $v(x)=u(x)-u(0)$ we obtain

$$\frac{1}{2}|u(0)|^2 \int_{G_0^d} r^{\alpha-4}\,dx \le \int_{G_0^d} r^{\alpha-4} u^2(x)\,dx + \int_{G_0^d} r^{\alpha-4} |v(x)|^2\,dx < \infty \tag{2.6}$$

(the first integral from the right is finite by (2.5) and the second is finite as well in virtue of Proposition 2.5). Since

$$\int_{G_0^d} r^{\alpha-4}\,dx = \operatorname{meas}\Omega \int_0^d r^{\alpha-3}\,dr = \infty$$

because of $\alpha - 2 < 0$, the assumption $u(0) \neq 0$ contradicts (2.6). Then $u(0) = 0$. Now using Hardy inequality (2.3) we obtain

$$\int\limits_{G_0^d} r^{\alpha-4} u^2(x)\, dx \leq \frac{4}{(2-\alpha)^2} \int\limits_{G_0^d} r^{\alpha-2} \Big(\frac{\partial u}{\partial r}\Big)^2\, dx. \tag{2.7}$$

Adding inequality (2.5) and (2.7) and using the formula $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}|\frac{\partial u}{\partial \omega}|^2$, we get the desired (2.4).

Lemma 2.7
Let $u(\varrho, \omega) \in C^0(\overline{\Omega})$ and $\nabla u(\varrho, \omega) \in L_2(\Omega)$ a.e. $\varrho \in (0, d)$. Assume that

$$U(\varrho) = \int\limits_{G_0^\varrho} |\nabla u|^2\, dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\, ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\, ds < \infty \tag{2.8}$$

for $\varrho \in (0, d)$. Then

$$\int\limits_\Omega \varrho u \frac{\partial u}{\partial r}\Big|_{r=\varrho}\, d\omega \leq \frac{\varrho\sqrt{q}}{2\lambda} U'(\varrho),$$

where q is defined by (1.3).

Proof. Writing $U(\varrho)$ in polar coordinates,

$$U(\varrho) = \int\limits_0^\varrho r \int\limits_\Omega \Big(\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \omega}\Big|^2\Big)\, d\omega\, dr + \beta_+ \int\limits_0^\varrho \frac{u^2(r, \frac{\omega_0}{2})}{r}\, dr + \beta_- \int\limits_0^\varrho \frac{u^2(r, -\frac{\omega_0}{2})}{r}\, dr$$

and differentiating with respect to ϱ we obtain

$$U'(\varrho) = \int\limits_\Omega \Big(\varrho\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{\varrho}\Big|\frac{\partial u}{\partial \omega}\Big|^2\Big)\Big|_{r=\varrho}\, d\omega + \beta_+ \frac{u^2(\varrho, \frac{\omega_0}{2})}{\varrho} + \beta_- \frac{u^2(\varrho, -\frac{\omega_0}{2})}{\varrho}. \tag{2.9}$$

Moreover, by Cauchy's inequality, we have

$$\rho u \frac{\partial u}{\partial r} \leq \frac{\varepsilon}{2} u^2 + \frac{1}{2\varepsilon}\rho^2 \Big(\frac{\partial u}{\partial r}\Big)^2$$

for all $\varepsilon > 0$. Thus, choosing $\varepsilon = \frac{\lambda}{\sqrt{q}}$, by Friedrichs–Wirtinger inequality (2.2), we obtain

$$\begin{aligned}
&\int\limits_\Omega \varrho u \frac{\partial u}{\partial r}\Big|_{r=\varrho}\, d\omega \\
&\leq \frac{\varepsilon q}{2\lambda^2}\Big\{ \int\limits_\Omega \Big|\frac{\partial u}{\partial \omega}\Big|^2_{r=\varrho} d\omega + \beta_+ u^2\Big(\varrho, \frac{\omega_0}{2}\Big) + \beta_- u^2\Big(\varrho, -\frac{\omega_0}{2}\Big)\Big\} + \frac{\varrho^2}{2\varepsilon} \int\limits_\Omega \Big|\frac{\partial u}{\partial r}\Big|^2_{r=\varrho} d\omega \\
&= \frac{\varrho\sqrt{q}}{2\lambda}\Big\{ \int\limits_\Omega \Big(\frac{1}{\varrho}\Big|\frac{\partial u}{\partial \omega}\Big|^2 + \varrho\Big|\frac{\partial u}{\partial r}\Big|^2\Big)\Big|_{r=\varrho} d\omega + \beta_+ \frac{u^2(\varrho, \frac{\omega_0}{2})}{\varrho} + \beta_- \frac{u^2(\varrho, -\frac{\omega_0}{2})}{\varrho}\Big\} \\
&= \frac{\varrho\sqrt{q}}{2\lambda} U'(\varrho).
\end{aligned}$$

We also need in the sequel well known inequalities (see e.g. (6.23), (6.24) Chapter I [6] or Lemma 6.36 [8])

$$\int\limits_{\Gamma} v\,ds \le C \int\limits_{G} (|v| + |\nabla v|)\,dx, \qquad \forall v(x) \in W^{1,1}(G),\ \forall \Gamma \subseteq \partial G,$$
$$\int\limits_{\partial G} v^2\,ds \le \int\limits_{G} \Big(\delta |\nabla v|^2 + \frac{1}{\delta} c_0 v^2\Big)\,dx, \qquad \forall v(x) \in W^{1,2}(G),\ \forall \delta > 0. \quad (2.10)$$

2.2. The Cauchy problem for differential inequality

THEOREM 2.8
Let $U(\varrho)$ be monotonically increasing, nonnegative differentiable function defined on $[0,d]$ and satisfying the problem

$$\begin{cases} U'(\varrho) - \mathcal{P}(\varrho)U(\varrho) + \mathcal{Q}(\varrho) \ge 0, & 0 < \varrho < d, \\ U(d) \le U_0, \end{cases} \qquad (CP)$$

where $\mathcal{P}(\varrho), \mathcal{Q}(\varrho)$ are nonnegative continuous functions defined on $[0,d]$, and U_0 is a constant. Then

$$U(\varrho) \le U_0 \exp\Big(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)\,d\tau\Big) + \int\limits_{\varrho}^{d} \mathcal{Q}(\tau) \exp\Big(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma\Big)\,d\tau. \qquad (2.11)$$

Proof. For the proof see §1.10 (Theorem 1.57) [1].

3. Local estimate at the boundary

Here we derive the local boundedness (near the boundary corner point) of a weak solution of problem (L).

THEOREM 3.1
Let $u(x)$ be a weak solution of problem (L) and assumptions $(a)-(c)$ be satisfied. Suppose, in addition, that $g(x) \in L_\infty(\Gamma_+)$, $h(x) \in L_\infty(\Gamma_-)$. Then the inequality

$$\sup_{G_0^{\varkappa\varrho}} |u(x)| \le \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}} \Big\{\varrho^{-1}||u||_{2,G_0^\varrho} + \varrho^{2(1-\frac{2}{p})}||f||_{\frac{p}{2},G_0^\varrho} + \varrho\big(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho}\big)\Big\}$$

holds for any $p > \tilde{n} > 2$, $\varkappa \in (0,1)$ and $\varrho \in (0,d)$, where C is a positive constant depending only on μ, ν, p, $\|\sum_{i=1}^{2} |b^i(x)|^2\|_{L_{\frac{p}{2}(G)}}$ and G.

Proof. We apply the Moser iteration method. We consider the integral identity (II) and make the coordinate transformation $x = \varrho x'$. Let G' be the image of

G, $\Gamma_+^{'}$ be the image of Γ_+, $\Gamma_-^{'}$ be the image of Γ_-, then we have $dx = \varrho^2 dx'$, $ds = \varrho ds'$. In addition, we denote

$$\begin{aligned} &v(x') = u(\varrho x'),\ \eta(x') = \eta(\varrho x'),\ \mathcal{F}(x') = \varrho^2 f(\varrho x'),\\ &\mathcal{G}(x') = \varrho g(\varrho x'),\ \mathcal{H}(x') = \varrho h(\varrho x'). \end{aligned} \tag{3.1}$$

Then from (II) we get

$$\begin{aligned} &\int\limits_{G'} \{a^{ij}(\varrho x') v_{x'_j} \eta_{x'_i} - \varrho b^i(\varrho x') v_{x'_i} \eta(x') - \varrho^2 c(\varrho x') v(x') \eta(x')\}\, dx' \\ &\quad + \int\limits_{\Gamma_+^{'}} \Big(\frac{\beta_+}{|x'|} v(x') + \frac{b}{|x'|} v(\gamma(x'))\Big) \eta(x')\, ds' + \beta_- \int\limits_{\Gamma_-^{'}} \frac{v(x')}{|x'|} \eta(x')\, ds' \\ &\quad = \int\limits_{\Gamma_+^{'}} \mathcal{G}(x') \eta(x')\, ds' + \int\limits_{\Gamma_-^{'}} \mathcal{H}(x') \eta(x')\, ds' - \int\limits_{G'} \mathcal{F}(x') \eta(x')\, dx' \end{aligned} \tag{$(II)'$}$$

for all $\eta(x') \in C^0(\overline{G'}) \cap \overset{\circ}{W}{}_0^1(G')$. We define quantity m by

$$m = m(\varrho) = \frac{1}{\nu}\Big(\|\mathcal{F}\|_{\frac{p}{2}, G_0^1} + \|\mathcal{G}\|_{\infty, \Gamma_{0+}^1} + \|\mathcal{H}\|_{\infty, \Gamma_{0-}^1} \Big) \tag{3.2}$$

and we set

$$\overline{v}(x') = |v(x')| + m. \tag{3.3}$$

We observe that

$$\begin{aligned} |\mathcal{F}(x')| \overline{v}(x') &= \frac{1}{m} |\mathcal{F}(x')| \cdot m \overline{v}(x') = \frac{1}{m} |\mathcal{F}(x')| (\overline{v}(x') - |v(x')|) \cdot \overline{v}(x') \\ &= \frac{1}{m} |\mathcal{F}(x')| \cdot \overline{v}^2(x') - \frac{1}{m} |\mathcal{F}(x')| \cdot |v(x')| \overline{v}(x') \\ &\le \frac{1}{m} |\mathcal{F}(x')| \cdot \overline{v}^2(x'); \\ |\mathcal{H}(x')| \overline{v}(x') &\le \frac{1}{m} |\mathcal{H}(x')| \cdot \overline{v}^2(x'); \\ |\mathcal{G}(x')| \overline{v}(x') &\le \frac{1}{m} |\mathcal{G}(x')| \cdot \overline{v}^2(x') \end{aligned} \tag{3.4}$$

in the same way. As the test function in the integral identity $(II)'$ we choose $\eta(x') = \zeta^2(|x'|) v(x')$, where $\zeta(|x'|) \in C_0^\infty([0,1])$ is nonnegative function to be further specified. By the chain and product rules $\eta(x)$ is a valid test function in $(II)'$ and also $\eta_{x'_i} = v_{x'_i} \zeta^2(|x'|) + 2\zeta(|x'|) \zeta_{x'_i} v(x')$, so that by substitution into $(II)'$ with regard to $c(\varrho x') \le 0$ in G' and $v \le |v| \le \overline{v}$, we obtain

$$\begin{aligned} &\int\limits_{G_0^1} a^{ij}(\varrho x') v_{x'_i} v_{x'_j} \zeta^2(|x'|)\, dx' + \beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|} \zeta^2(|x'|)\, ds' \\ &\quad + b \int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|} v(\gamma(x')) \zeta^2(|x'|)\, ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|} \zeta^2(|x'|)\, ds' \end{aligned}$$

$$\le \varrho \int\limits_{G_0^1} |b^i(\varrho x')v_{x_i}|\overline{v}(x')\zeta^2(|x'|)\,dx' + 2\int\limits_{G_0^1} |a^{ij}(\varrho x')\zeta_{x'_i}v_{x'_j}|\overline{v}(x')\zeta(|x'|)\,dx'$$
$$+ \int\limits_{\Gamma_{0-}^1} \mathcal{H}(x')\overline{v}(x')\zeta^2(|x'|)\,ds' + \int\limits_{\Gamma_{0+}^1} \mathcal{G}(x')\overline{v}(x')\zeta^2(|x'|)\,ds'$$
$$+ \int\limits_{G_0^1} \mathcal{F}(x')\overline{v}(x')\zeta^2(|x'|)\,dx'.$$

By the elliptic conditions and with regard to (3.4), hence it follows

$$\int\limits_{G_0^1} \nu|\nabla' v|^2\zeta^2(|x'|)\,dx' + \beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds'$$
$$+ b\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|}\zeta^2\,ds'$$
$$\le \int\limits_{G_0^1} \varrho\Big(\sum_{i=1}^2 |b^i(\varrho x')|^2\Big)^{\frac12} |\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \tag{3.5}$$
$$+ 2\mu\int\limits_{G_0^1} |\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}\|\mathcal{G}\|_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1} \overline{v}^2(x')\zeta^2(|x'|)\,ds'$$
$$+ \frac{1}{m}\|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1} \overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1} |\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.$$

We shall estimate the third integral on the left hand side inequality (3.5). Because of $v|_{\Gamma_{0+}^1} = v(r', \frac{\omega_0}{2})$ and, by Remark 1.1, $v(\gamma(x'))|_{\Gamma_{0+}^1} = v(r',0)$, using the representation $v(r',0) = v(r',\frac{\omega_0}{2}) - \int_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega$ we obtain:

$$\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds'$$
$$= \int\limits_0^1 \frac{v^2(r',\frac{\omega_0}{2})}{r'}\zeta^2(r')\,dr' - \int\limits_0^1 \frac{v(r',\frac{\omega_0}{2})}{r'}\zeta^2(r')\Bigg(\int\limits_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Bigg)\,dr'.$$

Next, by the Cauchy inequality, we have

$$\int\limits_0^1 \frac{v(r',\frac{\omega_0}{2})}{r'}\zeta^2(r')\Bigg(\int\limits_0^{\frac{\omega_0}{2}} \frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Bigg)\,dr'$$
$$\le \int\limits_{G_0^1} \frac{\zeta^2(r')}{r'^2}\Big|\frac{\partial v(r',\omega)}{\partial\omega}\Big|\Big|v\Big(r',\frac{\omega_0}{2}\Big)\Big|\,dx'$$

$$\leq \int\limits_{G_0^1} \frac{\zeta^2(r')}{r'^2}\Big(\frac{\varepsilon}{2}\Big|\frac{\partial v(r',\omega)}{\partial\omega}\Big|^2 + \frac{1}{2\varepsilon}v^2\Big(r',\frac{\omega_0}{2}\Big)\Big)\,dx' \tag{3.6}$$

$$\leq \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{1}{2\varepsilon}\int\limits_0^1\frac{\zeta^2(r')}{r'}\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} v^2\Big(r',\frac{\omega_0}{2}\Big)\,d\omega\,dr'$$

$$\leq \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\omega_0}{2\varepsilon}\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds', \qquad \forall\varepsilon>0.$$

Choosing in inequality (3.6) $\varepsilon = \frac{\nu}{b}$, from (3.5) we have

$$\begin{aligned}
&\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \\
&\quad + \Big(\beta_+ + b - \frac{b^2\omega_0}{2\nu}\Big)\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' + \beta_-\int\limits_{\Gamma_{0-}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' \\
&\leq \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \qquad (3.7)\\
&\quad + 2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}||\mathcal{G}||_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' \\
&\quad + \frac{1}{m}||\mathcal{H}||_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.
\end{aligned}$$

Thus, by the assumption (c) for β_+, from (3.7) it follows that

$$\begin{aligned}
&\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \\
&\leq \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{\frac{1}{2}}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \\
&\quad + 2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' \qquad (3.8)\\
&\quad + \frac{1}{m}||\mathcal{G}||_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}||\mathcal{H}||_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' \\
&\quad + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.
\end{aligned}$$

We estimate every term by the Cauchy inequality for any $\varepsilon > 0$:

$$2\mu|\nabla' v||\nabla'\zeta|\zeta(|x'|)\overline{v}(x') = 2(|\nabla' v|\cdot\zeta(|x'|))(\mu\overline{v}(x')|\nabla'\zeta|) \le \varepsilon|\nabla' v|^2\zeta^2(|x'|) + \frac{\mu^2}{\varepsilon}\overline{v}^2(x')|\nabla'\zeta|^2;$$

$$\begin{aligned}\varrho\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{\frac12}&|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\\ &= \zeta^2(|x'|)\bigg(\varrho\overline{v}(x')\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{\frac12}\bigg)\times|\nabla' v|\\ &\le \frac{\varrho^2}{2\varepsilon}\overline{v}^2(x')\zeta^2(|x'|)\cdot\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big) + \frac{\varepsilon}{2}|\nabla' v|^2\zeta^2(|x'|).\end{aligned}$$

For the estimating integrals over the boundaries on the right in (3.8) we apply inequality (2.10). Thus we get

$$\begin{aligned}\frac12\nu\int\limits_{G_0^1}&|\nabla' v|^2\zeta^2(|x'|)\,dx'\\ &\le \frac{3\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\mu^2}{\varepsilon}\int\limits_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx'\\ &+ \frac{\varrho^2}{2\varepsilon}\int\limits_{G_0^1}\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)\overline{v}^2(x')\zeta^2(|x'|)\,dx'\\ &+ \frac1m\int\limits_{G_0^1}\big|\mathcal{F}(|x'|)\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx'\\ &+ \frac1m\Big(\|\mathcal{G}\|_{\infty,\Gamma^1_{0-}} + \|\mathcal{H}\|_{\infty,\Gamma^1_{0+}}\Big)\int\limits_{G_0^1}\Big(\delta|\nabla'(\zeta\overline{v})|^2 + \frac1\delta c_0\overline{v}^2(x')\zeta^2(|x'|)\Big)\,dx',\\ &\qquad\forall\varepsilon,\delta>0.\end{aligned}\tag{3.9}$$

From relations

$$|\nabla'(\zeta\overline{v})|^2 \le 2(\zeta^2|\nabla'\overline{v}|^2 + \overline{v}^2(x')|\nabla'\zeta|^2), \qquad |\nabla'\overline{v}|^2 = |\nabla' v|^2 \tag{3.10}$$

it follows the inequality

$$|\nabla'(\zeta\overline{v})|^2 \le 2|\nabla' v|^2\zeta^2 + 2\overline{v}^2(x')|\nabla'\zeta|^2. \tag{3.11}$$

Now, by (3.9)–(3.11), choosing $\varepsilon = \frac{\nu}{6}$ in (3.9) and, by virtue of (3.2), we find that

$$\frac{\nu}{4}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx'$$

$$\leq \frac{6\mu^2}{\nu}\int\limits_{G_0^1} |\nabla'\zeta|^2\overline{v}^2(x')\,dx' + \frac{3\varrho^2}{\nu}\int\limits_{G_0^1}\bigg(\sum_{i=1}^{2}|b^i(\varrho x')|^2\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+ 2\delta\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + 2\delta\nu\int\limits_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx'$$
$$+ \frac{c_0\nu}{\delta}\int\limits_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx' + \frac{1}{m}\int\limits_{G_0^1}\big|\mathcal{F}(x')\big|\overline{v}^2(x')\zeta^2(|x'|)\,dx', \qquad \forall\delta > 0.$$

Now we choose $\delta = \frac{1}{16}$, then by (3.10), the last inequality means

$$\int\limits_{G_0^1}|\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx' \leq \frac{48\mu^2}{\nu^2}\int\limits_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx'$$
$$+ \frac{24\varrho^2}{\nu^2}\int\limits_{G_0^1}\bigg(\sum_{i=1}^{2}|b^i(\varrho x')|^2\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+ \int\limits_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx' + 128c_0\int\limits_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx'$$
$$+ \frac{8}{m\nu}\int\limits_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.$$

The above inequality we can rewrite as the following

$$\begin{aligned}\int\limits_{G_0^1}&|\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx'\\ &\leq C_1\int\limits_{G_0^1}(|\nabla'\zeta|^2 + \zeta^2(|x'|))\overline{v}^2(x')\,dx' \\ &\quad + C_2\int\limits_{G_0^1}\bigg(\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2 + \frac{|\mathcal{F}(x')|}{m}\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx',\end{aligned} \tag{3.12}$$

where constants C_1, C_2 depend only on c_0, μ, ν. The desired iteration process can now be developed from (3.12). By the Sobolev imbedding theorem (see §2 ch. II [7]) we have

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \leq C^*\int\limits_{G_0^1}((|\nabla'\zeta|^2 + \zeta^2)\overline{v}^2(x') + \zeta^2|\nabla'\overline{v}|^2)\,dx', \qquad \widetilde{n} > 2, \tag{3.13}$$

where constant C^* depends only on $\widetilde{n}$ and the domain G. Using the Hölder inequality for integrals

$$\int\limits_{G_0^1}\left(\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\right)\cdot\overline{v}^2(x')\zeta^2(x')\,dx' \\ \leq\left\|\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\right\|_{\frac{p}{2},G_0^1}\times\|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1},\qquad p>2 \tag{3.14}$$

and from (3.12)–(3.14) we get

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \\ \leq C_3\int\limits_{G_0^1}(|\nabla'\zeta|^2+\zeta^2(|x'|))\overline{v}^2(x')\,dx' \\ +C_4\left\|\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\right\|_{\frac{p}{2},G_0^1}\cdot\|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1},\qquad p>\widetilde{n}>2. \tag{3.15}$$

By the interpolation inequality for L_p-norms

$$\|\zeta\overline{v}\|_{\frac{2p}{p-2},G_0^1}\leq\varepsilon\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}+\widetilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}}\|\zeta\overline{v}\|_{2,G_0^1},\qquad p>\widetilde{n}>2,\ \forall\varepsilon>0,$$
$$\widetilde{c}=\frac{p-\widetilde{n}}{p}\left(\frac{\widetilde{n}}{p}\right)^{\frac{\widetilde{n}}{p-\widetilde{n}}},$$

and, by virtue of definition (3.2), from (3.16) it follows that

$$\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}\leq\sqrt{C_3}\cdot\|(\zeta+|\nabla'\zeta|)\overline{v}\|_{2,G_0^1} \\ +\sqrt{C_4}\left(\left\|\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2\right\|_{\frac{p}{2},G_0^1}+\nu\right)^{\frac12} \\ \times\left(\varepsilon||\zeta\overline{v}||_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}+\widetilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}}\|\zeta\overline{v}\|_{2,G_0^1}\right),\qquad p>\tilde{n},\ \forall\varepsilon>0. \tag{3.16}$$

Choosing

$$\varepsilon=\frac{1}{2\sqrt{C_4}}\left(\left\|\varrho^2\sum_{i=1}^{2}|b^i(\varrho x')|^2\right\|_{\frac{p}{2},G_0^1}+\nu\right)^{-\frac12}$$

from (3.16) we obtain

$$\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}\leq C\|(\zeta+|\nabla'\zeta|)\overline{v}\|_{2,G_0^1},\qquad 2\widetilde{n}\geq p>\widetilde{n}>2, \tag{3.17}$$

where C depends only on c_0, μ, ν, p, $\operatorname{diam}G$, $\|\sum_{i=1}^{2}|b^i(x)|^2\|_{\frac{p}{2},G}$. This inequality can now be iterated to yield the desired estimate.

For all $\varkappa\in(0,1)$ we define sets $G'_{(j)}\equiv G_0^{\varkappa+(1-\varkappa)2^{-j}}$, $j=0,1,2\dots$. It is easy to verify that $G_0^{\varkappa}\equiv G'_{(\infty)}\subset\ldots\subset G'_{(j+1)}\subset G'_j\subset\ldots\subset G'_{(0)}\equiv G_0^1$. Now we consider the sequence of cut-off function $\zeta_j(x')\in C^\infty(G'_{(j)})$ such that

$$0\leq\zeta_j(x')\leq1\ \text{in}\ G'_{(j)}\qquad\text{and}\qquad\zeta_j(x')\equiv1\ \text{in}\ G'_{(j+1)},$$

$$\zeta_j(x') \equiv 0 \qquad \text{for } |x'| > \varkappa + 2^{-j}(1-\varkappa),$$
$$|\nabla'\zeta_j| \le \frac{2^{j+1}}{1-\varkappa} \qquad \text{for } \varkappa + 2^{-j-1}(1-\varkappa) < |x'| < \varkappa + 2^{-j}(1-\varkappa).$$

We also define the number sequence $t_j = 2(\frac{\tilde{n}}{\tilde{n}-2})^j$, $j = 0, 1, 2 \ldots$. Now we rewrite inequality (3.17) replacing $\zeta(|x'|)$ by $\zeta_j(x')$; then, we obtain

$$\|\overline{v}\|_{\frac{2\tilde{n}}{\tilde{n}-2}, G'_{(j+1)}} \le C\frac{2^{j+2}}{1-\varkappa}\|\overline{v}\|_{2,G'_{(j)}}. \tag{3.18}$$

Putting $w = |\overline{v}|^{(\frac{\tilde{n}}{\tilde{n}-2})^j}$, by (3.18) and the definition on the number sequence t_j, we get

$$\begin{aligned} \|\overline{v}\|_{t_{j+1},G'_{(j+1)}} &= \left(\int\limits_{G_{(j+1)'}} w^{\frac{2\tilde{n}}{\tilde{n}-2}}\,dx'\right)^{\frac{\tilde{n}-2}{2\tilde{n}}\cdot(\frac{\tilde{n}-2}{\tilde{n}})^j} \\ &\le \left(C\frac{2^{j+2}}{1-\varkappa}\right)^{(\frac{\tilde{n}-2}{\tilde{n}})^j} \|w\|_{2,G'_{(j)}}^{(\frac{\tilde{n}-2}{\tilde{n}})^j} \\ &= \left(\frac{C}{1-\varkappa}\right)^{\frac{2}{t_j}} 4^{\frac{j+2}{t_j}}\|\overline{v}\|_{t_j,G'_{(j)}}. \end{aligned}$$

After iteration, we find that

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \left\{\frac{C}{1-\varkappa}\right\}^{2\sum_{j=0}^{\infty}\frac{1}{t_j}} \cdot 4^{\sum_{j=0}^{\infty}\frac{j+2}{t_j}} \cdot \|\overline{v}\|_{2,G_0^1}. \tag{3.19}$$

Notice that the series $\sum_{j=0}^{\infty}\frac{j+2}{t_j}$ is convergent by the d'Alembert ratio test, and the series $\sum_{j=0}^{\infty}\frac{1}{t_j} = \frac{\tilde{n}}{4}$ as a geometric series. Therefore from (3.19) we get

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\|\overline{v}\|_{2,G_0^1}.$$

Consequently, letting $j \to \infty$, we have

$$\sup_{x' \in G_0^{\varkappa}} |\overline{v}(x')| \le \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\|\overline{v}\|_{2,G_0^1}.$$

Hence, because of definition of function $\overline{v}(x')$ by (3.3) and definition of number m by (3.2), we get:

$$\sup_{x' \in G_0^{\varkappa}} |v(x')| \le \frac{C}{(1-\varkappa)^{\frac{\tilde{n}}{2}}}\left(\|v\|_{2,G_0^1} + \|\mathcal{F}\|_{\frac{p}{2},G_0^1} + \|\mathcal{G}\|_{\infty,\Gamma_{0+}^1} + \|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\right).$$

Returning to the variables x and u we obtain the required estimate (3.1).

4. Global integral estimate

In this section we obtain the global estimate for the weighted Dirichlet integral.

THEOREM 4.1
Let $u(x)$ be a weak solution of problem (L). Let assumptions $(a)-(c)$, (e) be satisfied. Suppose, in addition, that $g(x) \in L_2(\Gamma_+)$, $h(x) \in L_2(\Gamma_-)$. Then the inequality

$$\int\limits_G |\nabla u|^2\,dx + \int\limits_G \frac{u^2(x)}{r^2}\,dx + \int\limits_{\partial G} \frac{u^2(x)}{r}\,ds \\ \leq C\Big\{ \int\limits_G f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds \Big\} \tag{4.1}$$

holds, where constant $C>0$ depends only on b, β_+, ω_0, β_0, p, ν, M_0, G, $\| \sum_{i=1}^{2} |b^i(x)|^2 \|_{L_{\frac{p}{2}}(G)}$.

Proof. Setting in (II) $\eta(x)=u(x)$ and using the classical Hölder inequality, by assumptions (a), (c), we get

$$\nu \int\limits_G |\nabla u|^2\,dx + \int\limits_{\Gamma_+} \Big(\beta_+ \frac{u^2(x)}{r} + b\frac{u(x)}{r} u(\gamma(x))\Big)\,ds + \beta_- \int\limits_{\Gamma_-} \frac{u(x)}{r}\,ds \\ \leq \int\limits_G \sqrt{\sum_{i=1}^{2} |b^i(x)|^2} |u||\nabla u|\,dx \\ + \int\limits_{\Gamma_+} |u||g(x)|\,ds + \int\limits_{\Gamma_-} |u||h(x)|\,ds + \int\limits_G |u||f(x)|\,dx. \tag{4.2}$$

Now, by assumptions (b), (c), the Cauchy inequality and the Hölder inequality for integrals with $q=\frac{p}{2}$, $q'=\frac{p}{p-2}$, $p>2$, we have:

$$\int\limits_G \sqrt{\sum_{i=1}^{2} |b^i(x)|^2} |u||\nabla u|\,dx \\ = \int\limits_G |\nabla u| \Big(\sqrt{\sum_{i=1}^{2} |b^i(x)|^2} |u| \Big)\,dx \\ \leq \frac{\nu}{2} \int\limits_G |\nabla u|^2\,dx + \frac{1}{2\nu} \int\limits_G \sum_{i=1}^{2} |b^i(x)|^2 u^2\,dx \\ \leq \frac{\nu}{2} \int\limits_G |\nabla u|^2\,dx + \frac{1}{2\nu} \Big(\int\limits_G \Big(\sum_{i=1}^{2} |b^i(x)|^2 \Big)^{\frac{p}{2}}\,dx \Big)^{\frac{2}{p}} \cdot ||u||^2_{\frac{2p}{p-2}(G)}.$$

Next, we apply the inequality

$$\|u\|^2_{L_{\frac{2p}{p-2}}(G)} \le \delta\|\nabla u\|^2_{L_2(G)} + c(\delta,p,G)\|u\|^2_{L_2(G)}, \qquad p>2,\ \forall\delta>0$$

(see for example (2.19) §2, chapter II in [7]); hence it follows that

$$\begin{aligned}\int\limits_G \sqrt{\sum_{i=1}^2 |b^i(x)|^2}|u||\nabla u|\,dx \\ &\le \frac{\nu}{2}\int\limits_G |\nabla u|^2\,dx + \frac{1}{2\nu}\Big\|\sum_{i=1}^2 |b^i(x)|^2\Big\|_{L_{\frac{p}{2}}(G)} \\ &\quad\times\int\limits_G (\delta|\nabla u|^2 + c(\delta,p,G)u^2(x))\,dx, \qquad \forall\varepsilon>0,\ \forall\delta>0.\end{aligned} \tag{4.3}$$

We choose $\delta = \frac{\nu^2}{2\|\sum_{i=1}^2 |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}}$. As a result from (4.2)–(4.3) we obtain

$$\begin{aligned}&\frac{\nu}{4}\int\limits_G |\nabla u|^2\,dx + \int\limits_{\Gamma_+}\Big(\beta_+\frac{u^2(x)}{r} + b\frac{u(x)}{r}u(\gamma(x))\Big)\,ds + \beta_-\int\limits_{\Gamma_-}\frac{u(x)}{r}\,ds \\ &\quad\le C\int\limits_G u^2(x)\,dx + \int\limits_{\Gamma_+}|u||g(x)|\,ds + \int\limits_{\Gamma_-}|u||h(x)|\,ds + \int\limits_G |u||f(x)|\,dx,\end{aligned} \tag{4.4}$$

where $C = \mathrm{const}(p,\nu,\|\sum_{i=1}^2 |b^i(x)|^2\|_{L_{\frac{p}{2}}(G)}, G)$. Further, by the Cauchy inequality, in virtue of the assumption (c), we obtain

$$\begin{aligned}\int\limits_{\Gamma_+}|u||g(x)|\,ds &= \int\limits_{\Gamma_+}\Big(\sqrt{\frac{\beta_+}{r}}|u|\Big)\Big(\sqrt{\frac{r}{\beta_+}}|g(x)|\Big)\,ds \\ &\le \frac{1}{2}\beta_+\int\limits_{\Gamma_+}\frac{u^2(x)}{r}\,ds + \frac{1}{2\beta_0}\int\limits_{\Gamma_+} rg^2(x)\,ds; \\ \int\limits_{\Gamma_-}|u||h(x)|\,ds &= \int\limits_{\Gamma_-}\Big(\sqrt{\frac{\beta_-}{r}}|u|\Big)\Big(\sqrt{\frac{r}{\beta_-}}|h(x)|\Big)\,ds \\ &\le \frac{1}{2}\beta_-\int\limits_{\Gamma_-}\frac{u^2(x)}{r}\,ds + \frac{1}{2\beta_0}\int\limits_{\Gamma_-} rg^2(x)\,ds; \\ \int\limits_G |u||f(x)|\,dx &\le \frac{1}{2}\int\limits_G |u|^2\,dx + \frac{1}{2}\int\limits_G |f|^2\,dx.\end{aligned}$$

Hence and from (4.4) we have

$$\begin{aligned}\frac{\nu}{4}\int\limits_{G}|\nabla u|^2\,dx+\frac{1}{2}\beta_+\int\limits_{\Gamma_+}\frac{u^2(x)}{r}\,ds+b\int\limits_{\Gamma_+}\frac{u(x)}{r}u(\gamma(x))\,ds+\frac{1}{2}\beta_-\int\limits_{\Gamma_-}\frac{u(x)}{r}\,ds\\ \le C\int\limits_{G}u^2(x)\,dx+\frac{1}{2\beta_0}\int\limits_{\Gamma_+}rg^2(x)\,ds+\frac{1}{2\beta_0}\int\limits_{\Gamma_-}rh^2(x)\,ds+\frac{1}{2}\int\limits_{G}f^2(x)\,dx.\end{aligned}\tag{4.5}$$

Now we write $\Gamma_+=\Gamma^d_{0+}\cup\Gamma_{d+}$. At first, we estimate $b\int_{\Gamma^d_{0+}}\frac{u(x)}{r}u(\gamma(x))\,ds$. Because of $u|_{\Gamma^d_{0+}}=u(r,\frac{\omega_0}{2})$ and by Remark 1.1 $u(\gamma(x))|_{\Gamma^d_{0+}}=u(r,0)$, using the representation $u(r,0)=u(r,\frac{\omega_0}{2})-\int_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega$, we obtain:

$$\begin{aligned}b\int\limits_{\Gamma^d_{0+}}\frac{u(x)}{r}u(\gamma(x))\,ds&=b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})u(r,0)}{r}\,dr\\ &=b\int\limits_0^d\frac{u^2(r,\frac{\omega_0}{2})}{r}\,dr-b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})}{r}\left(\int\limits_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}d\omega\right)dr.\end{aligned}\tag{4.6}$$

Next, by the Cauchy inequality, we have

$$\begin{aligned}&b\int\limits_0^d\frac{u(r,\frac{\omega_0}{2})}{r}\left(\int\limits_0^{\frac{\omega_0}{2}}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega\right)dr\\ &\le b\int\limits_{G^d_0}\frac{1}{r^2}\Big|u\Big(r,\frac{\omega_0}{2}\Big)\Big|\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|\,dx\\ &\le b\int\limits_{G^d_0}\frac{1}{r^2}\left(\frac{\varepsilon}{2}\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|^2+\frac{1}{2\varepsilon}u^2\Big(r,\frac{\omega_0}{2}\Big)\right)dx\\ &\le\frac{b\varepsilon}{2}\int\limits_{G^d_0}|\nabla u|^2\,dx+\frac{b}{2\varepsilon}\int\limits_0^d\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}}\frac{u^2(r,\frac{\omega_0}{2})}{r}\,d\omega\,dr\\ &\le\frac{b\varepsilon}{2}\int\limits_{G^d_0}|\nabla u|^2\,dx+\frac{b\omega_0}{2\varepsilon}\int\limits_{\Gamma^d_{0+}}\frac{u^2(x)}{r}\,ds,\qquad\forall\varepsilon>0.\end{aligned}\tag{4.7}$$

By the assumption (e) the integral over Γ_{d+} we estimate as below:

$$b\int\limits_{\Gamma_{d+}}\frac{u(x)}{r}u(\gamma(x))\,ds\le b\frac{\operatorname{meas}\Gamma_+}{d}M_0^2.$$

Thus, from the assumption (e) and (4.5)–(4.7) we get

$$\Big(\frac{\nu}{4}-\frac{b\varepsilon}{2}\Big)\int\limits_{G}|\nabla u|^2\,dx+\Big(\frac{1}{2}\beta_++b-\frac{b\omega_0}{2\varepsilon}\Big)\int\limits_{\Gamma_+}\frac{u^2(x)}{r}\,ds+\frac{1}{2}\beta_-\int\limits_{\Gamma_-}\frac{u^2(x)}{r}\,ds$$

$$\leq C(M_0,\, b,\, d,\, G) + \frac{1}{2\beta_0}\int\limits_{\Gamma_+} rg^2(x)\,ds + \frac{1}{2\beta_0}\int\limits_{\Gamma_-} rh^2(x)\,ds + \frac{1}{2}\int\limits_{G} f^2(x)\,dx, \qquad \forall\varepsilon > 0.$$

If we choose $\varepsilon = \frac{\nu}{4b}$, then, in virtue of assumption (c) for β_+, we obtain

$$\int\limits_{G} |\nabla u|^2\,dx + \int\limits_{\partial G} \frac{u^2(x)}{r}\,ds \leq C\Big\{ \int\limits_{G} f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds \Big\}.$$

Finally, by Hardy–Friedrichs–Wirtinger inequality (2.4) with $\alpha = 2$, we get the desired estimate (4.1).

5. Local integral weighted estimates

Theorem 5.1

Let $u(x)$ be a weak solution of problem (L) and λ be as in (1.1). Let assumptions (a) − (e) be satisfied with $\mathcal{A}(r)$ being Dini-continuous at zero. Then there are $d \in (0, \frac{1}{e})$ and a constant $C > 0$ depending only on s, λ, ν, b, β_+, d, G, M_0 and on $\int_0^{\frac{1}{e}} \frac{\mathcal{A}(r)}{r}\,dr$ such that $\forall \varrho \in (0, d)$

$$\int\limits_{G_0^\varrho} \Big(|\nabla u|^2 + \frac{u^2(x)}{r^2}\Big)\,dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds$$
$$\leq C\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2) + \|f\|^2_{2,G} + \|g\|^2_{2,\Gamma_+} + \|h\|^2_{2,\Gamma_-}\Big)$$
$$\cdot \begin{cases} \varrho^{\frac{2\lambda k}{\sqrt{q}}}, & \textit{if } s > \dfrac{\lambda k}{\sqrt{q}}, \\ \varrho^{\frac{2\lambda k}{\sqrt{q}}} \ln^2\Big(\dfrac{1}{\varrho}\Big), & \textit{if } s = \dfrac{\lambda k}{\sqrt{q}}, \\ \varrho^{2s}, & \textit{if } 1 < s < \dfrac{\lambda k}{\sqrt{q}}, \end{cases} \tag{5.1}$$

where k and q are defined by (1.3).

Proof. Setting $\eta(x) = u(x)$ in $(II)_{loc}$, we obtain

$$\int\limits_{G_0^\varrho} |\nabla u|^2\,dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds$$
$$= \varrho \int\limits_{\Omega} u(x) \frac{\partial u}{\partial r}\Big|_{r=\varrho}\,d\omega + \int\limits_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)\,d\Omega_\varrho$$
$$+ \int\limits_{\Gamma_{0+}^\varrho} u(x)g(x)\,ds - b\int\limits_{\Gamma_{0+}^\varrho} \frac{u(x)}{r} u(\gamma(x))\,ds + \int\limits_{\Gamma_{0-}^\varrho} u(x)h(x)\,ds$$

$$+\int_{G_0^\varrho}\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\}\,dx.$$

To estimate the integral $b\cdot\int_{\Gamma_{0+}^\varrho}\frac{u(x)}{r}u(\gamma(x))\,ds$ we behave similarly to (4.6)–(4.7). Then we get:

$$\begin{aligned}
&\Big(1-\frac{b\varepsilon}{2}\Big)\int_{G_0^\varrho}|\nabla u|^2\,dx+\beta_+\Big(1+\frac{b}{\beta_+}-\frac{b\omega_0}{2\beta_+\varepsilon}\Big)\int_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds+\beta_-\int_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds\\
&\quad\le\varrho\int_\Omega u(x)\frac{\partial u}{\partial r}\Big|_{r=\varrho}d\omega+\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho \qquad (5.2)\\
&\quad+\int_{\Gamma_{0+}^\varrho}u(x)g(x)\,ds+\int_{\Gamma_{0-}^\varrho}u(x)h(x)\,ds\\
&\quad+\int_{G_0^\varrho}\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\}\,dx.
\end{aligned}$$

By assumption (c) $\beta_+>\frac{b^2\omega_0}{4}-b$. Therefore we can choose in (5.3) $\varepsilon=\frac{\sqrt{1+\omega_0\beta_+}-1}{\beta_+}$. Hence it follows that

$$0<1-\frac{b\varepsilon}{2}=1+\frac{b}{\beta_+}-\frac{b\omega_0}{2\varepsilon\beta_+}=1+\frac{b}{2\beta_+}-\frac{b\sqrt{1+\omega_0\beta_+}}{2\beta_+}=k \qquad (5.3)$$

(see (1.3)) and recalling (2.8) we obtain

$$\begin{aligned}
&kU(\varrho)\\
&\quad\le\varrho\int_\Omega u(x)\frac{\partial u}{\partial r}\Big|_{r=\varrho}d\Omega+\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho\\
&\quad+\int_{\Gamma_{0+}^\varrho}u(x)g(x)\,ds+\int_{\Gamma_{0-}^\varrho}u(x)h(x)\,ds \qquad (5.4)\\
&\quad+\int_{G_0^\varrho}\Big\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)-u(x)f(x)\Big\}\,dx.
\end{aligned}$$

Now, we shall derive an upper bound for the each integral from the right hand side of (5.4). The first integral we estimate by Lemma 2.7; next, in virtue of assumption (b) and the Cauchy inequality,

$$\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho\le\varrho\mathcal{A}(\varrho)\int_\Omega|u(x)||\nabla u|\,d\omega,$$

$$\int_{G_0^\varrho}\{(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u_{x_i}u(x)+c(x)u^2(x)\}\,dx \qquad (5.5)$$

$$\leq \mathcal{A}(\varrho) \int\limits_{G_0^\varrho} \left\{ |\nabla u|^2 + \frac{u^2(x)}{r^2} \right\} dx.$$

Thus, from (5.4)–(5.5) it follows that

$$\begin{aligned} kU(\varrho) \leq{}& \frac{\varrho\sqrt{q}}{2\lambda} U'(\varrho) + \varrho\mathcal{A}(\varrho) \int\limits_{\Omega} |u(x)||\nabla u|\, d\omega \\ &+ \int\limits_{\Gamma_{0+}^\varrho} |u(x)||g(x)|\, ds + \int\limits_{\Gamma_{0-}^\varrho} |u(x)||h(x)|\, ds \\ &+ \mathcal{A}(\varrho) \int\limits_{G_0^\varrho} \left(|\nabla u|^2 + \frac{u^2(x)}{r^2} \right) dx + \int\limits_{G_0^\varrho} |u(x)||f(x)|\, dx. \end{aligned} \tag{5.6}$$

Further, we derive an upper bound for each integral on the right hand side of (5.6). At first, applying the Cauchy and Friedrichs–Wirtinger inequalities (see (2.2)) with regard to (2.9), we have

$$\begin{aligned} &\mathcal{A}(\varrho) \int\limits_{\Omega} \varrho |u(x)||\nabla u|\, d\omega \\ &\leq \frac{1}{2}\mathcal{A}(\varrho) \int\limits_{\Omega} (\varrho^2 |\nabla u|^2 + |u(x)|^2)\, d\omega \\ &\leq \frac{1}{2}\mathcal{A}(\varrho) \int\limits_{\Omega} \varrho^2 \left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{\varrho^2}\left(\frac{\partial u}{\partial \omega}\right)^2 \right]\Big|_{r=\varrho} d\omega \\ &\quad + \frac{1}{2}\mathcal{A}(\varrho)\frac{q}{\lambda^2} \left\{ \int\limits_{\Omega} \left(\frac{\partial u}{\partial \omega}\right)^2 d\omega + \beta_+ u^2\left(\varrho, \frac{\omega_0}{2}\right) + \beta_- u^2\left(\varrho, -\frac{\omega_0}{2}\right) \right\} \qquad (5.7) \\ &\leq \frac{1}{2}\varrho\mathcal{A}(\varrho)\left(1 + \frac{q}{\lambda^2}\right) \left\{ \int\limits_{\Omega} \left[\varrho\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial u}{\partial \omega}\right)^2 \right]\Big|_{r=\varrho} d\omega \right. \\ &\quad \left. + \beta_+ \frac{u^2(\varrho, \frac{\omega_0}{2})}{\varrho} + \beta_- \frac{u^2(\varrho, -\frac{\omega_0}{2})}{\varrho} \right\} \\ &\leq c_1(b, \beta_+, \omega_0, \lambda)\varrho\mathcal{A}(\varrho)U'(\varrho). \end{aligned}$$

Next, using the Cauchy and Hardy–Friedrichs–Wirtinger (see (2.4) for $\alpha = 2$) inequalities, by (2.8), we obtain

$$\begin{aligned} &\mathcal{A}(\varrho) \int\limits_{G_0^\varrho} \left(|\nabla u|^2 + \frac{|u|^2}{r^2} \right) dx \\ &\leq c(b, \beta_+, \omega_0\lambda)\mathcal{A}(\varrho) \left\{ \int\limits_{G_0^\varrho} |\nabla u|^2\, dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\, ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\, ds \right\} \qquad (5.8) \\ &\leq c_2(b, \beta_+, \omega_0, \lambda)\mathcal{A}(\varrho)U(\varrho), \end{aligned}$$

and for all $\delta > 0$

$$
\begin{aligned}
\int\limits_{\Gamma_{0+}^{\varrho}} |u(x)||g(x)|\,ds &= \int\limits_{\Gamma_{0+}^{\varrho}} \left(\sqrt{\frac{\beta_+}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\beta_+}}|g(x)|\right) ds \\
&\le \frac{\delta\beta_+}{2}\int\limits_{\Gamma_{0+}^{\varrho}} \frac{u^2(x)}{r}\,ds + \frac{1}{2\delta\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds; \\
\int\limits_{\Gamma_{0-}^{\varrho}} |u(x)||h(x)|\,ds &= \int\limits_{\Gamma_{0-}^{\varrho}} \left(\sqrt{\frac{\beta_-}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\beta_-}}|h(x)|\right) ds \\
&\le \frac{\delta\beta_-}{2}\int\limits_{\Gamma_{0-}^{\varrho}} \frac{u^2(x)}{r}\,ds + \frac{1}{2\delta\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds; \\
\int\limits_{G_0^{\varrho}} |u(x)||f(x)|dx &\le \frac{\delta}{2}\int\limits_{G_0^{\varrho}} \frac{u^2(x)}{r^2}\,dx + \frac{1}{2\delta}\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx \\
&\le \frac{\delta}{2}c_3(b,\beta_+,\omega_0,\lambda)U(\varrho) + \frac{1}{2\delta}\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx
\end{aligned}
\tag{5.9}
$$

in virtue of inequality (2.4). From (5.6)–(5.9) it follows

$$
\begin{aligned}
&\langle k - c_4(\delta + \mathcal{A}(\varrho))\rangle\, U(\varrho) \\
&\quad\le \frac{\varrho\sqrt{q}}{2\lambda}(1 + c_5\mathcal{A}(\varrho))U'(\varrho) \\
&\quad+ \frac{1}{2\delta}\left\{\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx + \frac{1}{\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds + \frac{1}{\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds\right\}, \qquad \forall\delta > 0.
\end{aligned}
\tag{5.10}
$$

But, by condition (d),

$$
\int\limits_{G_0^{\varrho}} r^2 f^2(x)\,dx + \frac{1}{\beta_0}\int\limits_{\Gamma_{0+}^{\varrho}} rg^2(x)\,ds + \frac{1}{\beta_0}\int\limits_{\Gamma_{0-}^{\varrho}} rh^2(x)\,ds \le \frac{1}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_0}g_0^2 + \frac{1}{\beta_0}h_0^2\right)\cdot\varrho^{2s}.
$$

Now we take into account that, by (5.3), $0 < k < 1$ and therefore

$$
\begin{aligned}
\frac{k - c_4(\delta + \mathcal{A}(\varrho))}{1 + c_5\mathcal{A}(\varrho)} &= 1 - \frac{1 - k + c_4(\delta + \mathcal{A}(\varrho)) + c_5\mathcal{A}(\varrho)}{1 + c_5\mathcal{A}(\varrho)} \\
&\ge k[1 - c_6\delta - c_7\mathcal{A}(\varrho)], \qquad \forall\delta > 0.
\end{aligned}
$$

Thus, from (5.10) we have differential inequality (CP) of Subsection 2.2 with

$$
\begin{aligned}
\mathcal{P}(\varrho) &= \frac{2\lambda k}{\varrho\sqrt{q}}\cdot[1 - c_6\delta - c_7\mathcal{A}(\varrho)], \qquad \forall\delta > 0; \\
\mathcal{Q}(\varrho) &= \frac{\lambda}{2s\sqrt{q}}\left(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\right)\cdot\delta^{-1}\varrho^{2s-1}, \qquad \forall\delta > 0;
\end{aligned}
\tag{5.11}
$$

and, by (2.8) and Theorem 4.1,

$$U_0 = C(1+\beta_+ + \beta_-)\Big\{ \int\limits_G f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)ds + \int\limits_{\Gamma_-} h^2(x)\,ds \Big\}.$$

We shall consider three cases:

1) $\boldsymbol{s > \frac{\lambda k}{\sqrt{q}}}$.

Choosing $\delta = \varrho^{\varepsilon}$, $\forall \varepsilon > 0$,

$$\mathcal{P}(\varrho) = \frac{2\lambda k}{\varrho\sqrt{q}} \cdot [1 - c_6\varrho^{\varepsilon} - c_7\mathcal{A}(\varrho)];$$
$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \varrho^{2s-1-\varepsilon}.$$

Since $\mathcal{P}(\varrho) = \frac{2\lambda k}{\varrho\sqrt{q}} - \frac{\mathcal{K}(\varrho)}{\varrho}$, where $\mathcal{K}(\varrho)$ satisfies the Dini condition at zero, we have

$$-\int\limits_{\varrho}^{\tau} \mathcal{P}(s)\,ds = -\frac{2\lambda k}{\sqrt{q}}\ln\Big(\frac{\tau}{\varrho}\Big) + \int\limits_{\varrho}^{\tau}\frac{\mathcal{K}(s)}{s}\,ds \le \ln\Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}} + \int\limits_0^d \frac{\mathcal{K}(r)}{r}\,dr$$

$$\Longrightarrow \exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\Bigg) \le \Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}} \exp\Bigg(\int\limits_0^d \frac{\mathcal{K}(\tau)}{\tau}d\tau\Bigg) = K_0\Big(\frac{\varrho}{\tau}\Big)^{\frac{2\lambda k}{\sqrt{q}}};$$

$$\exp\Bigg(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\Bigg) \le \Big(\frac{\varrho}{d}\Big)^{\frac{2\lambda k}{\sqrt{q}}} \exp\Bigg(\int\limits_0^d \frac{\mathcal{K}(\tau)}{\tau}\,d\tau\Bigg) = K_0\Big(\frac{\varrho}{d}\Big)^{\frac{2\lambda k}{\sqrt{q}}}.$$

As well we have:

$$\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\Bigg(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\Bigg)d\tau$$
$$\le \frac{\lambda K_0}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\varrho^{\frac{2\lambda k}{\sqrt{q}}}\int\limits_{\varrho}^{d}\tau^{2s-\frac{2\lambda k}{\sqrt{q}}-\varepsilon-1}\,d\tau$$
$$\le \frac{\lambda K_0}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\frac{d^{s-\frac{\lambda k}{\sqrt{q}}}}{s-\frac{\lambda k}{\sqrt{q}}}\varrho^{2\frac{\lambda k}{\sqrt{q}}},$$

since $s > \frac{\lambda k}{\sqrt{q}}$ and we can choose $\varepsilon = s - \frac{\lambda k}{\sqrt{q}}$.

Now we apply Theorem 2.8: from (2.11), by virtue of the deduced inequalities and with regard to (2.4) for $\alpha = 2$, we obtain the required statement for $s > \frac{\lambda k}{\sqrt{q}}$.

2) $\boldsymbol{s = \frac{\lambda k}{\sqrt{q}}}$.

Taking in (5.11) any function $\delta(\varrho) > 0$ instead of $\delta > 0$, we obtain problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda k(1 - c_6\delta(\varrho))}{\varrho\sqrt{q}} - c_8\frac{\mathcal{A}(\varrho)}{\varrho};$$
$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2 + h_0^2)\Big) \cdot \delta^{-1}(\varrho)\varrho^{2\frac{\lambda k}{\sqrt{q}}-1}.$$

We choose $\delta(\varrho) = \frac{\sqrt{q}}{2c_6\lambda k\ln(\frac{ed}{\varrho})}$, $0 < \varrho < d$, where e is the Euler number. Then we obtain

$$-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)\,d\sigma \le -\frac{2\lambda k}{\sqrt{q}}\ln\frac{\tau}{\varrho} + \int\limits_{\varrho}^{\tau}\frac{d\sigma}{\sigma\ln(\frac{ed}{\sigma})} + c_8\int\limits_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma$$
$$= \ln\Big(\frac{\varrho}{\tau}\Big)^{2\frac{\lambda k}{\sqrt{q}}} + \ln\left(\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}}\right) + c_8\int\limits_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma$$
$$\Longrightarrow \exp\Big(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\Big) \le \Big(\frac{\varrho}{\tau}\Big)^{2\frac{\lambda k}{\sqrt{q}}}\cdot\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}}\cdot\exp\Big(c_8\int\limits_0^d\frac{\mathcal{A}(\sigma)}{\sigma}d\sigma\Big),$$
$$\exp\Big(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\Big) \le \Big(\frac{\varrho}{d}\Big)^{2\frac{\lambda k}{\sqrt{q}}}\cdot\ln\frac{ed}{\varrho}\cdot\exp\Big(c_8\int\limits_0^d\frac{\mathcal{A}(\tau)}{\tau}\,d\tau\Big).$$

In this case we also have

$$\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\Big(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\Big)d\tau$$
$$\le \frac{\lambda}{2s\sqrt{q}}\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\varrho^{2\frac{\lambda k}{\sqrt{q}}}\exp\Big(c_8\int\limits_0^d\frac{\mathcal{A}(\tau)}{\tau}\,d\tau\Big)\ln\frac{ed}{\varrho}$$
$$\times\int\limits_{\varrho}^{d}\delta^{-1}(\tau)\tau^{-1}\frac{1}{\ln(\frac{ed}{\tau})}\,d\tau$$
$$\le c_9\Big(\omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\varrho^{2\frac{\lambda k}{\sqrt{q}}}\ln^2\Big(\frac{ed}{\varrho}\Big).$$

Now we apply Theorem 2.8, and from (2.11), by virtue of the deduced inequalities, we obtain

$$U(\varrho) \le c_{10}\Big(U_0 + \omega_0 f_0^2 + \frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\varrho^{2\frac{\lambda k}{\sqrt{q}}}\ln^2\frac{1}{\varrho}, \qquad 0 < \varrho < d < \frac{1}{e}.$$

Thus, we proved the required statement for $s = \frac{\lambda k}{\sqrt{q}}$.

3) $\boldsymbol{0 < s < \frac{\lambda k}{\sqrt{q}}}$.

Now, similar to case 1) with regard to (5.11) we have

$$\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\right)\le\left(\frac{\varrho}{\tau}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\exp\left(\int\limits_0^d\frac{\mathcal{A}(\sigma)}{\sigma}\,d\tau\right)=c_{11}\left(\frac{\varrho}{\tau}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}},$$

and

$$\exp\left(-\int\limits_{\varrho}^{d}\mathcal{P}(\tau)\,d\tau\right)\le\left(\frac{\varrho}{d}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\exp\left(\int\limits_0^d\frac{\mathcal{A}(\tau)}{\tau}\,d\tau\right)=c_{11}\left(\frac{\varrho}{d}\right)^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}.$$

In this case we also have

$$\begin{aligned}&\int\limits_{\varrho}^{d}\mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)\,d\sigma\right)d\tau\\&\qquad\le\frac{\lambda}{2s\sqrt{q}}\Big(\omega_0f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\delta^{-1}\varrho^{\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}}\times\int\limits_{\varrho}^{d}\tau^{2s-\frac{2\lambda k(1-c_6\delta)}{\sqrt{q}}-1}d\tau\\&\qquad\le c_{12}\Big(\omega_0f_0^2+\frac{1}{\nu_0}g_0^2+\frac{1}{\nu_0}h_0^2\Big)\cdot\varrho^{2s},\end{aligned}$$

if we choose $\delta\in(0,\frac{1}{c_6}(1-\frac{s\sqrt{q}}{\lambda k}))$.

We again apply Theorem 2.8 and from (2.11), by virtue of the deduced inequalities, we obtain

$$\begin{aligned}U(\varrho)&\le c_{13}\Big\{U_0\varrho^{\frac{2\lambda k(1-c_5\delta)}{\sqrt{q}}}+\Big(\omega_0f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\cdot\varrho^{2s}\Big\}\\&\le c_{14}\Big(U_0+f_0^2+\frac{1}{\beta_0}(g_0^2+h_0^2)\Big)\varrho^{2s}.\end{aligned}$$

Thus, we proved the required statement of Theorem 5.1 for $0 < s < \frac{\lambda k}{\sqrt{q}}$.

6. The power modulus of continuity at the conical point for weak solutions

Proof of Theorem 1.5. We define the function

$$\psi(\varrho)=\begin{cases}\varrho^{\frac{\lambda k}{\sqrt{q}}}, & \text{if } s>\dfrac{\lambda k}{\sqrt{q}},\\ \varrho^{\frac{\lambda k}{\sqrt{q}}}\ln\left(\dfrac{1}{\varrho}\right), & \text{if } s=\dfrac{\lambda k}{\sqrt{q}},\\ \varrho^{s}, & \text{if } 1<s<\dfrac{\lambda k}{\sqrt{q}}\end{cases}$$

for $0 < \varrho < d$.

For the proof we apply theorem 3.1 about the local bound of the weak solution modulus

$$\sup_{G_0^{\varkappa\varrho}} |u(x)| \leq \frac{C}{(1-\varkappa)^{\frac{\widetilde{n}}{2}}} \Big\{ \varrho^{-1} ||u||_{2,G_0^\varrho} + \varrho^{2(1-\frac{2}{p})} ||f||_{\frac{p}{2},G_0^\varrho} + \varrho \Big(||g||_{\infty,\Gamma_{0+}^\varrho} + ||h||_{\infty,\Gamma_{0-}^\varrho} \Big) \Big\}.$$

Then, by Theorem 5.1, we obtain

$$\begin{aligned} \varrho^{-1} \|u\|_{2,G_0^\varrho} &\leq \Bigg(\int_{G_0^\varrho} \frac{u^2(x)}{r^2}\, dx \Bigg)^{\frac{1}{2}} \\ &\leq C(\|f\|_{2,G} + \|g\|_{2,\Gamma_+} + \|h\|_{2,\Gamma_-} + \sqrt{\omega_0} f_0 + \frac{1}{\sqrt{\beta_0}}(g_0 + h_0))\psi(\varrho). \end{aligned} \tag{6.1}$$

Further, by the assumption (d), we get

$$\begin{aligned} \varrho^{2(1-\frac{2}{p})} \|f\|_{\frac{p}{2},G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho}) \\ \leq c\Big(f_0 + \frac{1}{\sqrt{\beta_0}}(g_0 + h_0) \Big) \psi(\varrho), \end{aligned} \tag{6.2}$$

for $\widetilde{n} < p < 2\widetilde{n}$, $\forall \widetilde{n} > 2$. From (3.1), (6.1)–(6.2) it follows that

$$\sup_{G_{\varrho/4}^{\varrho/2}} |u(x)| \leq C\Big(\|f\|_{2,G} + \|g\|_{2,\Gamma_+} + \|h\|_{2,\Gamma_-} + f_0 + \frac{1}{\sqrt{\beta_0}}(g_0 + h_0) \Big) \psi(\varrho).$$

Putting $|x| = \frac{1}{3}\varrho$ we obtain finally the desired estimate (1.2).

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The research on the strong Markov property

Abstract. Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$. In this paper, we give the definitions of σ-algebras prior to α and post-α and discuss their properties. At the same time, we prove that the strong Markov property holds for an arbitrary Markov process, that is, we prove that the Markov property is equivalent to the strong Markov property.

1. Introduction

Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$. So for every $A \in \mathcal{E}$, $\{\omega:\ x_t(\omega) \in A\} \in \mathcal{F}$, where $(E, \mathcal{E})$ is an abstract space and t is the time parameter. The points of E are denoted as $x, y, \ldots$. The sets of $\mathcal{E}$ are denoted as $A, B, \ldots$. For convenience, suppose that $\mathcal{E}$ contains all sets of simple points of E, that is, $\{x\} \in \mathcal{E}$ for every $x \in E$.

Throughout this paper, suppose that $X(t,\omega)$ is non-interruptive Markov process unless mentioned. Otherwise, we may enlarge the state space E to $\tilde{E} = E \cup \{d\}$ by joining a single point d with $d \notin E$ into E, and change $X(t,\omega)$ into non-interruptive process $\tilde{X}(t,\omega)$ on $\tilde{E}$. It does not affect all conclusions in this paper.

Let $\alpha(\omega)$ be a random variable which might be ∞. In order to show that x_α is well defined when $\alpha = \infty$, choose a random variable $\beta(\omega)$ valued in $(E, \mathcal{E})$, and define $x_\infty(\omega) \triangleq \beta(\omega)$. Then $x_{\alpha(\omega)}(\omega)$ is well defined for all $\omega \in \Omega$. Now, define $\mathcal{F}(x_\alpha)$ as

$$\mathcal{F}(x_\alpha) \triangleq \{\{\omega:\ (x_{\alpha(\omega)}(\omega), \alpha(\omega)) \in A\}:\ A \in \mathcal{E} \times \mathcal{B}([0,\infty])\}, \tag{1.1}$$

where $\mathcal{B}([0,\infty])$ is a *Borel* σ-algebra generated by $[0,\infty]$. $\{(x,s)\}$ is an atom of $\mathcal{E} \times \mathcal{B}([0,\infty])$ for every $x \in E$, $s \in [0,\infty]$, namely, $\{(x,s)\} \in \mathcal{E} \times \mathcal{B}([0,\infty])$, and does not contain any proper subsets of $\mathcal{E} \times \mathcal{B}([0,\infty])$. It follows that $\{x_{\alpha(\omega)}(\omega) = x\} \cap \{\alpha(\omega) = s\}$ is an atom of $\mathcal{F}(x_\alpha)$. Since $\alpha(\omega)\colon \Omega \to \bar{\mathbb{R}}^+ \triangleq [0,\infty]$ is a mapping from Ω to $\bar{\mathbb{R}}^+$, and $x_t(\omega)\colon \Omega \to E$ is also a mapping from Ω to E for every fixed

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$t \geq 0$, it follows that $x_{\alpha(\omega)}(\omega)$ is a mapping from Ω to $E \times \bar{\mathbb{R}}^+$. Note that $\mathcal{E} \times \mathcal{B}([0,\infty])$ is a σ-algebra, therefore $\mathcal{F}(x_\alpha)$ is a σ-algebra by [2, Property 2.2.2].

Definition 1.1
$\mathcal{F}(x_\alpha)$ is called the σ-algebra generated by $x_{\alpha(\omega)}(\omega)$.

The core of the Markov process is the Markov property which is the base of theoretic and applied research on Markov process. But we often need a stronger property: "the strong Markov property". We know that "present" in the explanation of the Markov property is a fixed time t which has nothing to do with ω. But in many problems, "present" is required to be a random time $\alpha(\omega)$ which may take different values according to different ω, such as hitting time. Let $\eta_A(\omega)$ be the hitting time of $A \in \mathcal{E}$. Whether $X(t,\omega)$ satisfies Markov property at time $\eta_A(\omega)$. Note that $\eta_A(\omega)$ depends on ω. So the strong Markov property is distinct from the Markov property.

More precisely, this problem is explained as follows: Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on $(E, \mathcal{E})$, that is, for any *Borel* subset B of $(-\infty, \infty)$, we have

$$\{x:\ f(x) \in B\} \in \mathcal{E}. \tag{1.2}$$

Let $\alpha(\omega)$ be a random variable. Does the following equality

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|\mathcal{F}(x_\alpha)], \quad P_{\Omega_\alpha}\text{–a.e.}$$

hold? Here $\mathcal{N}_\alpha^+$ is a σ-algebra prior to α generated by $X(t,\omega)$, which is defined in Section 2.1; $\Omega_\alpha = \{\omega:\ \alpha(\omega) < \infty\}$; $\mathcal{F}(\cdot)$ denotes the smallest σ-algebra on Ω generated by all sets of bracket.

In order to prove (1.2), many scholars made great efforts, and obtained many fine results. The first one who thought (1.2) should be seriously proven is Doob (1945). To make (1.2) hold, what should we do?

(1) What restricted conditions should $\alpha(\omega)$ have?

(2) How to define the σ-algebra prior to $\alpha(\omega)$ so that it includes the special case $\alpha(\omega) \equiv$ constant?

(3) How to define the function $f(x)$ so that $f(x_t)$ is a random variable and $E[f(x_t)|\mathcal{F}(x_\alpha)]$ is $\mathcal{N}_\alpha^+$-measurable?

The questions above were mentioned in [1, P106].

2. σ-algebra prior to $\alpha(\omega)$ and its properties

2.1. The definition of the σ-algebra prior to $\alpha(\omega)$

Recall the σ-algebras $\mathcal{N}_T \triangleq \mathcal{F}(x_s(\omega);\ s < T)$ and $\mathcal{N}_T^+ \triangleq \mathcal{F}(x_s(\omega);\ s \leq T)$, generated by the trajectory of $X(t,\omega)$ prior to T, are defined by

$$\mathcal{N}_T \triangleq \mathcal{F}(x_s(\omega);\ s < T) \triangleq \mathcal{F}\Big(\bigcup_{s<T} x_s^{-1}(\mathcal{E})\Big) \tag{2.1}$$

and

$$\mathcal{N}_T^+ \triangleq \mathcal{F}(x_s(\omega);\ s \leq T) \triangleq \mathcal{F}\Big(\bigcup_{s\leq T} x_s^{-1}(\mathcal{E})\Big), \tag{2.2}$$

respectively. In particular, taking $T = \infty$, we have

$$\mathcal{N}_\infty = \mathcal{F}\Big(\bigcup_{s<\infty} x_s^{-1}(\mathcal{E})\Big) \qquad \text{and} \qquad \mathcal{N}_\infty^+ = \mathcal{F}\Big(\bigcup_{s\leq\infty} x_s^{-1}(\mathcal{E})\Big).$$

Here $x_s^{-1}(\mathcal{E}) \triangleq \{\{x_s(\omega) \in B\} :\ B \in \mathcal{E}\}$. Intuitively, $\mathcal{F}(x_s(\omega);\ s < T)$ or $\mathcal{F}(x_s(\omega);\ s \leq T)$ is the σ-algebra generated by the stochastic process prior to T of $X(t,\omega)$, that is, generated by the two stochastic precesses $(x_s(\omega);\ s < T)$ and $(x_s(\omega);\ s \leq T)$, respectively. Of course, here T is a constant that has nothing to do with ω. How to define the $\mathcal{N}_{\alpha(\omega)}$ and $\mathcal{N}_{\alpha(\omega)}^+$ if α is a random variable? Similarly to the way of defining $\mathcal{N}_T$ and $\mathcal{N}_T^+$, they are defined as follows: Let $y_t(\omega) = x_t(\omega)$ if $t < \alpha(\omega)$; $\bar{y}_t(\omega) = x_t(\omega)$ if $t \leq \alpha(\omega)$. Put $Y(t,\omega) = \{y_t(\omega);\ t \geq 0\}$ and $\bar{Y}(t,\omega) \triangleq \{\bar{y}_t(\omega);\ t \geq 0\}$. Then they satisfy:

$$Y(t,\omega) = (X(t,\omega);\ t < \alpha(\omega))$$

and

$$\bar{Y}(t,\omega) = (X(t,\omega);\ t \leq \alpha(\omega)).$$

That is, $\{y_t(\omega) \in B\} = \{x_t(\omega) \in B,\ t < \alpha(\omega)\}$ and $\{\bar{y}_t(\omega) \in B\} = \{x_t(\omega) \in B,\ t \leq \alpha(\omega)\}$ for any $t \geq 0$ and $B \in \mathcal{E}$, where when $t = \infty$, $\{x_t(\omega) \in B,\ t < \alpha(\omega)\} = \emptyset$ and $\{x_t(\omega) \in B,\ t \leq \alpha(\omega)\} = \{\beta(\omega) \in B,\ \alpha(\omega) = \infty\}$, respectively. By the definition of a stochastic process, $Y(t,\omega)$ and $\bar{Y}(t,\omega)$ are two stochastic processes prior to $\alpha(\omega)$ of $x(t,\omega)$, that is, the two processes end at time $t < \alpha(\omega)$ and $t \leq \alpha(\omega)$, respectively.

From (2.1), (2.2) it follows that the σ-algebras prior to α of $X(t,\omega)$ are defined by

$$\mathcal{N}_\alpha \triangleq \mathcal{F}(y_t(\omega);\ t < \infty) \triangleq \mathcal{F}\Big(\bigcup_{t<\infty} y_t^{-1}(\mathcal{E})\Big) \tag{2.3}$$

and

$$\mathcal{N}_\alpha^+ \triangleq \mathcal{F}(\bar{y}_t(\omega);\ t \leq \infty) \triangleq \mathcal{F}\Big(\bigcup_{t\leq\infty} \bar{y}_t^{-1}(\mathcal{E})\Big), \tag{2.4}$$

respectively. When $t = \infty$, $\bar{y}_t^{-1}(\mathcal{E})$ is defined by $\bar{y}_t^{-1}(\mathcal{E}) \triangleq \{\{\omega :\ \beta(\omega) \in B,\ \alpha(\omega) = \infty\} :\ B \in \mathcal{E}\}$. Obviously, if $\alpha(\omega) \equiv T$ (constant), $\mathcal{N}_\alpha = \mathcal{N}_T$ and $\mathcal{N}_\alpha^+ = \mathcal{N}_T^+$.

Definition 2.1
$\mathcal{N}_\alpha$ and $\mathcal{N}_\alpha^+$ defined by (2.3) and (2.4), respectively, are called σ-algebras prior to α of $X(t,\omega)$.

2.2. The properties of σ-algebra prior to $\alpha(\omega)$

We now discuss the properties of $\mathcal{N}_\alpha$ and $\mathcal{N}_\alpha^+$, which are the foundations of studying the strong Markov property.

Theorem 2.2

$$\mathcal{F}(\alpha) \subseteq \mathcal{N}_\alpha; \qquad \mathcal{F}(\alpha) \subseteq \mathcal{N}_\alpha^+.$$

Proof. The proofs of both statements are similar, we only prove the first relation. Since $\{x_s(\omega) \in E\} = \Omega$, we have

$$\{\alpha(\omega) > s\} = \{x_s(\omega) \in E,\ \alpha(\omega) > s\} = \{y_s(\omega) \in E\} \in \mathcal{N}_\alpha.$$

It is well known that $\mathcal{F}(\alpha) = \mathcal{F}(\alpha(\omega) > s;\, s \geq 0)$. Hence, the theorem is valid.

Theorem 2.3
Let

$$\begin{aligned}\Pi = \{\{&x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha > s\} :\\ &n \geq 1;\, t_1 \leq \ldots \leq t_n \leq s;\, A_1, \ldots, A_n \in \mathcal{E}\};\\ \Pi^+ = \{\{&x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha \geq s\} :\\ &n \geq 1;\, t_1 \leq \ldots \leq t_n \leq s \leq \infty;\, A_1, \ldots, A_n \in \mathcal{E}\},\end{aligned}$$

where for $s = \infty$, $\{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha > s\} = \emptyset$ and $\{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha \geq s\} = \{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha = \infty\}$. Then

$$\mathcal{F}(\Pi) = \mathcal{N}_\alpha \qquad \textit{and} \qquad \mathcal{F}(\Pi^+) = \mathcal{N}_\alpha^+.$$

Proof. $\{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha > s\} = \{y_{t_1} \in A_1, \ldots, y_{t_n} \in A_n,\, y_s \in E\} \in \mathcal{N}_\alpha$, hence, $\mathcal{F}(\Pi) \subseteq \mathcal{N}_\alpha$. Again, for every $t \geq 0$ and $A \in \mathcal{E}$, obviously, $\{y_t \in A\} = \{x_t \in A,\ \alpha > t\} \in \Pi$. Therefore, $\mathcal{N}_\alpha = \mathcal{F}(\bigcup_{t<\infty} y_t^{-1}(\mathcal{E})) \subseteq \mathcal{F}(\Pi)$, from which and above it follows that $\mathcal{F}(\Pi) = \mathcal{N}_\alpha$. Similarly as above we obtain $\mathcal{F}(\Pi^+) = \mathcal{N}_\alpha^+$.

Theorem 2.4
Let $\alpha(\omega)$ be a nonnegative random variable. Then

$$\mathcal{N}_\alpha \subseteq \mathcal{N}_\alpha^+.$$

Proof. For any $t_1 \leq t_2 \leq \ldots \leq t_m \leq s < t$, from $\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m,\, \alpha \geq t\} \in \mathcal{N}_\alpha^+$ we get

$$\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m,\, \alpha > s\} = \lim_{t \downarrow s} \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\} \in \mathcal{N}_\alpha^+.$$

Here $\lim_{t\downarrow s}\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\}$ is defined by $\bigcup_{n=1}^\infty \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq a_n\}$ for an arbitrary sequence of number $\{a_n\}_{n\geq 1}$,

$a_n \downarrow s$ as $n \uparrow \infty$. We easily verify that $\lim_{t\downarrow s}\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\}$ has nothing to do with the chosen $\{a_n\}_{n\geq 1}$. Hence, by Theorem 2.3, the theorem is proven.

THEOREM 2.5
Let $\alpha(\omega)$ be a stopping time with respect to $\mathcal{N}_t^+$, that is, $\{\alpha \leq t\} \in \mathcal{N}_t^+$ for every $t \geq 0$. Then $A \cap \{\alpha \leq t\} \in \mathcal{N}_t^+$ and $A \cap \{\alpha < t\} \in \mathcal{N}_t^+$ for every $A \in \mathcal{N}_\alpha^+$.

Proof. Suppose that A has the following shape

$$A = \{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\ \alpha \geq s\}$$

for any $n \geq 1$ and $t_1 \leq \ldots \leq t_n \leq s$ and $A_1, \ldots, A_n \in \mathcal{E}$. Obviously, $A \cap \{\alpha \leq t\} = \{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \cap \{s \leq \alpha \leq t\} \in \mathcal{N}_t^+$. So, by λ-π-system method and Theorem 2.3, the first assertion is obtained. Again, $A \cap \{\alpha < t\} = \lim_{u\uparrow t} A \cap \{\alpha \leq u\} \in \mathcal{N}_t^+$, which is the other assertion.

3. σ-algebra post-$\alpha(\omega)$ and its properties

3.1. The definition of the σ-algebra post-$\alpha(\omega)$

Let $w_t(\omega) = x_t(\omega)$ if $\alpha(\omega) < t$ and $\bar{w}_t(\omega) = x_t(\omega)$ if $\alpha(\omega) \leq t$. Set $W(t,\omega) \triangleq \{w_t(\omega);\ t \geq 0\} = (X(t,\omega); \alpha(\omega) < t)$. $\bar{W}(t,\omega) \triangleq \{\bar{w}_t(\omega);\ t \geq 0\} = (X(t,\omega); \alpha(\omega) \leq t)$. That is, $\{w_t(\omega) \in B\} = \{x_t(\omega) \in B,\ \alpha(\omega) < t\}$ and $\{\bar{w}_t(\omega) \in B\} = \{x_t(\omega) \in B,\ \alpha(\omega) \leq t\}$ for any $t \geq 0$ and $B \in \mathcal{E}$. Here for $t = \infty$, $\{x_t(\omega) \in B,\ \alpha(\omega) < t\} = \{\beta(\omega) \in B,\ \alpha(\omega) < \infty\}$ and $\{x_t(\omega) \in B,\ \alpha(\omega) \leq t\} = \{\beta(\omega) \in B\}$, respectively. We adjoin a point Δ with $\Delta \notin E$ to E to expand E into $\hat{E} = E \cup \{\Delta\}$, and set $\hat{\mathcal{E}} \triangleq \mathcal{F}(\mathcal{E},\ \{\Delta\})$. Let

$$\tilde{w}_t(\omega) \triangleq \begin{cases} w_t(\omega), & t > \alpha(\omega), \\ \Delta, & t \leq \alpha(\omega) \end{cases} = \begin{cases} x_t(\omega), & t > \alpha(\omega), \\ \Delta, & t \leq \alpha(\omega); \end{cases}$$

$$\tilde{\bar{w}}_t(\omega) \triangleq \begin{cases} \bar{w}_t(\omega), & t \geq \alpha(\omega), \\ \Delta, & t < \alpha(\omega) \end{cases} = \begin{cases} x_t(\omega), & t \geq \alpha(\omega), \\ \Delta, & t < \alpha(\omega). \end{cases}$$

Then $\tilde{W}(t,\omega) \triangleq \{\tilde{w}_t(\omega);\ t \geq 0\}$ and $\tilde{\bar{W}}(t,\omega) \triangleq \{\tilde{\bar{w}}_t(\omega);\ t \geq 0\}$ are changed into non-interruptive processes on $(\hat{E}, \hat{\mathcal{E}})$, respectively. The state Δ is the starting point of $\tilde{W}(t,\omega)$ and $\tilde{\bar{W}}(t,\omega)$, that is, for all $\omega \in \Omega$, $\tilde{W}(t,\omega)$ and $\tilde{\bar{W}}(t,\omega)$ start from state Δ, and stay time at Δ is $\alpha(\omega)$, then the ω enter into E to move according to the primary trajectory. The σ-algebras post-α ${}_\alpha\mathcal{N}$ and ${}_\alpha\mathcal{N}^+$ are defined by

$$ {}_\alpha\mathcal{N} \triangleq \mathcal{F}(\tilde{w}_t(\omega); t \leq \infty) \triangleq \mathcal{F}\Bigg(\bigcup_{t\leq\infty} \tilde{w}_t^{-1}(\mathcal{E})\Bigg) \tag{3.1}$$

and

$$ {}_\alpha\mathcal{N}^+ \triangleq \mathcal{F}(\tilde{\bar{w}}_t(\omega); t \leq \infty) \triangleq \mathcal{F}\Bigg(\bigcup_{t\leq\infty} \tilde{\bar{w}}_t^{-1}(\mathcal{E})\Bigg), \tag{3.2}$$

respectively. Here when $t = \infty$, $\tilde{w}_t^{-1}(\mathcal{E})$ and $\tilde{\bar{w}}_t^{-1}(\mathcal{E})$ are defined by $\tilde{w}_t^{-1}(\mathcal{E}) \triangleq$

$\{\{\beta \in B,\, \alpha < \infty\}:\ B \in \mathcal{E}\}$ and $\tilde{\bar{w}}_t^{-1}(\mathcal{E}) \triangleq \{\{\beta \in B\}:\ B \in \mathcal{E}\}$, respectively. By the definition of $\mathcal{F}(\cdot)$ on Ω, obviously,

$$_\alpha\mathcal{N} = \mathcal{F}\Big(\bigcup_{t\le\infty} w_t^{-1}(\hat{\mathcal{E}})\Big); \qquad _\alpha\mathcal{N}^+ = \mathcal{F}\Big(\bigcup_{t\le\infty} \bar{w}_t^{-1}(\hat{\mathcal{E}})\Big).$$

Definition 3.1
$_\alpha\mathcal{N}$ and $_\alpha\mathcal{N}^+$ defined by (3.1) and (3.2) are called σ-algebras post-α of $X(t,\omega)$, respectively.

Intuitively, $_\alpha\mathcal{N}$ or $_\alpha\mathcal{N}^+$ is the σ-algebra generated by the stochastic process post-α of $X(t,\omega)$.

3.2. The properties of the σ-algebra post-$\alpha(\omega)$

Similarly to the proof of Theorem 2.2 we obtain the following theorem.

Theorem 3.2

$$\mathcal{F}(\alpha) \subseteq {_\alpha\mathcal{N}}; \qquad \mathcal{F}(\alpha) \subseteq {_\alpha\mathcal{N}^+}.$$

Theorem 3.3

$$\mathcal{F}(\alpha) \subseteq \mathcal{F}(x_\alpha).$$

Proof. Since $\{x_{\alpha(\omega)}(\omega) \in E\} = \Omega$, from (1.1), it follows that $\{\alpha \in B\} \in \mathcal{F}(x_\alpha)$.

Theorem 3.4
Let

$$\Gamma = \{\{\alpha < s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\}:\ n \ge 1,\, s \le t_1 \le \dots \le t_n,\, A_1, \dots, A_n \in \mathcal{E}\}$$

and

$$\Gamma^+ = \{\{\alpha \le s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\}:\ n \ge 1,\, s \le t_1 \le \dots \le t_n,\\ A_1, \dots, A_n \in \mathcal{E}\},$$

where when $s = \infty$, $\{\alpha < s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} = \{\alpha < \infty,\, \beta \in A_1, \dots, \beta \in A_n\}$ and $\{\alpha \le s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} = \{\beta \in A_1, \dots, \beta \in A_n\}$. Then

$$\mathcal{F}(\Gamma) = {_\alpha\mathcal{N}} \qquad \textit{and} \qquad \mathcal{F}(\Gamma^+) = {_\alpha\mathcal{N}^+}.$$

Proof. The proof is analogous to the proof of Theorem 2.3.

Theorem 3.5

$$_\alpha\mathcal{N} \subseteq {_\alpha\mathcal{N}^+}.$$

Proof. By Theorem 3.4, $\{\alpha \le u,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} \in {_\alpha\mathcal{N}^+}$. So

$$\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} = \lim_{u\uparrow s}\{\alpha \le u,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \in {}_\alpha\mathcal{N}^+,$$

where $\lim_{u\uparrow s}\{\alpha \le u,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\}$ is defined by $\bigcup_{i=1}^{\infty}\{\alpha \le a_i,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\}$ for any sequence of number $\{a_n\}_{n\ge1}$, $a_n \uparrow s$ as $n \uparrow \infty$. When $s = \infty$, $\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} = \{\alpha < \infty,\, \beta \in A_1, \ldots, \beta \in A_n\} \in {}_\alpha\mathcal{N}^+$ from the definition of ${}_\alpha\mathcal{N}^+$ and Theorem 3.2. So $\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \in {}_\alpha\mathcal{N}^+$ for every $s \le \infty$. By Theorem 3.4 the proof is accomplished.

Theorem 3.6
Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in measurable space $(E, \mathcal{E})$. Then

$$\mathcal{F}(x_\alpha) \subseteq \mathcal{N}_\alpha^+; \qquad \mathcal{F}(x_\alpha) \subseteq {}_\alpha\mathcal{N}^+.$$

Proof. For any $A \in \mathcal{E}$, obviously,

$$\begin{aligned}\{\omega : x_{\alpha(\omega)}(\omega) \in A\} &= \{\omega : x_{\alpha(\omega)}(\omega) \in A\} \cap \{\omega : \alpha(\omega) \le \infty\}\\ &= \bigcup_{s<\infty}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \qquad (3.3)\\ &\quad + \{\omega : \beta(\omega) \in A\} \cap \{\omega : \alpha(\omega) = \infty\}.\end{aligned}$$

By Theorem 2.3 and Theorem 2.4, for every $s \ge 0$,

$$\begin{aligned}&\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}\\ &\quad = \{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) \ge s\} \qquad (3.4)\\ &\qquad - \{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) > s\} \in \mathcal{N}_\alpha^+.\end{aligned}$$

Now we prove

$$\bigcup_{s<\infty}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \in \mathcal{N}_\alpha^+ \qquad (3.5)$$

by virtue of transfinite induction. Suppose that $\preceq$ is well ordering on $[0, \infty)$ with the first element a_0. By (3.4),

$$\begin{aligned}&\bigcup_{s\preceq a_0}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\})\\ &\qquad = \{\omega : x_{a_0}(\omega) \in A\} \cap \{\omega : \alpha(\omega) = a_0\} \in \mathcal{N}_\alpha^+.\end{aligned}$$

Suppose that

$$\bigcup_{s\preceq a}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \in \mathcal{N}_\alpha^+$$

for any a with $a \prec T$. Chosen an increasing sequence $\{a_i : \ i \ge 1,\, a_i \prec T\}$ satisfying that for any given number $t \prec T$, there exists a_i such that $t \preceq a_i$. So

$$\begin{aligned}&\bigcup_{s\prec T}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\})\\ &\qquad = \bigcup_{i=1}^{\infty}\left[\bigcup_{s\preceq a_i}(\{\omega : x_s \in A\} \cap \{\omega : \alpha = s\})\right] \in \mathcal{N}_\alpha^+.\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigcup_{s\preceq T}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\
&\qquad=\bigcup_{s\prec T}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\
&\qquad\quad+\{\omega:\ x_T(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=T\}\in\mathcal{N}_\alpha^+.
\end{aligned}\tag{3.6}$$

Transfinite induction implies that (3.6) holds for any $T\in[0,\infty)$. Again, take an increasing sequence $\{T_i:\ i\geq 1\}$ satisfying that for any given number $s\in[0,\infty)$, there exists a T_i such that $s\preceq T_i$. So

$$\begin{aligned}
&\bigcup_{s<\infty}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\
&\qquad=\bigcup_{i=1}^{\infty}\left[\bigcup_{s\preceq T_i}(\{\omega:\ x_s\in A\}\cap\{\omega:\ \alpha=s\})\right]\in\mathcal{N}_\alpha^+.
\end{aligned}$$

Again, $\{\omega:\ \beta(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=\infty\}\in\mathcal{N}_\alpha^+$ from the definition of $\mathcal{N}_\alpha^+$. Hence, by (3.3), $\{\omega:\ x_{\alpha(\omega)}(\omega)\in A\}\in\mathcal{N}_\alpha^+$. By Theorem 2.2,

$$\{\omega:\ (x_\alpha(\omega),\alpha(\omega))\in A\times B\}=\{x_{\alpha(\omega)}(\omega)\in A\}\cap\{\alpha(\omega)\in B\}\in\mathcal{N}_\alpha^+$$

for every $B\in\mathcal{B}([0,\infty])$. Note that $\mathcal{E}\times\mathcal{B}([0,\infty])=\mathcal{F}(A\times B;\ A\in\mathcal{E},\ B\in\mathcal{B}([0,\infty])$, and $\{A\times B;\ A\in\mathcal{E},\ B\in\mathcal{B}([0,\infty])\}$ is a π-system. So, by λ-π-system method, it follows that $\mathcal{F}(x_\alpha)\subseteq\mathcal{N}_\alpha^+$. Again, $\{\omega:\ \beta(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=\infty\}\in{}_\alpha\mathcal{N}^+$ from the definition of ${}_\alpha\mathcal{N}^+$ and Theorem 3.2. Similarly to the proof of the fact $\mathcal{F}(x_\alpha)\subseteq\mathcal{N}_\alpha^+$, we get $\mathcal{F}(x_\alpha)\subseteq{}_\alpha\mathcal{N}^+$.

4. The strong Markov property

Suppose that Θ_s denotes the shift operator, that is

$$\Theta_s(f(t_1+s,\ldots,t_n+s))=f(t_1,\ldots,t_n)$$

for any natural number n and function $f(t_1,\ldots,t_n)$ of n-variables defined on n-dimensional real number space $\mathbb{R}^n$. Generally, if $s=s(\omega)$ is a function of ω, then, for $|s(\omega)|<\infty$, $\Theta_{s(\omega)}$ denotes the shift as follows:

$$\begin{aligned}
\Theta_{s(\omega)}(f(t_1,\ldots,t_n))&=f(t_1-s(\omega),\ldots,t_n-s(\omega))\\
&=\sum_{-\infty<u<\infty}f(t_1-u,\ldots,t_n-u)\mathcal{X}_{\{s(\omega)=u\}}.
\end{aligned}$$

More generally, if $f=f(t_1,\ldots,t_n,x_{s(\omega)}(\omega))$ is also a function of $x_{s(\omega)}(\omega)$, then, for $|s(\omega)|<\infty$, $\Theta_{s(\omega)}$ denotes the shift as follows:

$$\begin{aligned}
&\Theta_{s(\omega)}(f(t_1,\ldots,t_n,x_{s(\omega)}(\omega)))\\
&\quad=f(t_1-s(\omega),\ldots,t_n-s(\omega),x_{s(\omega)}(\omega))\\
&\quad=\sum_{-\infty<u<\infty}\sum_{x\in E}f(t_1-u,\ldots,t_n-u,x)\mathcal{X}_{\{s(\omega)=u,x_{s(\omega)}(\omega)=x\}}(\omega),
\end{aligned}$$

where $\mathcal{X}_{\{s(\omega)=u,x_{s(\omega)}(\omega)=x\}}(\omega)$ is an indicator relative to $\{s(\omega) = u,\, x_{s(\omega)}(\omega) = x\}$, that is, $X_{\{s(\omega)=u,\, x_{s(\omega)}(\omega)=x\}}(\omega) = 1$ if $\omega \in \{s(\omega) = u,\, x_{s(\omega)}(\omega) = x\}$ and $X_{\{s(\omega)=u,\, x_{s(\omega)}(\omega)=x\}}(\omega) = 0$ otherwise.

Definition 4.1
$X(t, \omega)$ is called a homogeneous Markov process if

$$p(s, t + s; x, A) = p(0, t; x, A),$$

where $p(s, t; x, A)$ is the transition probability function of $X(t, \omega)$.

Let $\omega_0 \in \{x_s = x\}$. For an arbitrary $\mathcal{E}$-measurable bounded real-valued function $f(x)$, by [2, Theorem 5.2.5], $E[f(x_{t+s})|x_s](\omega_0)(\overset{\triangle}{=} E[f(x_{t+s})|\mathcal{F}(x_s)](\omega_0))$ may be denoted by $K(s, t + s; x, f(x_{t+s}))$. So $E[f(x_{t+s})|x_s](\omega)$ may be denoted as $\sum_{x\in E} K(s, t + s; x, f(x_{t+s}))\mathcal{X}_{\{x_s=x\}}(\omega) = K(s, t + s; x_s(\omega), f(x_{t+s}))$.

Definition 4.1'
$X(t, \omega)$ is called a homogeneous Markov process if for an arbitrary $\mathcal{E}$-measurable bounded real-valued function $f(x)$, such that

$$E[f(x_{t+s})|x_s] = \Theta_s E[f(x_{t+s})|x_s] = K(0, t; x_s(\omega), f(x_t)) \overset{\triangle}{=} E_{x_s}[f(x_t)]. \qquad (4.1)$$

Let $f = \mathcal{X}_A(x), A \in \mathcal{F}$. By Markov property, (4.1) holds if and only if

$$\begin{aligned} E[\mathcal{X}_A(x_{t+s})|x_s](\omega) &= p(s, t + s; x_s(\omega), A) = \sum_{x\in E} p(s, t + s; x, A)\mathcal{X}_{\{x_s=x\}}(\omega) \\ &= \sum_{x\in E} p(0, t; x, A)\mathcal{X}_{\{x_s=x\}}(\omega) \\ &= p(0, t; x_s(\omega), A), \quad P_{\mathcal{F}(x_s)}\text{–a.e..} \end{aligned}$$

So, by $\mathcal{L}$-system method (Appendix B, Theorem B.5), it follows that the two definitions are equivalent.

Lemma 4.2
$f(x_t)$ is $x_t^{-1}(\mathcal{E})$-measurable real-valued function if and only if $f(x)$ is a $\mathcal{E}$-measurable real-valued function defined on a measurable space $(E, \mathcal{E})$. So $f(x_t)$ is a random variable if $f(x)$ is a $\mathcal{E}$-measurable real-valued function defined on a measurable space $(E, \mathcal{E})$.

Proof. Let $g(\omega) \overset{\triangle}{=} x_t(\omega)$. Then $g(\omega)$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ for any fixed $t \geq 0$. If we rewrite $f(x_t(\omega))(\omega) \overset{\triangle}{=} f \circ g(\omega)$, from [2, Theorem 2.2.13] $f(x_t(\omega))$ is a $x_t^{-1}(\mathcal{E})$-measurable (so it is also $\mathcal{F}$-measurable) mapping from Ω to $\bar{\mathbb{R}}(\overset{\triangle}{=} \mathbb{R} \cup \{\infty\})$ if and only if there exists a $\mathcal{E}$-measurable real-valued function $f(x)$ such that $f(x_t(\omega)) = f \circ g(\omega)$.

LEMMA 4.3
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$. Put $z_s(\omega) \triangleq E[f(x_{t+s})|x_s] = K(s,t+s;x_s(\omega),f(x_{t+s}))$. Set

$$H_s = \{z_s(\omega)\}, \qquad H = \bigcup_{s\le\infty} H_s.$$

Let $\mathcal{B}(H)$ denote the σ-algebra generated by all Borel subsets in H. Then:

(1) *$Z(s,\omega) \triangleq \{z_s(\omega):\ s\ge 0\}$ is a stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(H,\mathcal{B}(H))$.*

(2) *$Z(s,\omega)$ is a martingale relative to σ-algebra filtration $\{\mathcal{N}_s^+;\ 0\le s\le t\}$.*

Proof. (1) First, we prove $z_s(\omega)$ is a random variable for any fixed s. $z_s(\omega)\colon \Omega \to \bar{\mathbb{R}} \triangleq \{\infty\}\cup\mathbb{R}$ is a $\mathcal{F}(x_s)$-measurable real-valued function (here assume without loss of generality that the mathematical expectation may only value $+\infty$) by the definition of conditional mathematical expectation, namely, for every *Borel* subset A of $\bar{\mathbb{R}}$,

$$\{\omega:\ z_s(\omega)\in A\} \in x_s^{-1}(\mathcal{E}) \subseteq \mathcal{F}. \tag{4.2}$$

Let $\mathcal{B}(\bar{\mathbb{R}})$ be the *Borel* σ-algebra generated by $\mathbb{R}\cup\{\infty\}$. Then $\mathcal{B}(H)\subseteq\mathcal{B}(\bar{\mathbb{R}})$, from which and (4.2) it follows that

$$\{\omega:\ z_s(\omega)\in A\}\in\mathcal{F} \tag{4.3}$$

for every $A\in\mathcal{B}(H)$. From (4.3) it follows $z_s(\omega)$ is a random variable valued in a measurable space $(H,\mathcal{B}(H))$ for every fixed $s\ge 0$. Therefore, $Z(s,\omega)$ is a stochastic process valued in a measurable space $(H,\mathcal{B}(H))$ from the definition of stochastic process.

(2) Since $Z(s,\omega)=E[f(x_t)|\mathcal{N}_s^+](\omega)$ by Markov property, for any $s\le u$,

$$E[Z(u,\omega)|\mathcal{N}_s^+] = E\{E[f(x_t)|\mathcal{N}_u^+]|\mathcal{N}_s^+\} = E[f(x_t)|\mathcal{N}_s^+] = Z(s,\omega), \quad P_{\mathcal{N}_s^+}\text{–a.e.},$$

from which it follows (2) is valid.

Note. $z_s(\omega)$ is also regarded as a composite mapping with $x_s(\omega)$ as intermediate variable and ω as independent variable.

LEMMA 4.4
Let $\alpha(\omega)$ be an arbitrary nonnegative random variable. Set

$$\alpha^{(n)}(\omega) = \sum_{k=1}^{n2^n} \frac{k}{2^n}\mathcal{X}_{\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha>n\}}(\omega);$$

$$\alpha_-^{(n)}(\omega) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\mathcal{X}_{\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}}(\omega) + n\mathcal{X}_{\{\alpha>n\}}(\omega).$$

Then

(1) $\alpha^{(n)}(\omega) \downarrow \alpha(\omega)$, $\alpha_-^{(n)}(\omega) \uparrow \alpha(\omega)$ *as* $n \uparrow \infty$,

(2) $\mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha^{(n+1)})$, $\mathcal{F}(\alpha_-^{(n)}) \subseteq \mathcal{F}(\alpha_-^{(n+1)})$ *for every* $n \geq 1$,

(3) $\mathcal{F}(\alpha) = \mathcal{F}(\alpha^{(\infty)}) = \mathcal{F}(\alpha_-^{(\infty)})$.

Here $\mathcal{F}(\alpha^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\alpha^{(n)}))$; $\mathcal{F}(\alpha_-^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\alpha_-^{(n)}))$.

For the convenience of representation, $[0, \frac{1}{2^n}]$ and $\mathcal{X}_{\{\frac{0}{2^n} \leq \alpha \leq \frac{1}{2^n}\}}$ are marked by $(0, \frac{1}{2^n}]$ and $\mathcal{X}_{\{\frac{0}{2^n} < \alpha \leq \frac{1}{2^n}\}}$, respectively throughout this paper.

Proof. (1) By the property of construction of measurable function it follows (1).

(2) Set $A_k^{(n)} = \{\frac{k-1}{2^n} < \alpha \leq \frac{k}{2^n}\}$ for every $k = 1, 2, \ldots, n2^n$, $A_{n2^n+1}^{(n)} = \{\alpha > n\}$. Obviously,

$$\mathcal{F}(\alpha^{(n)}) = \mathcal{F}\big(A_k^{(n)};\ 1 \leq k \leq n2^n + 1\big). \tag{4.4}$$

For $1 \leq k \leq n2^n$,

$$A_k^{(n)} = \Big\{\frac{2(k-1)}{2^{n+1}} < \alpha \leq \frac{2k-1}{2^{n+1}}\Big\} + \Big\{\frac{2k-1}{2^{n+1}} < \alpha \leq \frac{2k}{2^{n+1}}\Big\} \in \mathcal{F}(\alpha^{(n+1)}), \tag{4.5}$$

$$\{\alpha > n\} = \sum_{k=n2^{n+1}+1}^{(n+1)2^{n+1}+1} A_k^{(n+1)} \in \mathcal{F}(\alpha^{(n+1)}). \tag{4.6}$$

The first assertion of (2) follows from (4.4)–(4.6). Similarly we get the second assertion of (2).

(3) Obviously, $\mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha)$ for every $n \geq 1$, hence, $\mathcal{F}(\alpha^{(\infty)}) \subseteq \mathcal{F}(\alpha)$. Next we prove $\mathcal{F}(\alpha) \subseteq \mathcal{F}(\alpha^{(\infty)})$. It is well known that $\mathcal{F}(\alpha) = \mathcal{F}(\{\alpha \geq s\}; s \geq 0)$. Hence, it is sufficient to prove $\{\alpha \geq s\} \in \mathcal{F}(\alpha^{(\infty)})$ for any $s \geq 0$. Set $a_n = \min(s - \frac{k}{2^n};\ s - \frac{k}{2^n} \geq 0,\ 1 \leq k \leq n2^n)$ and $K_n = s - a_n$, Obviously, $(K_n, \infty) \downarrow [s, \infty)$ as $n \uparrow \infty$, where $(K_n, \infty) \downarrow [s, \infty)$ is defined by $\bigcap_{n=1}^{\infty}(K_n, \infty) = [s, \infty)$. Hence $\{\alpha \geq s\} = \bigcap_{n=1}^{\infty}\{\alpha > K_n\} \in \mathcal{F}(\alpha^{(\infty)})$ since $\{\alpha > K_n\} \in \mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha^{(\infty)})$. In the same manner as above it follows the rest part of (3).

The intuitive idea of Lemma 4.4 is that: the interval $[0, \infty)$ is partitioned into $n2^n + 1$ many pairwise disjoint little intervals $[0, \frac{1}{2^n}], (\frac{1}{2^n}, \frac{2}{2^n}], \ldots, (\frac{n2^n-1}{2^n}, n]$, (n, ∞). We then construct two simple function $\alpha^{(n)}(\omega)$ and $\alpha_-^{(n)}(\omega)$, whose values in every little interval are taken the maximum and the infimum values of $\alpha(\omega)$ in the corresponding little interval. (But $\alpha^{(n)}(\omega)$ take value $n+1$ in little interval (n, ∞)), respectively. We have the same conclusion as Lemma 4.4 if $[0, n] = \bigcup_{k=1}^{n2^n}(a_{k-1}^{(n)}, a_k^{(n)}]$ is an arbitrary partition of $[0, n]$ into a sequence of pairwise disjoint little intervals. This partition method is given a token "$B(2^{(n)})$", called it partition method "$B(2^{(n)})$". Let $d^{(n)} = \max_{1 \leq k \leq n2^n}\ (a_k^{(n)} - a_{k-1}^{(n)})$. $d^{(n)}$ is called the distance of $B(2^{(n)})$.

Lemma 4.4'
Let $\alpha(\omega)$ be a nonnegative random variable. For every $n \geq 1$, $[0,n]$ is partitioned into $n2^n$ many pairwise disjoint little intervals $[0, a_1^{(n)}], (a_1^{(n)}, a_2^{(n)}], \ldots, (a_{n2^n-1}^{(n)}, n] \triangleq (a_{n2^n-1}^{(n)}, a_{n2^n}^{(n)}]$, and these partitions satisfy the following conditions:

(a) *For every $n \geq 1$, every such a little interval of partition method "$B(2^{(n)})$" is equal to the sum of such two disjoint little intervals of partition method $B(2^{(n+1)})$.*

(b) $\lim_{n\to\infty} d^{(n)} = 0$.

Let

$$\bar{\alpha}^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_k^{(n)} \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha > n\}}(\omega);$$

$$\bar{\alpha}_-^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_{k-1}^{(n)} \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}}(\omega) + n\mathcal{X}_{\{\alpha > n\}}(\omega).$$

Then

(1) $\bar{\alpha}^{(n)}(\omega) \downarrow \alpha(\omega)$, $\bar{\alpha}_-^{(n)}(\omega) \uparrow \alpha(\omega)$ *as* $n \uparrow \infty$,

(2) $\mathcal{F}(\bar{\alpha}^{(n)}) \subseteq \mathcal{F}(\bar{\alpha}^{(n+1)})$, $\mathcal{F}(\bar{\alpha}_-^{(n)}) \subseteq \mathcal{F}(\bar{\alpha}_-^{(n+1)})$ *for every* $n \geq 1$,

(3) $\mathcal{F}(\alpha) = \mathcal{F}(\bar{\alpha}^{(\infty)}) = \mathcal{F}(\bar{\alpha}_-^{(\infty)})$.

Here $\mathcal{F}(\bar{\alpha}^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\bar{\alpha}^{(n)}))$; $\mathcal{F}(\bar{\alpha}_-^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\bar{\alpha}_-^{(n)}))$.

Lemma 4.5
Let $\alpha(\omega)$ be a nonnegative random variable. Then

(1) $\mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$ *for every* $n \geq 1$,

(2) $\mathcal{N}_{\bar{\alpha}_-^{(n)}} \subseteq \mathcal{N}_{\bar{\alpha}_-^{(n+1)}}$ *for every* $n \geq 1$,

(3) $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}} = \mathcal{N}_{\bar{\alpha}_-^{(\infty)}}$.

Proof. (1) Let $\Pi_n = \{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} : m \geq 1; t_1 \leq \ldots \leq t_m \leq s; A_1, \ldots, A_m \in \mathcal{E}\}$ for every $n = 1, 2, \ldots$. By Theorem 2.3 it follows that $\mathcal{N}_{\alpha_-^{(n)}} = \mathcal{F}(\Pi_n)$ for every $n = 1, 2, \ldots$. Suppose, without loss of generality, that $s \in (\frac{k-1}{2^n}, \frac{k}{2^n}]$. Then s must lie in either the interval $(\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$ or the interval $[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}]$.

If $s \in (\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$, then

$$\{\alpha_-^{(n)} > s\} = \left\{\alpha_-^{(n)} \geq \frac{k}{2^n}\right\} = \left\{\alpha > \frac{k}{2^n}\right\},$$

$$\{\alpha_-^{(n+1)} > s\} = \left\{\alpha_-^{(n+1)} \geq \frac{2k-1}{2^{n+1}}\right\} = \left\{\alpha > \frac{2k-1}{2^{n+1}}\right\}.$$

Hence,

$$\{\alpha_-^{(n+1)} > s\} = \{\alpha_-^{(n)} > s\} + \Big\{\alpha_-^{(n+1)} = \frac{2k-1}{2^{n+1}}\Big\},$$

from which it follows that

$$\begin{aligned}\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} &\cap \{\alpha_-^{(n)} > s\} \\ = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} &\cap \{\alpha_-^{(n+1)} > s\} \\ - \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} &\cap \Big\{\alpha_-^{(n+1)} = \frac{2k-1}{2^{n+1}}\Big\} \\ = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} &\cap \{\alpha_-^{(n+1)} > s\} \\ - \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} &\cap \Big\{s < \alpha_-^{(n+1)} \leq \frac{2k-1}{2^{n+1}}\Big\}.\end{aligned} \tag{4.7}$$

By Theorem 2.2 and Theorem 2.3 as $t_m \leq s < \frac{2k-1}{2^{n+1}}$ it follows

$$\begin{aligned}&\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \Big\{s < \alpha_-^{(n+1)} \leq \frac{2k-1}{2^{n+1}}\Big\} \\ &= \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n+1)} > s\} \cap \Big\{\alpha_-^{(n+1)} \leq \frac{2k-1}{2^{n+1}}\Big\} \\ &\in \mathcal{N}_{\alpha_-^{(n+1)}},\end{aligned}$$

this and (4.7) yield

$$\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}} \tag{4.8}$$

for every $s \in (\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$.

If $s \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})$, obviously,

$$\{\alpha_-^{(n+1)} > s\} = \Big\{\alpha_-^{(n+1)} \geq \frac{2k}{2^{n+1}}\Big\} = \Big\{\alpha > \frac{k}{2^n}\Big\} = \Big\{\alpha_-^{(n)} \geq \frac{k}{2^n}\Big\} = \{\alpha_-^{(n)} > s\},$$

from which it follows that

$$\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}} \tag{4.9}$$

for every $s \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})$.

If $s = \frac{k}{2^n}$, an analogous treatment of (4.8) implies

$$\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}}, \tag{4.10}$$

from (4.8)–(4.10) it follows that $\Pi_n \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$. Hence, by Theorem 2.3, we get $\mathcal{N}_{\alpha_-^{(n)}} = \mathcal{F}(\Pi_n) \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$ for every $n \geq 1$.

(2) By an analogous treatment of (1) we complete the proof of (2).

(3) Obviously, $\mathcal{N}_{\alpha_-^{(\infty)}} \subseteq \mathcal{N}_\alpha$ by $\mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_\alpha$ for every n. Next we prove $\mathcal{N}_\alpha \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$. Set $a_n = \min(\frac{k}{2^n} - s : \frac{k}{2^n} - s \geq 0, 1 \leq k \leq n2^n)$; $K_n = s + a_n$, from which it follows $(K_n, \infty) \uparrow (s, \infty)$ as $n \uparrow \infty$, where $(K_n, \infty) \uparrow (s, \infty)$ is defined by $\bigcup_{n=1}^\infty (K_n, \infty) = (s, \infty)$. Therefore, by $\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > K_n\} \in \mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$ and $\{\alpha_-^{(n)} > K_n\} \uparrow$ as $n \uparrow$, it follows that

$$\begin{aligned}
&\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha > s\} \\
&\quad = \lim_{n \uparrow \infty} \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha_-^{(n)} > K_n\} \in \mathcal{N}_{\alpha_-^{(\infty)}},
\end{aligned}$$

this and Theorem 2.3 gives $\mathcal{N}_\alpha \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$. Finally, $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}}$. In the same manner one can prove $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}}$, $\mathcal{N}_\alpha = \mathcal{N}_{\bar{\alpha}_-^{(\infty)}}$.

LEMMA 4.6
Let $X(t, \omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$, $\alpha(\omega)$ be a stopping time, that is, $\{\alpha \leq t\} \in \mathcal{N}_t^+$ for every $t \geq 0$. Put

$$Z(s, \omega) \stackrel{\triangle}{=} E[f(x_t)|x_s](\omega).$$

Suppose that $\bar{Z}(s, \omega)$ is a $\mathcal{N}_{s+}^+ (\stackrel{\triangle}{=} \bigcap_{u>s} \mathcal{N}_u^+)$-adaptive process which is uniquely determined by $Z(s, \omega)$ according to [7, Theorem 3.5]. Then

$$E[f(x_t)|x_\alpha] = \bar{Z}(\alpha(\omega), \omega), \quad P_{\{\alpha \leq t\}}\text{–a.e.}$$

that is,

$$\mathcal{X}_{\{\alpha \leq t\}} E[f(x_t)|x_\alpha] = \mathcal{X}_{\{\alpha \leq t\}} \bar{Z}(\alpha(\omega), \omega), \quad P_{\mathcal{F}(x_\alpha)}\text{–a.e.}.$$

Proof. Take

$$\bar{\alpha}^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_k^{(n)} \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}}(\omega) + (n+1) \mathcal{X}_{\{\alpha(\omega) > n\}}(\omega),$$

and the corresponding to partition of $[0, n] = \sum_{k=1}^{n2^n} (a_{k-1}^{(n)}, a_k^{(n)}]$ satisfies that t is a partition point when $n > t$. So there exists K_n with $1 \leq K_n \leq n2^n$ such that $\{\alpha \leq t\} = \sum_{k=1}^{K_n} \{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}$. Take $\{a_k^{(n)} : n \geq 1, 1 \leq k \leq n2^n\} \subseteq D$, where D is defined as that in [7, Theorem 3.5], for every $A \in \mathcal{E}$,

$$\begin{aligned}
&\int\limits_{x_\alpha^{-1}(A)\{\alpha \leq t\}} E[f(x_t)|x_\alpha] \, P(\mathrm{d}\omega) \\
&\quad = \int\limits_{x_\alpha^{-1}(A)\{\alpha \leq t\}} f(x_t) \, P(\mathrm{d}\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K_n} \int\limits_{x_\alpha^{-1}(A)\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}} E\big[f(x_t)|\mathcal{N}^+_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega)\\
&= \sum_{k=1}^{K_n} \int\limits_{x_\alpha^{-1}(A)\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}} E\big[f(x_t)|x_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \sum_{k=1}^{K_n} Z\big(a_k^{(n)},\omega\big)\mathcal{X}_{\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}}(\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \sum_{k=1}^{K_n} Z\big(a_k^{(n)},\omega\big)\mathcal{X}_{\{\bar\alpha^{(n)}=a_k^{(n)}\}}(\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} Z(\bar\alpha^{(n)},\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \lim_{n\uparrow\infty} Z(\bar\alpha^{(n)},\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega),
\end{aligned}
$$

where $x_\alpha^{-1}(A)=\{\omega:\ x_\alpha\in A\}$. The first equality follows from the definition of conditional expectation; the second equality follows from $x_\alpha^{-1}(A)\cap\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}\in\mathcal{F}(x_\alpha)\subseteq\mathcal{N}_\alpha^+$ and Theorem 2.5; the third equality follows from Markov property; the seventh equality follows from dominated convergence theorem; the last equality follows from [7, Theorem 3.5]. Similarly to the above proof we obtain

$$
\begin{aligned}
&\int\limits_{x_\alpha^{-1}(A)\{\alpha\le u\}} \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega)\\
&\qquad= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le u\}} \mathcal{X}_{\{\alpha\le t\}}\bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega)
\end{aligned}
$$

for every $u\ge 0$, from which and λ-π-system method it follows

$$
\int\limits_{x_\alpha^{-1}(A)\{\alpha\in B\}} \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) = \int\limits_{x_\alpha^{-1}(A)\{\alpha\in B\}} \mathcal{X}_{\{\alpha\le t\}}\bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega)
$$

for every $B\in\mathcal{B}([0,\infty])$, Note that $\mathcal{F}(x_\alpha)=\mathcal{F}(x_\alpha^{-1}(A)\{\alpha\in B\};\ A\in\mathcal{E},\ B\in\mathcal{B}([0,\infty]))$. From which and λ-π-system method it follows, for every $C\in\mathcal{F}(x_\alpha)$,

$$
\int\limits_C \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) = \int\limits_C \mathcal{X}_{\{\alpha\le t\}}\bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega). \tag{4.11}
$$

Since $E[f(x_t)|x_{a_k^{(n)}}]$ is a measurable function with $x_{a_k^{(n)}}(\omega)$ as intermediate variable and ω as independent variable, that is, there exists a function $K(a_k^{(n)}, t; x, f(x_t))$ on $(E, \mathcal{E})$ such that $E[f(x_t)|x_{a_k^{(n)}}] = K(a_k^{(n)}, t; x_{a_k^{(n)}}, f(x_t))$. So, by [2, Theorem 2.2.13], we have that $E[f(x_t)|x_{a_k^{(n)}}]$ is both $\mathcal{E}$-measurable (in this case, $E[f(x_t)|x_{a_k^{(n)}}]$ is regarded as defined on space $(E, \mathcal{E})$) and $\mathcal{F}(x_{a_k^{(n)}})$-measurable (in this case, $E[f(x_t)|x_{a_k^{(n)}}]$ is regarded as defined on space $(\Omega, \mathcal{F}(x_{a_k^{(n)}}))$). Rewrite $Z(x, a_k^{(n)}) \triangleq K(a_k^{(n)}, t; x, f(x_t))$. Let $Z^{(n)}(x, s) = \sum_{k=1}^{K_n} Z(x, a_k^{(n)}) \mathcal{X}_{\{a_{k-1}^{(n)} < s \leq a_k^{(n)}\}}$. Hence $Z^{(n)}(x, s)$ is $\mathcal{E} \times \mathcal{B}([0, \infty])$-measurable (see [9, Section 2.6, Problem 8]). So $\lim_{n\uparrow\infty} Z^{(n)}(x, s)$ is also $\mathcal{E} \times \mathcal{B}([0, \infty])$-measurable. Since $\{a_k^{(n)} : n \geq 1, 1 \leq k \leq n2^n\} \subseteq D$, and $\bar{Z}(s, \omega)$ is right continuous and $\bar{Z}(\bar{\alpha}^{(n)}(\omega), \omega) = \sum_{k=1}^{K_n} Z(x, a_k^{(n)}) \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha(\omega) \leq a_k^{(n)}\}}$ by [7, Theorem 3.5], then

$$\bar{Z}(\alpha(\omega), \omega) = \lim_{n\uparrow\infty} \bar{Z}(\bar{\alpha}^{(n)}(\omega), \omega) = \lim_{n\uparrow\infty} Z^{(n)}(x, \alpha(\omega)).$$

Thus $\bar{Z}(\alpha(\omega), \omega)$ is $\mathcal{F}(x_\alpha)$-measurable by [2, Theorem 2.2.13]. By Radon–Nikodym Theorem and (4.11) the proof of theorem is comleted.

Remark 4.7
We easily verify $\alpha^{(n)}(\omega)$ and $\bar{\alpha}^{(n)}(\omega)$ are stopping times if $\alpha(\omega)$ is a stopping time. In fact, for any $t \geq 0$, if $\frac{1}{2^n} \leq t < n+1$, letting $a_n = \max(a_k^{(n)} : t \geq \frac{k}{2^n}, 1 \leq k \leq n2^n)$, then $\{\alpha^{(n)} \leq t\} = \{\alpha^{(n)} \leq a_n\} = \{\alpha \leq a_n\} \in \mathcal{N}_t^+$; if $t \geq n+1$, obviously, $\{\alpha^{(n)} \leq t\} = \Omega \in \mathcal{N}_t^+$; if $t < \frac{1}{2^n}$, obviously, $\{\alpha^{(n)} \leq t\} = \emptyset \in \mathcal{N}_t^+$. So $\alpha^{(n)}(\omega)$ is a stopping time. Similarly as above we obtain that $\bar{\alpha}^{(n)}(\omega)$ is also a stopping time.

Lemma 4.8
Let $X(t, \omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, let $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on $(E, \mathcal{E})$, $\alpha(\omega)$ be a stopping time. Then

$$E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big], \quad P_{\{\alpha \leq \bar{T}_n\}}\text{-a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = \mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big], \quad P_{\mathcal{N}_{\bar{\alpha}^{(n)}_-}}\text{-a.e..} \tag{4.12}$$

In particular, if $X(t, \omega)$ is a homogeneous Markov process,

$$E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = E\big\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big\}, \quad P_{\{\alpha \leq \bar{T}_n\}}\text{-a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = \mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big\}, \quad P_{\mathcal{N}_{\bar{\alpha}^{(n)}_-}}\text{-a.e..}$$

Here $\bar{T}_n = \max(a_k^{(n)} : t \geq a_k^{(n)}, 1 \leq k \leq n2^n)$; $\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega) = \sum_{x \in E} K(0, t - \bar{\alpha}^{(n)}(\omega); x, f(x_{t-\bar{\alpha}^{(n)}(\omega)}) \mathcal{X}_{\{x_{\bar{\alpha}^{(n)}} = x\}}(\omega) = K(0, t - \bar{\alpha}^{(n)}(\omega); x_{\bar{\alpha}^{(n)}}, f(x_{t-\bar{\alpha}^{(n)}(\omega)})$.

Remark 4.9
If $\bar{\alpha}^{(n)}(\omega) > t$ for some ω, then $\Theta_{\bar{\alpha}^{(n)}(\omega)} Z(\bar{\alpha}^{(n)}(\omega), \omega)$ might not be well defined. In this case, we may give $\Theta_{\bar{\alpha}^{(n)}(\omega)} Z(\bar{\alpha}^{(n)}(\omega), \omega)$ an arbitrary value. Obviously, it does not affect our conclusion. So we plight it in this way throughout this paper.

Proof. For every $n \geq 1$, set

$$B^{(n)}_{(t_1,A_1)(t_2,A_2)\ldots(t_m,A_m)s} = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\bar{\alpha}^{(n)}_- > s\},$$

$$\begin{aligned}\Pi_n = \{B^{(n)}_{(t_1,A_1)(t_2,A_2)\ldots(t_m,A_m)s} :\ & m \geq 1,\, t_1 \leq \ldots \leq t_m \leq s,\\ & A_1, \ldots, A_m \in \mathcal{E}\},\end{aligned}$$

$$N = \max\big(k :\ t \geq a^{(n)}_k,\, 1 \leq k \leq n2^n\big).$$

Using the abbreviation $B \triangleq \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\bar{\alpha}^{(n)}_- > s\}$ we have

$$\int\limits_{B\{\bar{\alpha}^{(n)}_- < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} f(x_t)\, P(\mathrm{d}\omega). \tag{4.13}$$

Since $\alpha(\omega)$ is a stopping time,

$$\big\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\big\} = \big\{\alpha \leq a^{(n)}_k\big\} - \big\{\alpha \leq a^{(n)}_{k-1}\big\} \in \mathcal{N}^+_{a^{(n)}_k}. \tag{4.14}$$

Let $K_n = \min(k :\ 1 \leq k \leq n2^n,\, a^{(n)}_k \geq s)$. When $k \geq K_n$, by $t_1 \leq t_2 \leq \ldots t_m \leq s \leq a^{(n)}_k$, it follows that $\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \in \mathcal{N}^+_{a^{(n)}_k}$. This and (4.14) give $\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\} \in \mathcal{N}^+_{a^{(n)}_k}$ for every $k \geq K_n$. Again, for $k < K_n$, we obviously have $B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\} = \emptyset \in \mathcal{N}^+_{a^{(n)}_k}$. Therefore, using Markov property, (4.13) is changed into

$$\begin{aligned}
&\int\limits_{B\{\bar{\alpha}^{(n)}_- < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega)\\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} E\big[f(x_t)|\mathcal{N}^+_{a^{(n)}_k}\big]\, P(\mathrm{d}\omega)\\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} E\big[f(x_t)|x_{a^{(n)}_k}\big]\, P(\mathrm{d}\omega) \qquad (4.15)\\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{B\{\bar{\alpha}^{(n)}_- < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega).
\end{aligned}$$

Let

$$\Lambda = \left\{ B : \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega),\ B \in \mathcal{N}_{\bar{\alpha}_-^{(n)}} \right\}.$$

If $B = \Omega$,

$$\begin{aligned} \int\limits_{\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) &= \sum_{k=1}^{N} \int\limits_{\{a_{k-1}^{(n)} < \alpha \le a_k^{(n)}\}} E\big[f(x_t) | x_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega) \\ &= \int\limits_{\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega), \end{aligned}$$

where the first equality follows from the Markov property and the definition of conditional expectation and (4.14). Again it could be easily verified that Λ satisfies the other conditions of λ-system. Therefore, Λ is a λ-system. Hence, by λ-π-system method, it follows that $\Lambda \supseteq \mathcal{F}(\Pi_n) = \mathcal{N}_{\bar{\alpha}_-^{(n)}}$, namely,

$$\int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega)$$

for any $B \in \mathcal{N}_{\bar{\alpha}_-^{(n)}}$. From which and definition of conditional expectation we get

$$\begin{aligned} &\int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} E\big[f(x_t) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]\, P(\mathrm{d}\omega) \\ &\qquad = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]\, P(\mathrm{d}\omega). \end{aligned} \tag{4.16}$$

If $X(t, \omega)$ satisfies homogeneity, the last integrand of (4.15) is $\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)$. So the last integrand of (4.16) is changed into $E\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\}$. By Radon–Nikodym Theorem we obtain the lemma.

Lemma 4.10
Let $X(t, \omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then, for any $t_1 \le \ldots \le t_m \le s$ and $A_1, \ldots, A_m \in \mathcal{E}$,

$$\begin{aligned} &\int\limits_{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha = s\}} E[f(x_t) | x_s]\, P(\mathrm{d}\omega) \\ &\qquad = \int\limits_{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha = s\}} E[f(x_t) | x_\alpha]\, P(\mathrm{d}\omega). \end{aligned}$$

Proof. Since $E[f(x_t)|x_s](\omega) = Z(s,\omega)$ is a martingale relative to σ-algebra filtration $\{\mathcal{N}_s^+;\ s \le t\}$, by [7, Theorem 3.5], $E[f(x_t)|x_s] = E[\bar{Z}(s,\omega)|\mathcal{N}_s^+]$. Again, $\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m,\ \alpha = s\} \in \mathcal{N}_s^+$, so

$$\begin{aligned}
&\int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} E[f(x_t)|x_s]\,P(\mathrm{d}\omega) \\
&\quad= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} E[\bar{Z}(s,\omega)|\mathcal{N}_s^+]\,P(\mathrm{d}\omega) \\
&\quad= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} \bar{Z}(s,\omega)\,P(\mathrm{d}\omega) \\
&\quad= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} E[f(x_t)|x_\alpha]\,P(\mathrm{d}\omega),
\end{aligned}$$

where the last equality follows from Lemma 4.6.

Lemma 4.11
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_t)|\mathcal{N}_\alpha] = E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha], \quad P_{\{\alpha\le t\}}\text{–a.e..} \tag{4.17}$$

Namely,

$$\mathcal{X}_{\{\alpha\le t\}}E[f(x_t)|\mathcal{N}_\alpha] = \mathcal{X}_{\{\alpha\le t\}}E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha], \quad P_{\mathcal{N}_\alpha}\text{–a.e..} \tag{4.18}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_t)|\mathcal{N}_\alpha] = E\{\Theta_{\alpha(\omega)}\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}, \quad P_{\{\alpha\le t\}}\text{–a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha\le t\}}E[f(x_t)|\mathcal{N}_\alpha] = \mathcal{X}_{\{\alpha\le t\}}E\{\Theta_{\alpha(\omega)}\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}, \quad P_{\mathcal{N}_\alpha}\text{–a.e..}$$

Proof. $\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}} \subseteq \mathcal{N}_{\underline{\bar{\alpha}}^{(n+1)}}$ for every $n \ge 1$ by Lemma 4.5. Set

$$Z_n = \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[f(x_t)|\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}}\big]; \qquad X_n = \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}}\big].$$

Then $\{Z_n;\ n \ge N\}$ is a martingale with respect to σ-algebra family $\{\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}};\ n \ge N\}$. From above and (4.12) it follows that $\{X_n;\ n \ge N\}$ is also a martingale with respect to σ-algebra family $\{\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}};\ n \ge N\}$. So, by the property of conditional expectation we get, for any $n \ge m \ge N$,

$$\begin{aligned}
X_m &= E\big[X_n|\mathcal{N}_{\underline{\bar{\alpha}}^{(m)}}\big] = E\big\{\mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\underline{\bar{\alpha}}^{(n)}}\big]|\mathcal{N}_{\underline{\bar{\alpha}}^{(m)}}\big\} \\
&= \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\underline{\bar{\alpha}}^{(m)}}\big].
\end{aligned} \tag{4.19}$$

Here the third equality is a consequence of the fact that $\mathcal{X}_{\{\alpha\leq\bar{T}_N\}}$ is $\mathcal{N}_{\bar{\alpha}_-^{(m)}}$-measurable if $m\geq N$. Next, take $\{a_k^{(n)}:\ n\geq 1,\, 1\leq k\leq n2^n\}\subseteq D$, where D is given by [7, Theorem 3.5], then

$$\lim_{n\to\infty}\mathcal{X}_{\{\alpha\leq\bar{T}_N\}}Z(\bar{\alpha}^{(n)}(\omega),\omega)=\mathcal{X}_{\{\alpha\leq\bar{T}_N\}}\bar{Z}(\alpha(\omega),\omega),\quad P\text{–a.e..}\tag{4.20}$$

By the convergence theorem of a martingale (see [3, Corollary 2.13]) and Lemma 4.5,

$$\lim_{N\to\infty}\lim_{n\to\infty}Z_n=\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|\mathcal{N}_\alpha],\quad P\text{–a.e..}\tag{4.21}$$

Again, from (4.19), (4.20) and the convergence theorem of a martingale it follows that

$$\begin{aligned}&\lim_{N\to\infty}\lim_{m\to\infty}X_m\\&\quad=\lim_{N\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\mathcal{X}_{\{\alpha\leq\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big]\\&\quad=\mathcal{X}_{\{\alpha\leq t\}}E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha].\end{aligned}\tag{4.22}$$

By (4.12), (4.21), (4.22) we obtain (4.17). By (4.17), for any $B\in\mathcal{N}_\alpha$,

$$\int_B\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|\mathcal{N}_\alpha]\,P(\mathrm{d}\omega)=\int_B\mathcal{X}_{\{\alpha\leq t\}}E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha]\,P(\mathrm{d}\omega),$$

this yields (4.18). If $X(t,\omega)$ is a homogeneous Markov process, then the right-hand side of (4.19) is changed into

$$\mathcal{X}_{\{\alpha\leq\bar{T}_N\}}E\big\{\Theta_{\bar{\alpha}^{(n)}}\bar{Z}(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big\}.$$

Note that $\bar{Z}(s,\omega)$ is right continuous, so (4.22) is changed into

$$\lim_{N\to\infty}\lim_{m\to\infty}X_m=\mathcal{X}_{\{\alpha\leq t\}}E\{\Theta_{\alpha(\omega)}\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}.$$

Theorem 4.12 (the strong Markov property)
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_t)|\mathcal{N}_\alpha^+]=E[f(x_t)|x_\alpha],\quad P_{\{\alpha\leq t\}}\text{–a.e..}$$

Further,

$$\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|\mathcal{N}_\alpha^+]=\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|x_\alpha],\quad P_{\mathcal{N}_\alpha^+}\text{–a.e..}\tag{4.23}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_t)|\mathcal{N}_\alpha^+]=[E_{x_\alpha}(f(x_{t-\alpha}))],\quad P_{\{\alpha\leq t\}}\text{–a.e..}\tag{4.24}$$

Further,

$$\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|\mathcal{N}_\alpha^+]=\mathcal{X}_{\{\alpha\leq t\}}[E_{x_\alpha}(f(x_{t-\alpha}))],\quad P_{\mathcal{N}_\alpha^+}\text{–a.e.,}\tag{4.25}$$

where $E_{x_{\alpha(\omega)}}(f(x_{t-\alpha(\omega)}))=\sum_{x\in E}K(0,t-\alpha(\omega);x,f(x_{t-\alpha(\omega)}))\mathcal{X}_{\{x_{\alpha(\omega)}=x\}}(\omega)=K(0,t-\alpha(\omega);x_{\alpha(\omega)},f(x_{t-\alpha(\omega)}))$ for every $\omega\in\Omega$ with $\alpha(\omega)\leq t$.

Proof. For any $t_1 \leq t_2 \leq \ldots \leq t_m \leq s \leq t$ set

$$\begin{aligned} B^+ &= \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq s\}; \\ C &= \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha = s\}; \\ B &= \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha > s\}. \end{aligned}$$

Then $B^+ = B + C$. From Theorem 2.3 and Theorem 2.2 it follows that $B \cap \{\alpha \leq t\} \in \mathcal{N}_\alpha$. Again, $C \cap \{\alpha \leq t\} = C \in \mathcal{N}_s^+$, from above we have

$$\begin{aligned} &\int\limits_{B^+\{\alpha\leq t\}} E[f(x_t)|\mathcal{N}_\alpha^+]\, P(\mathrm{d}\omega) \\ &= \int\limits_{B\{\alpha\leq t\}} f(x_t)\, P(\mathrm{d}\omega) + \int\limits_{C\{\alpha\leq t\}} f(x_t)\, P(\mathrm{d}\omega) \\ &= \int\limits_{B\{\alpha\leq t\}} E[f(x_t)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega) + \int\limits_{C\{\alpha\leq t\}} E[f(x_t)|\mathcal{N}_s^+]\, P(\mathrm{d}\omega) \\ &= \int\limits_{B\{\alpha\leq t\}} E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega) + \int\limits_{C\{\alpha\leq t\}} E[f(x_t)|x_s]\, P(\mathrm{d}\omega) \\ &= \int\limits_{B\{\alpha\leq t\}} \bar{Z}(\alpha(\omega),\omega)\, P(\mathrm{d}\omega) + \int\limits_{\{x_{t_1}\in A_1,\ldots,x_{t_m}\in A_m\}\cap\{\alpha(\omega)=s\}} E[f(x_t)|x_s]\, P(\mathrm{d}\omega) \\ &= \int\limits_{B\{\alpha\leq t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) + \int\limits_{C\{\alpha\leq t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) \\ &= \int\limits_{B^+\{\alpha\leq t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega), \end{aligned}$$

where the third equality follows from Lemma 4.11 and Markov property; the fifth equality follows from Lemma 4.6 and Lemma 4.10. By the λ-π-system method,

$$\int\limits_{B\{\alpha\leq t\}} E[f(x_t)|\mathcal{N}_\alpha^+]\, P(\mathrm{d}\omega) = \int\limits_{B\{\alpha\leq t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega)$$

for any $B \in \mathcal{N}_\alpha^+$. Next, it is required to verify that $E[f(x_t)|x_\alpha]$ is $\mathcal{N}_\alpha^+$-measurable by the definition of conditional expectation. Since $E[f(x_t)|x_\alpha]$ is $\mathcal{F}(x_\alpha)$-measurable by the definition of conditional expectation, $E[f(x_t)|x_\alpha]$ is $\mathcal{N}_\alpha^+$-measurable from $\mathcal{F}(x_\alpha) \subseteq \mathcal{N}_\alpha^+$ according to Theorem 3.6. Again, if $X(t,\omega)$ is a homogeneous Markov process, similarly to the above proof we obtain (4.24) and (4.25).

Note that if $\alpha(\omega) \equiv s$ (constant), then $\mathcal{F}(x_\alpha) = \mathcal{F}(x_s)$ and $\mathcal{N}_\alpha^+ = \mathcal{N}_s^+$. The following corollary is a consequence of Theorem 4.12.

Corollary (Markov property)
Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded

real-valued function defined on a measurable space $(E, \mathcal{E})$. *If* $X(t,\omega)$ *satisfies* (4.23), *then* $X(t,\omega)$ *is a Markov process, that is,* $X(t,\omega)$ *satisfies*

$$E[f(x_t)|\mathcal{N}_s^+] = E[f(x_t)|x_s], \quad P_{\mathcal{N}_s^+}\text{-}a.e.$$

for any $0 \leq s \leq t$.

In particular, if $X(t,\omega)$ *satisfies* (4.25), *then* $X(t,\omega)$ *is a homogeneous Markov process, that is,* $X(t,\omega)$ *has property:*

$$E[f(x_t)|\mathcal{N}_s^+] = E_{x_s}[f(x_{t-s})], \quad P_{\mathcal{N}_s^+}\text{-}a.e.$$

for any $0 \leq s \leq t$.

By the same method used in the proof of Theorem 4.12, Theorem 4.12 is extended as follows:

Theorem 4.12' (the strong Markov property)
Let $X(t,\omega)$ *be an arbitrary Markov process defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and valued in measurable space* $(E, \mathcal{E})$, $f(x_1, \ldots, x_n)$ *be a* n*-dimensional* $\mathcal{E}^n$*-measurable bounded real-valued function defined on a measurable space* $(E^n, \mathcal{E}^n)$ *and let* $\alpha(\omega)$ *be a stopping time. Then*

$$E[f(x_{t_1}, \ldots, x_{t_n})|\mathcal{N}_\alpha^+] = E[f(x_{t_1}, \ldots, x_{t_n})|x_\alpha], \quad P_{\{\alpha \leq \min(t_1, \ldots, t_n)\}}\text{-}a.e..$$

Further,

$$\begin{aligned}&\mathcal{X}_{\{\alpha \leq \min(t_1, \ldots, t_n)\}} E[f(x_{t_1}, \ldots, x_{t_n})|\mathcal{N}_\alpha^+] \\ &\quad = \mathcal{X}_{\{\alpha \leq \min(t_1, \ldots, t_n)\}} E[f(x_{t_1}, \ldots, x_{t_n})|x_\alpha], \quad P_{\mathcal{N}_\alpha^+}\text{-}a.e..\end{aligned}$$

In particular, if $X(t,\omega)$ *is a homogeneous Markov process, then*

$$E[f(x_{t_1}, \ldots, x_{t_n})|\mathcal{N}_\alpha^+] = [E_{x_\alpha}(f(x_{t_1-\alpha}, \ldots, x_{t_n-\alpha}))], \quad P_{\{\alpha \leq \min(t_1, \ldots, t_n)\}}\text{-}a.e..$$

Further,

$$\begin{aligned}&\mathcal{X}_{\{\alpha \leq \min(t_1, \ldots, t_n)\}} E[f(x_{t_1}, \ldots, x_{t_n})|\mathcal{N}_\alpha^+] \\ &\quad = \mathcal{X}_{\{\alpha \leq \min(t_1, \ldots, t_n)\}} [E_{x_\alpha}(f(x_{t_1-\alpha}, \ldots, x_{t_n-\alpha}))], \quad P_{\mathcal{N}_\alpha^+}\text{-}a.e..\end{aligned}$$

Theorem 4.13 (the strong Markov property)
Let $X(t,\omega)$ *be an arbitrary Markov process defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and valued in a measurable space* $(E, \mathcal{E})$, $\xi(\omega)$ *be* ${}_\alpha\mathcal{N}^+$*-measurable, and* $E|\xi| < \infty$. *Then*

$$E[\xi|\mathcal{N}_\alpha^+] = E[\xi|x_\alpha], \quad P_{\Omega_\alpha}\text{-}a.e.. \tag{4.26}$$

Proof. If $\xi(\omega) = \mathcal{X}_{\{\alpha \leq s\}} \mathcal{X}_{\{x_{t_1} \in A_1\} \ldots \{x_{t_n} \in A_n\}}$, where $s \leq t_1 \leq \ldots \leq t_n$, taking $f(x_{t_1}, \ldots, x_{t_n}) = \mathcal{X}_{\{x_{t_1} \in A_1\} \ldots \{x_{t_n} \in A_n\}}$ in Theorem 4.12' yields

$$\begin{aligned}&\mathcal{X}_{\{\alpha \leq s\}} E[\mathcal{X}_{\{x_{t_1} \in A_1\} \ldots \{x_{t_n} \in A_n\}}|\mathcal{N}_\alpha^+] \\ &\quad = \mathcal{X}_{\{\alpha \leq s\}} E[\mathcal{X}_{\{x_{t_1} \in A_1\} \ldots \{x_{t_n} \in A_n\}}|x_\alpha], \quad P_{\Omega_\alpha}\text{-a.e..}\end{aligned}$$

By Theorem 2.2 and Theorem 3.3,

$$\begin{aligned} &E[\mathcal{X}_{\{\alpha\le s, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}}|\mathcal{N}_\alpha^+] \\ &\quad = E[\mathcal{X}_{\{\alpha\le s, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}}|x_\alpha],\ P_{\Omega_\alpha}\text{–a.e..} \end{aligned} \tag{4.27}$$

Set

$$\mathcal{L} = \{\text{all integrable functions}\};$$
$$\mathcal{H} = \{\text{all } \xi(\omega) \text{ which satisfy (4.26)}\}.$$

Then $\mathcal{H}$ is $\mathcal{L}$-system. Since $\mathcal{X}_{\{\alpha\le s,\, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}} \in \mathcal{H}$ for any $n \ge 1$ and $0 \le s \le t_1 \le \ldots \le t_n$ and $A_1, \ldots, A_n \in \mathcal{E}$ from (4.27), again, ${}_\alpha\mathcal{N}^+ = \mathcal{F}(\{\alpha \le s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} :\ n \ge 1,\, 0 \le s \le t_1 \le \ldots \le t_n,\, A_1, \ldots, A_n \in \mathcal{E})$ from Theorem 3.4, by $\mathcal{L}$-system method it follows that $\mathcal{H}$ includes all ${}_\alpha\mathcal{N}^+$-measurable functions in $\mathcal{L}$.

Theorem 4.14 (the strong Markov property)
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|x_\alpha], \quad P_{\Omega_\alpha}\text{–a.e..} \tag{4.28}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E_{x_\alpha}[f(x_t)], \quad P_{\Omega_\alpha}\text{–a.e..} \tag{4.29}$$

Proof. By Theorem 3.2 and Theorem 3.4, similarly to the proof of (3.5), it follows that

$$\begin{aligned} \{x_{t+\alpha} \in A\} &= \{\alpha \le t+\alpha,\, x_{t+\alpha} \in A\} \\ &= \bigcup_{s<\infty} (\{\alpha \le t+s,\, x_{t+s} \in A\} \cap \{\alpha = s\}) + \{\beta \in A,\, \alpha = \infty\} \\ &\in {}_\alpha\mathcal{N}^+ \end{aligned}$$

for every $A \in \mathcal{E}$ and $t \ge 0$, that is, $x_{t+\alpha}$ is ${}_\alpha\mathcal{N}^+$-measurable. Therefore, $f(x_{t+\alpha})$ is ${}_\alpha\mathcal{N}^+$-measurable from [2, Theorem 2.2.13]. So $f(x_{t+\alpha})$ is also $\mathcal{F}$-measurable. Hence $f(x_{t+\alpha})$ is a random variable, that is, for every $B \in \mathcal{B}((-\infty, \infty))$,

$$\{\omega :\ f(x_{t+\alpha}) \in B\} \in \mathcal{F}. \tag{4.30}$$

Again, $E|f(x_{t+\alpha})| < \infty$, which follows from $f(x)$ is bounded. Hence, from Theorem 4.13 we get (4.28). Next, if $X(t,\omega)$ is a homogeneous Markov process, we shall prove (4.29). Set

$$f^{(n)}(x) = \sum_{k=-n2^n+1}^{n2^n} \frac{k}{2^n} \mathcal{X}_{\{\frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\}} + (n+1)\mathcal{X}_{\{f(x)>n\}} - n\mathcal{X}_{\{f(x)\le -n\}};$$

$$\begin{aligned} A_k^{(n)} &= \left\{x:\ \frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\right\} \quad (-n2^n+1 \le k \le n2^n);\\ A_{n2^n+1}^{(n)} &= \{x:\ f(x) > n\};\\ A_{-n2^n}^{(n)} &= \{x:\ f(x) \le -n\}. \end{aligned}$$

Since $f(x)$ is $\mathcal{E}$-measurable, then $A_k^{(n)} \in \mathcal{E}$ for every $-n2^n \le k \le n2^n+1$. Again, because $x_t(\omega)$ values in a measurable space $(E,\mathcal{E})$, if $f(x)$ is replaced by $\mathcal{X}_{A_k^{(n)}}(x)$ in (4.30), it follows that $\mathcal{X}_{\{x_{t+\alpha}(\omega)\in A_k^{(n)}\}}$ is $\mathcal{F}$-measurable. Again, by (4.28), for every n and $-n2^n \le k \le n2^n+1$,

$$E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] = E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|x_\alpha\big]$$

for every $\omega \in \Omega_\alpha - N_{nk}$, where N_{nk} is a P-null measurable set and satisfies $N_{nk} \subseteq \Omega_\alpha$, from which it follows that

$$\begin{aligned} \mathcal{X}_{\{\alpha=s\}}E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] &= E\big[\mathcal{X}_{\{\alpha=s\}}\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= E\big[\mathcal{X}_{\{\alpha=s\}}\mathcal{X}_{\{x_{t+s}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= \mathcal{X}_{\{\alpha=s\}}E\big[\mathcal{X}_{\{x_{t+s}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= \mathcal{X}_{\{\alpha=s\}}E_{x_\alpha}\big[\mathcal{X}_{\{x_t\in A_k^{(n)}\}}\big] \end{aligned}$$

for every $\omega \notin N_{nk}$, where the first equality follows from (4.28) and $\mathcal{X}_{\{\alpha=s\}}$ is $\mathcal{F}(x_\alpha)$-measurable according to Theorem 3.3; the last equality follows from (4.25). Note that N_{nk} does not depend on s. Then

$$\mathcal{X}_{\{\alpha<\infty\}}E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}\big[\mathcal{X}_{\{x_t\in A_k^{(n)}\}}\big]$$

for every $\omega \in \Omega_\alpha - N_{nk}$. Hence,

$$\mathcal{X}_{\{\alpha<\infty\}}E[f^{(n)}(x_{t+\alpha})|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}[f^{(n)}(x_t)]$$

for every $\omega \in \Omega_\alpha - N^{(n)}$, where $N^{(n)}$ is defined by $\bigcup_{k=-n2^n}^{n2^n+1} N_{nk}$. Further, by monotone convergence theorem we obtain

$$\mathcal{X}_{\{\alpha<\infty\}}E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}[f(x_t)]$$

for every $\omega \in \Omega_\alpha - N$, where $N = \bigcup_{n=1}^\infty N^{(n)}$, thus yields (4.29).

By the above theorem and corollary we have the following statements.

Theorem 4.15

Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then the following statements are equivalent:

(1) *(Markov property) For any* $t \geq 0$,

$$E[f(x_t)|\mathcal{N}_s^+] = E[f(x_t)|x_s], \quad P_{\mathcal{N}_s^+}\text{–}a.e.$$

for any $0 \leq s \leq t$.

(2) *(the strong Markov property) For any* $t \geq 0$,

$$E[f(x_t)|\mathcal{N}_\alpha^+] = E[f(x_t)|x_\alpha], \quad P_{\{\alpha\leq t\}}\text{–}a.e..$$

Further, we have

$$\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|x_\alpha], \quad P_{\mathcal{N}_\alpha^+}\text{–}a.e.,$$

(3) *(the strong Markov property) Let* $\xi(\omega)$ *be* ${}_\alpha\mathcal{N}^+$*-measurable, and* $E|\xi| < \infty$. *Then*

$$E[\xi|\mathcal{N}_\alpha^+] = E[\xi|x_\alpha], \quad P_{\Omega_\alpha}\text{–}a.e..$$

(4) *(the strong Markov property) For any* $t \geq 0$,

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|x_\alpha], \quad P_{\Omega_\alpha}\text{–}a.e..$$

Appendix A. Theorems and concepts cited in this paper

For convenience of the reader, we list all theorems used in this paper.

Theorem A.1 ([2] Property 2.2.2)
Let f *be a mapping from* Ω *to* E, $\mathcal{H}$ *be a* σ*-algebra of* E. *Then* $f^{-1}(\mathcal{H})$ *is a* σ*-algebra of* Ω.

Theorem A.2 ([2] Theorem 2.2.13)
Let Ω *be a set,* $(E, \mathcal{E})$ *be a measurable space,* f *be a mapping from* Ω *to* E. *Then* φ *is a* $f^{-1}(\mathcal{E})$*-measurable function from* Ω *to* $\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty\}$ *if and only if there exists a* $\mathcal{E}$*-measurable real-valued function* g *on* $(E, \mathcal{E})$ *such that* $\varphi = g \circ f$. *And if* φ *is bounded or finite, then* g *is bounded or finite.*

Theorem A.3 ([2] Theorem 5.2.5)
Let ξ *be a random variable defined on a probability space* $(\Omega, \mathcal{F}, P)$, $\mathcal{C}$ *be a* σ*-subalgebra of* $\mathcal{F}$, B *be an arbitrary atom of* $\mathcal{C}$. *Then, for any* $\omega \in B$,

$$E(\xi|\mathcal{C})(\omega) \equiv constant.$$

Further, if $P(B) > 0$, *then*

$$E(\xi|\mathcal{C})(\omega) = \frac{1}{P(B)} \int_B \xi \, \mathrm{d}P$$

for every $\omega \in B$.

Theorem A.4 ([2] Theorem 5.3.1)
Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, $E\xi$ exist, f be a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$. Then, there exists a $\mathcal{E}$-measurable function g, which is P_f-almost everywhere uniquely determined by $E(\xi|\mathcal{F}(f))$, defined on $(E, \mathcal{E})$ such that

$$E(\xi|\mathcal{F}(f)) = g \circ f, \quad P_{\mathcal{F}(f)}\text{-a.e.},$$

where g satisfies

$$\int_A g\, P_f(\mathrm{d}x) = \int_{f^{-1}(A)} \xi\, P(\mathrm{d}\omega)$$

for every $A \in \mathcal{E}$, where P_f is a probability measure derived by f, that is, P_f satisfies $P_f(A) = P(f^{-1}(A))$ for every $A \in \mathcal{E}$.

Theorem A.5 (Integrable Transform Theorem; [2] Theorem 3.4.1)
Let f be a measurable transformation from the a measurable space $(\Omega, \mathcal{F})$ to the measurable space $(E, \mathcal{E})$; g be a measurable function defined on $(E, \mathcal{E})$; μ be a measure on $(\Omega, \mathcal{F})$; μ_f be a derived measure on $(E, \mathcal{E})$ by f, that is, $\mu_f(B) \triangleq \mu(f^{-1}(B))$ for every $B \in \mathcal{E}$. Then

$$\int_{f^{-1}(B)} g \circ f\, \mathrm{d}\mu = \int_B g\, \mathrm{d}\mu_f,$$

which means: if one of the two integrals exists, then the other also exists, and the two integrals are equal.

Theorem A.6 (Extended Föllmer Lemma; [7] Theorem 3.5)
Let $X(t, \omega)$ be a martingale with respect to σ-algebra filtration $\{\mathcal{F}_t;\ t \geq 0\}$, D be a countable dense subset of $\mathbb{R}_+$. Then there exists a $\mathcal{F}_{t+}$-adaptive process $\bar{X}(t, \omega)$, which satisfies the following properties:

(1) *The every trajectory of $\bar{X}(t, \omega)$ is right continuous, and there exists a null measurable ω-set N such that*

$$\bar{X}(t, \omega) = \lim_{s \in D, s \downarrow t} X(s, \omega)$$

for every $t \geq 0$ and $\omega \in \Omega - N$.

(2) *There exists a null measurable ω-set N_1 such that, for every $t > 0$ and $\omega \in \Omega - N_1$,*

$$\bar{X}(t-, \omega) = \lim_{s \in R_+, s \uparrow t} \bar{X}(s, \omega)$$

exists and is finite, and

$$\bar{X}(t-, \omega) = \lim_{s \in D, s \uparrow t} X(s, \omega).$$

(3) *For every* $t \geq 0$, $X(t,\omega) = E[\bar{X}(t,\omega)|\mathcal{F}_t]$, P*–a.e..*

(4) $\bar{X}(t,\omega)$ *is a martingale with respect to* σ*-algebra filtration* $\mathcal{F}_{t+}$.

Here $\mathbb{R}_+ = [0,\infty)$*;* $\{\mathcal{F}_t;\ t \geq 0\}$ *is a* σ*-algebra filtration, that is, if* $s \leq t$*, then* $\mathcal{F}_s \subseteq \mathcal{F}_t$*;* $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

Theorem A.7 ([3] Corollary 2.13)
Let $\{\mathcal{F}_n;\ n \geq 0\}$ *be a monotone increasing* σ*-subalgebra family of* $\mathcal{F}$*,* Y *be an integrable random variable,* $\mathcal{F}_\infty \triangleq \mathcal{F}(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$*. Set*

$$X_n = E[Y|\mathcal{F}_n]$$

for every $n \geq 0$*. Then we have*

(1) $\{X_n,\ n \geq 0\}$ *is uniformly integrable.*

(2) $X_n \to E(Y|\mathcal{F}_\infty)$*,* P*–a.e., and* $E|X_n - E(Y|\mathcal{F}_\infty)| \to 0$ *as* $n \to \infty$.

Theorem A.8 (Radon–Nikodym Theorem; [2] Theorem 3.7.6)
Let μ *be a* σ*-finite measure on* σ*-algebra* $\mathcal{A}$ *of* Ω*. If the set function* φ *defined on* $\mathcal{A}$ *is* σ*-finite and* σ*-additive and* μ*-continuous, then there exists a* $\mathcal{A}$*-measurable finite function* f *defined on* $(\Omega,\mathcal{A})$ *such that* φ *is the indefinite integral of* f *on a measurable space* $(\Omega,\mathcal{A},\mu)$*, and* f *is* $\mu_{\mathcal{A}}$*-almost surly uniquely determined by* φ.

Theorem A.9 (Tulcea Theorem; [2] theorem 5.4.5)
Let $(\Omega_n,\mathcal{A}_n)$*,* $n = 1,2,\ldots$ *be sequence of measurable spaces. Set* $\Omega^{(n)} = \prod_{k=1}^{n} \Omega_k$*,* $\mathcal{A}^{(n)} = \prod_{k=1}^{n} \mathcal{A}_k$*,* $\Omega^{(\infty)} = \prod_{k=1}^{\infty} \Omega_k$*,* $\mathcal{A}^{(\infty)} = \prod_{k=1}^{\infty} \mathcal{A}_k$*. Let* $P_n(\omega_1,\ldots,\omega_{n-1},A_n)$*,* $(\omega_1,\ldots,\omega_{n-1},A_n) \in \Omega^{(n-1)} \times \mathcal{A}_n$*,* $n = 2,3,\ldots$ *be the transition probabilities;* $P_1(A), A \in \mathcal{A}_1$ *be the probability on* $\mathcal{A}_1$*. Then there exists only one probability measure* $P^{(\infty)}$ *on* $\mathcal{A}^{(\infty)}$ *such that*

$$P^{(\infty)}(C(B^{(n)})) = P^{(n)}(B^{(n)})$$

and

$$P^{(n)}(B^{(n)}) = \int\limits_{\Omega_1} \ldots \int\limits_{\Omega_n} \mathcal{X}_{B^{(n)}}(\omega_1,\ldots,\omega_n) P_n(\omega_1,\ldots,\omega_{n-1},\mathrm{d}\omega_n) \ldots P_1(\mathrm{d}\omega_1).$$

Here $C(B^{(n)})$ *indicates the cylinder set based on* $B^{(n)}$*;* $B^{(n)} \in \mathcal{A}^{(n)}$.

Theorem A.10 (Fubini Theorem; [2] Theorem 4.2.1)
Let $(\Omega_i,\mathcal{A}_i,\mu_i)$*,* $i = 1,2$ *be two* σ*-finite measurable spaces,* f *be nonnegative* $\mathcal{A}_1 \times \mathcal{A}_2$*-measurable function. Then*

$$\begin{aligned}\int\limits_{\Omega_1\times\Omega_2} f\,\mathrm{d}\mu_1 \times \mu_2 &= \int\limits_{\Omega_1}\left(\int\limits_{\Omega_2} f(\omega_1,\omega_2)\,\mathrm{d}\mu_2(\omega_2)\right)\mathrm{d}\mu_1(\omega_1)\\ &= \int\limits_{\Omega_2}\left(\int\limits_{\Omega_1} f(\omega_1,\omega_2)\,\mathrm{d}\mu_1(\omega_1)\right)\mathrm{d}\mu_2(\omega_2).\end{aligned}$$

Dfinition A.11 ([2] Definition 5.1.3)
Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\{B_n\} \subseteq \mathcal{A}$ be a countable subdivision of Ω, that is, $\Omega = \sum_{n=1}^{\infty} B_n$ and $B_i \cap B_j = \emptyset$, $i \neq j$. Put $\mathcal{G} = \mathcal{F}(B_n; n = 1, 2, \ldots)$. Suppose $E\xi$ exists. The following $\mathcal{G}$-measurable function in the sense of equivalence (that is, we may give an arbitrary value on null measurable set of $\mathcal{G}$, such as. If $P(B_n) = 0$, then $E(\xi|B_n)$ may be given arbitrarily.)

$$E(\xi|\mathcal{G}) = \sum_{n=1}^{\infty} E(\xi|B_n)\mathcal{X}_{B_n}(\omega)$$

is called the conditional expectation of ξ given $\mathcal{G}$.

Appendix B. The concepts of λ-system and $\mathcal{L}$-system

Here we will introduce the concepts of λ-system and $\mathcal{L}$-system, the λ-π-system method and $\mathcal{L}$-system method mentioned in this paper, which are taken from [1, Appendix].

Dfinition B.1
A system Π of subsets of a set Ω is called a π-system, if $A_1 \in \Pi, A_2 \in \Pi \implies A_1 A_2 \in \Pi$.

Dfinition B.2
A system Λ of subsets of a set Ω is called a λ-system, if it has the following properties:

(1) $\Omega \in \Lambda$;

(2) $A_1 \in \Lambda, A_2 \in \Lambda, A_1 \cap A_2 = \emptyset \implies A_1 \cup A_2 \in \Lambda$;

(3) $A_1 \in \Lambda, A_2 \in \Lambda, A_1 \supset A_2 \implies A_1 - A_2 \in \Lambda$;

(4) $A_n \in \Lambda, A_n \uparrow A, n = 1, 2, \ldots \implies A \in \Lambda$.

Theorem B.3
(1) *If the system $\mathcal{M}$ of subsets of a set Ω is a π-system, and is also a λ-system, then $\mathcal{M}$ is a σ-algebra.*

(2) *If λ-system Λ contains π-system Π, then $\Lambda \supseteq \mathcal{F}(\Pi)$.*

When we make use of Theorem B.3, we call this method λ-π-system method.
Let $\mathcal{L}$ be a family of functions defined on Ω, and satisfies:
if $\xi(\omega) \in \mathcal{L}$, set

$$\eta(\omega) = \begin{cases} \xi(\omega) & \text{if } \xi(\omega) \geq 0, \\ 0 & \text{if } \xi(\omega) < 0, \end{cases}$$

then $\eta(\omega)$ and $\eta(\omega) - \xi(\omega)$ lie in $\mathcal{L}$.

Dfinition B.4
A set L of functions is called $\mathcal{L}$-system, if it satisfies the following conditions:

(1) $\mathbb{1} \in L$, where the $\mathbb{1}$ is the function whose functional value is equal to 1;

(2) For two arbitrary functions in L, their linear combination lies in L;

(3) If $\xi_n(\omega) \in L, 0 \leq \xi_n(\omega) \uparrow \xi(\omega)$, and $\xi(\omega)$ is bounded or lies in $\mathcal{L}$, then $\xi(\omega) \in L$.

Theorem B.5
If a $\mathcal{L}$-system L contains the indicator function $\mathcal{X}_A(\omega)$ of every set A of π-system Π, then L contains all $\mathcal{F}(\Pi)$-measurable function in $\mathcal{L}$.

When we make use of Theorem B.5, we call this method $\mathcal{L}$-system method.

Appendix C. The concepts of partial ordering

We recall the concepts of partial ordering and three important theorems from real analysis (such as [8]).

Dfinition C.1
Let $\mathcal{S}$ be an arbitrary set. $\mathcal{S}$ is said to be a partially ordered set, if there is a binary relation "$\preceq$" called a partial ordering, defined on $\mathcal{S}$ with the following properties:

(1) $x \preceq x$ for all $x \in \mathcal{S}$ (reflexive),

(2) $x \preceq y, y \preceq z \implies x \preceq z$ for all $x, y, z \in \mathcal{S}$ (transitive),

(3) $x \preceq y, y \preceq x \implies x = y$ for all $x, y \in \mathcal{S}$ (antisymmetric).

Dfinition C.2
A partially ordered set $\mathcal{S}$ is called a totally ordered set if it follows $x \preceq y$ or $x \preceq y$ for any $x, y \in \mathcal{S}$.

Dfinition C.3
Let $\mathcal{S}$ be a partially ordered set, x_0 lies in $\mathcal{S}$. x_0 is said to be the maximal element of $\mathcal{S}$ if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x_0 \preceq x$; x_0 is said to be the minimal element of $\mathcal{S}$ if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x \preceq x_0$.

Dfinition C.4
Let $\mathcal{S}$ be a partially ordered set, $\mathcal{M}$ be a subset of $\mathcal{S}$, α lies in $\mathcal{S}$. α is said to be an upper bound of $\mathcal{M}$ in $\mathcal{S}$ if it follows $x \preceq \alpha$ for all $x \in \mathcal{M}$; α is said to be a lower bound of $\mathcal{M}$ in $\mathcal{S}$ if it follows $\alpha \preceq x$ for all $x \in \mathcal{M}$.

Dfinition C.5
Let $\mathcal{S}$ be a partially ordered set, $\mathcal{A}$ be a subset of $\mathcal{S}$. α is called a minimum element of $\mathcal{A}$ if α is a lower bound of $\mathcal{A}$ and α lies in $\mathcal{A}$; α is called a maximum element of $\mathcal{A}$ if α is an upper bound of $\mathcal{A}$ and α lies in $\mathcal{A}$.

Dfinition C.6
A partial ordering "$\preceq$" on $\mathcal{S}$ is said to be a well ordering if for every nonempty subset of $\mathcal{S}$ has the minimum element. $\mathcal{S}$ is called well-ordered set if there is a well ordering defined on $\mathcal{S}$.

Theorem C.7 (Zorn Lemma)
Let $\mathcal{S}$ be a partially ordered set. If every totally ordered subset $\mathcal{A}$ of $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal element.

Theorem C.8 (Well Order Theorem)
Every set can be well ordered.

Theorem C.9 (Principle of Transfinite Induction)
Let $(W, \preceq)$ be a well-ordered set. For any $a \in W$, let

$$I(a) = \{x \in W : \ x \prec a\}.$$

If A is a subset of W such that $a \in A$ whenever $I(a) \subset A$, then $A = W$.

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A more colorful hat problem

Abstract. The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. We consider a generalized hat problem with $q \geq 2$ colors. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

1. Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [13]. The hat problem was also the subject of articles in The New York Times [22], Die Zeit [7], and abcNews [21]. It is also one of the Berkeley Riddles [5].

The hat problem with $2^k - 1$ players was solved in [15], and for 2^k players in [12]. The problem with n players was investigated in [8]. The hat problem and Hamming codes were the subject of [9]. The generalized hat problem with n people and q colors was investigated in [20].

There are known many variations of the hat problem. For example in the papers [1, 11, 19] there was considered a variation in which passing is not allowed, thus

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everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [17] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [18] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [3] investigated a problem similar to the hat problem, in that paper there are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [6], linear programming [17], genetic programming [10], economics [1, 19], biology [18], approximating Boolean functions [3], and autoreducibility of random sequences [4, 13–16].

In this paper we consider a generalized hat problem with $q \geq 2$ colors which was first investigated in [20]. Every player has got a hat of one from q possible colors, and the probabilities of getting hats of all colors are equal. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

2. Preliminaries

First, let us observe that we can confine to deterministic strategies (that is, strategies such that the decision of each player is determined uniquely by the hat colors of the other players). We can do this since for any randomized (not deterministic) strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

Let $\{v_1, v_2, \ldots, v_n\}$ mean a set of players. By $Sc = \{1, 2, \ldots, q\}$ we denote the set of colors.

By a case for the hat problem with n players and q colors we mean a function $c\colon \{v_1, v_2, \ldots, v_n\} \to \{1, 2, \ldots, q\}$, where $c(v_i)$ means the hat color of player v_i. The set of all cases for the hat problem with n players and q colors we denote by $C(n, q)$, of course $|C(n, q)| = q^n$. If $c \in C(n, q)$, then to simplify notation, we write $c = c(v_1)c(v_2)\ldots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \ldots, (v_n, c(v_n))\}$. For example, if a case $c \in C(4, 3)$ is such that $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, and $c(v_4) = 2$, then we write $c = 2312$.

By a situation of a player v_i we mean a function $s_i\colon \{v_1, v_2, \ldots, v_n\} \to Sc \cup \{0\}$, where $s_i(v_j) \in Sc$ if $i \neq j$, while $s_i(v_i) = 0$. The set of all possible situations of v_i in the hat problem with n players and q colors we denote by $St_i(n, q)$, of course $|St_i(n, q)| = q^{n-1}$. If $s_i \in St_i(n, q)$, then for simplicity of notation, we write $s_i = s_i(v_1)s_i(v_2)\ldots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \ldots, (v_n, s_i(v_n))\}$. For example, if $s_2 \in St_2(4, 3)$ is such that $s_2(v_1) = 3$, $s_2(v_3) = 4$, and $s_2(v_4) = 2$, then we write $s_2 = 3042$.

We say that a case c corresponds to a situation s_i of player v_i if $c(v_j) = s_i(v_j)$, for every $j \neq i$. This implies that a case corresponds to a situation of v_i if every player excluding v_i in the case has a hat of the same color as in the situation. Of course, to every situation correspond exactly q cases.

By a guessing instruction of a player v_i we mean a function $g_i\colon St_i(n,q) \to Sc \cup \{*\}$, which for a given situation gives the color v_i guesses his hat is if $g_i(s_i) \neq *$, otherwise v_i passes. Thus a guessing instruction is a rule determining the behavior of a player in every situation.

Let c be a case, and let s_i be the situation (of player v_i) corresponding to this case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($* \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one player guesses his hat color correctly, and no player guesses his hat color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $* \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of player v_i. The family of all strategies for the hat problem with n players and q colors we denote by $\mathcal{F}(n,q)$.

If $S \in \mathcal{F}(n,q)$, then the set of cases for which the team wins using the strategy S we denote by $W(S)$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|W(S)|}{|C(n,q)|}$. We define $h(n,q) = \max\{p(S) : \ S \in \mathcal{F}(n,q)\}$. We say that a strategy S is optimal for the hat problem with n players and q colors if $p(S) = h(n,q)$.

By solving the hat problem with n players and q colors we mean finding the number $h(n,q)$.

3. Hat problem with three players and three colors

In this section we solve the hat problem with three players and three colors.

We say that a strategy is symmetric if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is nonsymmetric if it is not symmetric.

The authors of [18] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal. We solve this problem by proving the optimality of a nonsymmetric strategy found without using computer.

Let us consider the following strategy for the hat problem with three players and three colors.

Strategy 1

Let $S = (g_1, g_2, g_3) \in \mathcal{F}(3,3)$ be the strategy as follows:

$$g_1(s_1) = \begin{cases} s_1(v_3), & \text{if } s_1(v_2) \neq s_1(v_3), \\ *, & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} s_2(v_3), & \text{if } s_2(v_1) \neq s_2(v_3), \\ *, & \text{otherwise;} \end{cases}$$

$$g_3(s_3) = \begin{cases} s_3(v_1), & \text{if } s_3(v_1) = s_3(v_2), \\ *, & \text{otherwise.} \end{cases}$$

It means that players proceed as follows.

- **The player v_1.** If v_2 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.
- **The player v_2.** If v_1 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.
- **The player v_3.** If v_1 and v_2 have hats of the same color, then he guesses he has a hat of the color they have, otherwise he passes.

All cases we present in table, where the symbol + means correct guess (success), − means wrong guess (loss), and blank square means passing.

No	The color of the hat of			The guess of			Result
	v_1	v_2	v_3	v_1	v_2	v_3	
1	1	1	1			+	+
2	1	1	2	−	−	−	−
3	1	1	3	−	−	−	−
4	1	2	1	+			+
5	1	2	2		+		+
6	1	2	3	−	−		−
7	1	3	1	+			+
8	1	3	2	−	−		−
9	1	3	3		+		+
10	2	1	1		+		+
11	2	1	2	+			+
12	2	1	3	−	−		−
13	2	2	1	−	−	−	−
14	2	2	2			+	+
15	2	2	3	−	−	−	−
16	2	3	1	−	−		−
17	2	3	2	+			+
18	2	3	3		+		+
19	3	1	1		+		+
20	3	1	2	−	−		−
21	3	1	3	+			+
22	3	2	1	−	−		−
23	3	2	2		+		+
24	3	2	3	+			+
25	3	3	1	−	−	−	−
26	3	3	2	−	−	−	−
27	3	3	3			+	+

For example, in the first case the player v_1 sees two hats of the same color, so he passes. By the same reason the player v_2 also passes. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. Since v_3 has a hat of the first color, the guess is correct, and the result of the case is a win.

In the second case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by the same reason as v_1. Moreover, the guess of v_3 is also wrong. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. The guess is wrong, as v_3 has a hat of the second color.

In the fourth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of the same color, the guess is correct. The player v_2 sees two hats of the same color, so he passes. The player v_3 sees two hats of different colors, so he passes. This implies that the result of the case is a win.

In the sixth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by reasons similar as v_1. The player v_3 passes, as he sees two hats of different colors.

Counting the plusses in the last column, we get the following observation.

Observation 2
Using Strategy 1 the team wins for 15 of 27 cases.

Now, we solve the hat problem with three players and three colors.

Fact 3
$h(3,3) = \frac{5}{9}$.

Proof. Since using Strategy 1 the team wins for 15 of 27 cases, we have $h(3,3) \geq \frac{15}{27} = \frac{5}{9}$. Suppose that $h(3,3) > \frac{5}{9}$, that is, there exists a strategy such that the team wins for more than 15 cases. Let S be any strategy for the hat problem with three players and three colors. Any guess made by any player in any situation is wrong in exactly two cases, because to any situation of any player correspond three cases, and in exactly two of them this player has a hat of a color different than the one he guesses. In the strategy S every player guesses his hat color in at most 5 situations, because if some player guesses his hat color in at least 6 situations, then the team loses for at least 12 cases, and wins for at most 15 cases, a contradiction. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond three cases, and in exactly one of them this player has a hat of the color he guesses. There are three players, every one of them guesses his hat color in at most five cases, and every guess is correct in exactly one case. Therefore using the strategy S the team wins for at most 15 cases, a contradiction.

4. Hat problem with n players and q colors

Now we consider the generalized hat problem with n players and q colors. Noga Alon [2] has proven that for this problem there exists a strategy such that the chance of success is greater than or equal to

$$1-\frac{1+(q-1)\log n}{n}-\Big(1-\frac{1}{q}\Big)^{n}.$$

First we prove an upper bound on the number of cases for which the team wins using any strategy for the problem.

Theorem 4
If S is a strategy for the hat problem with n players and q colors, then

$$|W(S)|\leq n\left\lfloor\frac{q^{n}-|W(S)|}{q-1}\right\rfloor.$$

Proof. Any guess made by any player in any situation is wrong in exactly $q-1$ cases, because to any situation of any player correspond q cases, and in exactly $q-1$ of them this player has a hat of a color different than the one he guesses. Let us consider any player. The number of situations in which he guesses his hat color in the strategy S cannot be neither greater than nor equal to

$$\left\lfloor\frac{q^{n}-|W(S)|}{q-1}\right\rfloor+1,$$

otherwise the number of cases in which he guesses his hat color wrong is greater than or equal to

$$(q-1)\left(\left\lfloor\frac{q^{n}-|W(S)|}{q-1}\right\rfloor+1\right).$$

It is more than

$$(q-1)\left(\frac{q^{n}-|W(S)|}{q-1}\right)=q^{n}-|W(S)|.$$

This implies that the team loses for more than $q^{n}-|W(S)|$ cases, and therefore the number of cases for which the team wins is less than

$$|C(n,q)|-(q^{n}-|W(S)|)=q^{n}-q^{n}+|W(S)|=|W(S)|.$$

This is a contradiction, as $|W(S)|$ is the number of cases for which the team wins. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond q cases, and in exactly one of them this player has a hat of the color he guesses. This implies that the number of cases for which the team wins using the strategy S is at most

$$n\left\lfloor\frac{q^{n}-|W(S)|}{q-1}\right\rfloor.$$

Now we give an equivalent upper bound on the chance of success of any strategy for the hat problem with n players and q colors, which is easy to prove.

Theorem 5

Let S be any strategy for the hat problem with n players and q colors. Then

$$p(S) \leq \frac{n}{q^n}\left\lfloor \frac{q^n - q^n \cdot p(S)}{q-1} \right\rfloor.$$

Now we see that Fact 3 follows from Theorem 4, as well as from Theorem 5. We show that it follows from Theorem 4.

Proof of Fact 3. Since using Strategy 1 the team wins for 15 of 27 cases, by definition we get $h(3,3) \geq p(S) = \frac{15}{27} = \frac{5}{9}$. Now we prove that $h(3,3) \leq \frac{5}{9}$. Let S be an optimal strategy for the hat problem with three players and three colors. By Theorem 4 we have

$$|W(S)| \leq 3\left\lfloor \frac{27 - |W(S)|}{2} \right\rfloor.$$

This implies that

$$|W(S)| \leq 3 \cdot \frac{27 - |W(S)|}{2} = 40.5 - \frac{3|W(S)|}{2}.$$

Now we easily get $|W(S)| \leq \frac{81}{5} = 16.2$. Since $|W(S)|$ is an integer, we have $|W(S)| \leq 16$. If $|W(S)| = 16$, then $16 \leq 3\lfloor\frac{27-16}{2}\rfloor = 3 \cdot 5 = 15$, a contradiction. This implies that $|W(S)| \leq 15$. Since $|C(3,3)| = 27$, we get $p(S) \leq \frac{15}{27} = \frac{5}{9}$. Since S is an optimal strategy for the hat problem with three players and three colors, by definition we get $h(3,3) = p(S) \leq \frac{5}{9}$.

The next result proven in [20, Proposition 3] is a corollary from Theorem 4 or 5.

Corollary 6 ([20, Proposition 3])

If S is a strategy for the hat problem with n players and q colors, then

$$p(S) \leq \frac{n}{n+q-1}.$$

Proof. By Theorem 4 we have

$$|W(S)| \leq n\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

This implies that

$$|W(S)| \leq n \cdot \frac{q^n - |W(S)|}{q-1} = \frac{nq^n}{q-1} - |W(S)|\Big(\frac{n}{q-1}\Big).$$

Consequently,

$$|W(S)|\Big(1+\frac{n}{q-1}\Big) \le \frac{nq^n}{q-1} \iff |W(S)| \le \frac{q-1}{n+q-1}\cdot\frac{nq^n}{q-1}$$
$$\iff p(S) = \frac{|W(S)|}{q^n} \le \frac{n}{n+q-1}.$$

Now we show that the previous corollary is weaker than Theorem 4, that is, Theorem 4 does not follow from Corollary 6. Let S be any strategy for the hat problem with three players and three colors. By Theorem 4 we have $|W(S)| \le 15$ (it is shown in the proof of Fact 3 using Theorem 4). Thus

$$p(S) = \frac{|W(S)|}{|C(3,3)|} \le \frac{15}{3^3} = \frac{5}{9}.$$

By Corollary 6 we get

$$p(S) \le \frac{n}{n+q-1} = \frac{3}{5}.$$

Since $\frac{3}{5} > \frac{5}{9}$, Corollary 6 is weaker than Theorem 4.

Now let us consider the hat problem with two colors ($q = 2$), and any strategy S for this problem. By Corollary 6 we get the upper bound

$$p(S) \le \frac{n}{n+1}$$

previously given in [15], which is sharp for $n = 2^k - 1$, where k is a positive integer.

5. Number of strategies that suffice to be examined

In this section we consider the number of strategies the examination of which suffices to solve the hat problem, and the generalized hat problem with q colors.

First, we count all possible strategies for the hat problem. We have n players, there are 2^{n-1} possible situations of each one of them, and in each situation there are three possibilities of behavior (to guess the first color, to guess the second color, or to pass). This implies that the number of possible strategies is equal to

$$\big(3^{2^{n-1}}\big)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 7
To solve the hat problem with n players, it suffices to examine

$$\big(3^{2^{n-1}-2}\big)^n = \big(3^{2^{n-1}}\big)^n \cdot \frac{1}{9^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\ldots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. This guess is wrong in the case $11\ldots1$, causing the loss of the team. Thus the result of this case cannot be made worse. If some player other than v_1, say v_i, guesses he has the second color when he sees only hats of the first color, then his guess is wrong in the case $11\ldots1$, and is correct in the case when v_i has the second color and all the remaining vertices have the first color. Since it cannot make worse the chance of success, we may assume that every player excluding v_i guesses he has a hat of the second color when he sees hats only of the first color. Assume that some player, say v_i, guesses his hat color when he sees one hat of the second color and $n-2$ hats of the first color. If in this situation he guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of the player who has a hat of the second color. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation he guesses he has a hat of the second color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of the player who has a hat of the second color is correct. Since it makes the guess of this player pointless, we may assume that in that situation v_i does not guess he has a hat of the second color. This implies that we may assume that every player who sees one hat of the second color and $n-2$ hats of the first color, passes. Now we conclude that for each player we can assume his behavior in two situations. This implies that for each player there are two situations less to consider. In this way we get the desired number.

Now, we count all possible strategies for the generalized hat problem with q colors. We have n players, there are q^{n-1} possible situations of each one of them, and in each situation there are $q+1$ possibilities of behavior (to guess one of the q colors, or to pass). This implies that the number of possible strategies is equal to

$$\left((q+1)^{q^{n-1}}\right)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 8

To solve the hat problem with n players and q colors, it suffices to examine

$$\left((q+1)^{q^{n-1}-1}\right)^n = \left((q+1)^{q^{n-1}}\right)^n \cdot \frac{1}{(q+1)^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players and q colors. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\ldots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. Let v_i be any player other than v_1. If in this situation v_i guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of v_1. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation v_i guesses he has a hat of any color other than the first, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of v_1 is correct. Since it makes the guess of v_1 pointless, we may assume that in that situation v_i does not guess any color other that the first. This implies that we may assume that every player other than v_1 in the situation in which v_1 has a hat of the second color, and all the remaining players have hats of the first color, passes. Now we conclude that for each player we can assume his behavior in one situation. This implies that for each player there is one situation less to consider. In this way we get the desired number.

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The regular density on the plane

Abstract. In the note [1] the notion of the regular density point of the measurable subset of the real line was introduced. Then it was shown that the new definition is equivalent to the definition of O'Malley points, which has been examined in [2]. In this note we demonstrate that the analogous definitions for measurable subsets of the plane are not equivalent.

1. Notation

In the sequel we use the following symbols:
χ_A – the characteristic function of the set A,
$\mu(A)$ – the two dimensional Lebesgue measure of the set A,
$\mu_1(B)$ – the linear Lebesgue measure of the set $B \subset \mathbb{R}$,
$\bigvee_a^b f$ – the total variation of the function f on $[a,b]$,
$\mathrm{D}(A,B) = \mu(A \cap B)/\mu(B)$ – the average density of A on the set B,
$A_{x_0} = \{y : (x_0, y) \in A\}$ – the vertical cut of the set A,
$A - (x,y) = \{(p-x, q-y) : (p,q) \in A\}$ – the translation of the set A,
$\lambda A = \{(\lambda p, \lambda q) : (p,q) \in A\}$ – the homothety of the set A.

2. Introduction

In the paper [2] W. Poreda and W. Wilczyński considered the operator Φ_{OM} defined on the class $\mathcal{S}$ of Lebesgue measurable subsets of the real line. The definition of Φ_{OM} was suggested by R. O'Malley in oral communication:

Definition 1
Let A be a measurable subset of $\mathbb{R}$, $x \in \mathbb{R}$. We say that x is an O'Malley point of A iff

$$\int_0^1 \frac{\chi_{A'}(x+t) + \chi_{A'}(x-t)}{t}\, dt < \infty.$$

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The set of all O'Malley points of A we denote by $\Phi_{OM}(A)$. In [2] the authors established, between others that

- every O'Malley point of A is the density point of A,
- the operator Φ_{OM} has the properties similar to the properties of the density operator Φ, however the analogue of the Lebesgue Theorem does not hold,
- the family $\mathcal{T}_0 = \{A \in \mathcal{S} :\ A \subset \Phi_{OM}(A)\}$ forms the topology stronger than the natural topology but coarser than the density topology on the real line,
- the analogue of the Lusin–Menchoff Theorem for Φ_{OM} holds.

In the note [1] the notion of the regular density was introduced:

Definition 2
Let $A \in \mathcal{S}$ and $x \in \mathbb{R}$. Put $f_x(h) = \frac{\mu_1(A \cap [x-h, x+h])}{2h}$ for $h > 0$ and $f_x(0) = 1$. We say that x is the regular density point of the set A if and only if the following conditions are satisfied:

1. $x_0 \in \Phi(A)$,
2. $\bigvee_0^1 f_x < +\infty$.

The main result of the paper [1] states, that the notions of the regular density point and the O'Malley point are equivalent.

In this note we are going to demonstrate, that the situation on the plane is not analogous. In order to do this we are going to redefine the notions of the regular density point and the O'Malley point for the planar sets.

3. The two dimensional case

Definition 3
Let $A \subset \mathbb{R}^2$ be a measurable set. Let us define the function $f\colon [0,1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} D(A, [-x, x]^2) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

We say, that $(0,0)$ is the point of the ordinary regular density of A ($(0,0) \in \Phi_R(A)$) if

1. the function f is continuous at the point 0,
2. $\bigvee_0^1 f < +\infty$.

We say that $(x,y) \in \Phi_R(A)$ iff $(0,0) \in \Phi_R(A - (x,y))$.

The first condition means that $(0,0)$ is the ordinary density point of A. Let us examine the second condition:

Proposition 1
For each positive ε the function f restricted to the interval $[\varepsilon, 1]$ satisfies the Lipschitz condition, so it is absolutely continuous.

Proof. For $x, x+h \in [\varepsilon, 1]$, $h > 0$ we have

$$\begin{aligned} f(x+h) - f(x) &= \frac{\mu(A \cap [-x-h, x+h]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x,x]^2)}{4x^2} \\ &\leq \frac{\mu(A \cap [-x,x]^2) + 4h(2x+2h)}{4x^2} - \frac{\mu(A \cap [-x,x]^2)}{4x^2} \\ &= \frac{4h(2x+2h)}{4x^2} \leq 2\frac{h}{x^2} \\ &\leq \frac{2h}{\varepsilon^2}. \end{aligned}$$

At the same time

$$\begin{aligned} f(x+h) - f(x) &= \frac{\mu(A \cap [-x-h, x+h]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x,x]^2)}{4x^2} \\ &\geq \frac{\mu(A \cap [-x,x]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x,x]^2)}{4x^2} \\ &= \frac{x^2 - (x+h)^2}{(x+h)^2} = h\frac{-2x-h}{(x+h)^2} \\ &\geq -\frac{2h}{\varepsilon^2}. \end{aligned}$$

Corollary 1

As f is absolutely continuous on $[\varepsilon, 1]$ and $f([0,1]) \subset [0,1]$, we have

$$\bigvee_{\varepsilon}^{1} f = \int_{\varepsilon}^{1} |f'(x)|\, dx$$

and, consequently

$$\bigvee_{0}^{1} f \leq \int_{0}^{1} |f'(x)|\, dx + 1.$$

In order to simplify calculations divide the square $[-x,x]^2$ into four triangles: $T_1(x)$ having the vertices $(0,0)$, (x,x) and $(x,-x)$; $T_2(x)$ having the vertices $(0,0)$, $(-x,x)$ and (x,x); $T_3(x)$ having the vertices $(0,0)$, $(-x,-x)$ and $(-x,x)$ and $T_4(x)$ having the vertices $(0,0)$, $(x,-x)$ and $(-x,-x)$. It is easy to observe, that for $x \in (0,1]$, $f(x)$ is the arithmetic average of the average densities of A on the corresponding triangles.

Let us consider the triangle $T_1(x)$. Let

$$g_1(x) = \begin{cases} D(A, T_1(x)) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Our objective is to estimate from above the number $\int_0^1 |g'{}_1(x)|\, dx$.

Proposition 2
Let $x, x+h \in (0,1]$, $h>0$. *Let* $P(x,h) = T_1(x+h) \setminus T_1(x)$. *Then*

$$\lim_{h\to 0+} \frac{\mu(A\cap P(x,h))}{h} = \mu_1(A_{(x)} \cap [-x,x])$$

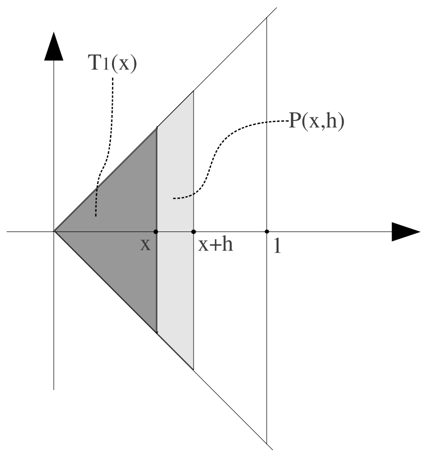
for μ_1*–a.e.* $x \in (0,1)$.

Proof. Let $s(x) = \mu_1(A_{(x)} \cap [-x,x]) = \mu_1((A\cap T_1(1))_{(x)})$. From the Fubini Theorem the function s is measurable. As the set $A\cap T_1(1)$ is bounded, the function s is also summable. Let

$$S(x) = \int_0^x s(t)\,dt.$$

From the Lebesgue Differentiation Theorem $S'(x) = s(x)$ for μ_1–a.e. $x\in(0,1)$. But

$$S'(x) = \lim_{h\to 0+} \frac{S(x+h)-S(x)}{h} = \lim_{h\to 0+} \frac{\mu(A\cap P(x,h))}{h}.$$



Now we are ready to give a more convenient formula for the function g_1':

Proposition 3
The following formula holds for μ_1*–almost every* $x\in[0,1]$*:*

$$g_1'(x) = \frac{2}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{2x} - g_1(x)\right).$$

Proof. Let the point $x\in(0,1]$ fulfill the thesis of the previous proposition. Then

$$g_1'(x) = \lim_{h\to 0+} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0+} \frac{1}{h}\left(\frac{\mu(A\cap T_1(x+h))}{(x+h)^2} - \frac{\mu(A\cap T_1(x))}{x^2}\right)$$
$$= \lim_{h\to 0+} \frac{1}{h}\frac{1}{x^2}\left(\mu(A\cap T_1(x+h))\left(1-\frac{h}{(x+h)}\right)^2 - \mu(A\cap T_1(x))\right)$$
$$= \lim_{h\to 0+} \frac{1}{h}\frac{1}{x^2}\left(\mu(A\cap P(x,h)) - \left(\frac{2h}{(x+h)} + \frac{h^2}{(x+h)^2}\right)\mu(A\cap T_1(x+h))\right).$$

By virtue of the previous proposition, and the fact that $\mu(A\cap T_1(x+h))$ tends to $\mu(A\cap T_1(x))$, when h tends to 0 we have that

$$g_1'(x) = \frac{1}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{x} - 2g_1(x)\right) = \frac{2}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{2x} - g_1(x)\right).$$

Using the last proposition we are able to estimate the number $\int_0^1 |g_1'(x)|\,dx$:

$$\int_0^1 |g_1'(x)|\,dx$$
$$= \int_0^1 \left|\frac{2}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{2x} - g_1(x)\right)\right| dx = \int_0^1 \left|\frac{2}{x}\left(\int_{-x}^{x} \frac{\chi_A(x,t)}{2x}\,dt - g_1(x)\right)\right| dx$$
$$= \int_0^1 \left|\frac{2}{x}\left(\int_{-x}^{x} \frac{\chi_A(x,t)-g_1(x)}{2x}\,dt\right)\right| dx = \int_0^1 \left|\left(\int_{-x}^{x} \frac{\chi_A(x,t)-g_1(x)}{x^2}\,dt\right)\right| dx$$
$$\le \int_0^1\int_{-x}^{x} \left|\frac{\chi_A(x,t)-g_1(x)}{x^2}\right| dt\,dx = \int_{T_1(1)} \left|\frac{\chi_A(x,t)-g_1(x)}{x^2}\right| d\mu$$
$$= \int_{T_1(1)\cap A} \frac{1-g_1(x)}{x^2}\,d\mu + \int_{T_1(1)\cap A'} \frac{g_1(x)}{x^2}\mathrm{d}\mu.$$

Let us denote the last two integrals by C and D, respectively.

Proposition 4

$$C < +\infty \iff D < +\infty.$$

In fact

$$1 \ge g(1) = \int_0^1 g_1'(x)\,dx = \int_0^1\int_{-x}^{x} \frac{\chi_A(x,t)-g_1(x)}{x^2}\,dt\,dx = C - D.$$

Definition 4

Let $A\subset\mathbb{R}^2$ be a measurable set. Let $\|(x,y)\| = \max(|x|,|y|)$. We say that $(0,0)$ is an O'Malley point of the set A ($(0,0)\in\Phi_{OM}(A)$) iff

$$\int_{[-1,1]^2\cap A'} \|(x,y)\|^{-2}\,d\mu < +\infty.$$

We say that $(x,y) \in \Phi_{OM}(A)$ iff $(0,0) \in \Phi_{OM}(A-(x,y))$.

Theorem 1
Let $A \subset \mathbb{R}^2$ be a measurable set. Then

$$\Phi_{OM}(A) \subset \Phi_R(A).$$

Proof. Assume that $(0,0) \in \Phi_{OM}(A)$. At first we are going to show, that $(0,0)$ is the ordinary density point of A. Suppose conversely that there exist the sequence (h_n) of positive numbers, tending to 0, and $\varepsilon > 0$ such that $D(A,[-h_n,h_n]^2) < 1-\varepsilon$ for $n = 1,2,\ldots$. We can assume that $(\frac{h_{n+1}}{h_n})^2 < \frac{\varepsilon}{2}$ and $h_0 = 1$. Let $Z_n = [-h_n,h_n]^2 \setminus [-h_{n+1},h_{n+1}]^2$. Then

$$\int\limits_{[-1,1]^2\cap A'} \|(x,y)\|^{-2}\,d\mu = \sum_{n=0}^{\infty} \int\limits_{Z_n\cap A'} \|(x,y)\|^{-2}\,d\mu.$$

But

$$\int\limits_{Z_n\cap A'} \|(x,y)\|^{-2}\,d\mu > h_n^{-2} \int\limits_{Z_n\cap A'} d\mu \geq h_n^{-2}\cdot h_n^2\frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Hence

$$\int\limits_{[-1,1]^2\cap A'} \|(x,y)\|^{-2}\,d\mu = +\infty,$$

the contradiction.

From the assumption that $\int_{[-1,1]^2\cap A'} \|(x,y)\|^{-2}\,d\mu < +\infty$ it follows that $\int_{T_1(1)\cap A'} \|(x,y)\|^{-2}\,d\mu < +\infty$. But for $(x,y) \in T_1(1)$ we have $\|(x,y)\| = x$. Hence

$$D = \int\limits_{T_1(1)\cap A'} \frac{g_1(x)}{x^2}\,d\mu \leq \int\limits_{T_1(1)\cap A'} \frac{1}{x^2}\,d\mu < +\infty.$$

Then, by virtue of Proposition 4, $C < +\infty$, and consequently

$$\int_0^1 |g_1'(x)|\,dx < +\infty.$$

The same holds for the functions g_i corresponding with triangles T_i, $i = 2,3,4$. From

$$f(x) = \frac{1}{4}(g_1(x) + g_2(x) + g_3(x) + g_4(x))$$

it follows that

$$\bigvee_0^1 f \leq \int_0^1 |f'(x)|\,dx + 1 \leq \int_0^1 \frac{1}{4}(|g_1'(x)| + |g_2'(x)| + |g_3'(x)| + |g_4'(x)|)\,dx + 1 < +\infty.$$

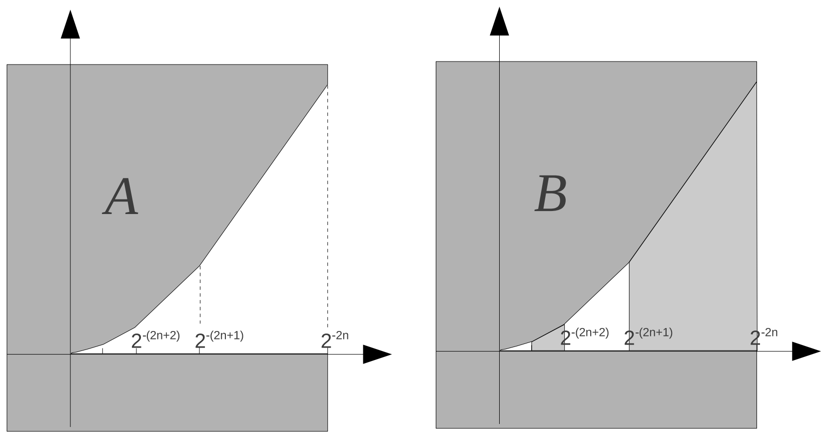
Remark 1

In the one dimensional case the theorem analogous to the Theorem 1 gives the necessary and sufficient condition for x to be the point of regular density. The following example shows that in two dimensions the opposite implication does not hold.

Example 1

In the following example we construct two planar sets, A and B such that

1. $A \subset B$,
2. $(0,0)$ is the regular density point of A,
3. $(0,0)$ is not the regular density of B.



Let

$$x_n = \frac{1}{2^n}, \quad y_n = \frac{1}{(n+1)2^n} \qquad \text{for } n = 0, 1, 2 \ldots.$$

Let $h\colon [0,1] \to [0,1]$ be a function such that

a. $h(0) = 0$,

b. $h(x_n) = y_n$ for $n = 0, 1, 2, \ldots,$

c. h is linear on each interval $[x_{n+1}, x_n]$.

It is easy to observe, that the function h is convex and continuous on the interval $[0,1]$. Let

$$A = [-1,1]^2 \setminus \{(x,y) : \ x \in (0,1) \wedge y \in (0, h(x))\}$$

and

$$P_n = \Big\{(x,y) : \ x \in \Big[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}\Big]\Big\}.$$

Finally, let $B = A \cup \bigcup_{n=0}^{\infty} P_n$.

Since the function h is convex, for every $t \in (0,1)$ the set $\{(x,y): x \in (0,t) \wedge y \in (0,h(x))\}$ is included in the triangle Z_t with vertices $(0,0)$, $(t,0)$ and $(t,h(t))$. Hence

$$f_A(t) = D(A,[-t,t]^2) \geq \frac{1}{4t^2}(4t^2 - \mu(Z_t)) \longrightarrow 1$$

when t tends to 0.

We shall show that the function f_A is decreasing on the interval $(0,1)$. Let $t \in (0,1)$ and $\lambda \in (0,1)$. We have

$$f_A(\lambda t) = D(A,[-\lambda t,\lambda t]^2) = D\Big(\frac{1}{\lambda}A,[-t,t]^2\Big)$$

and

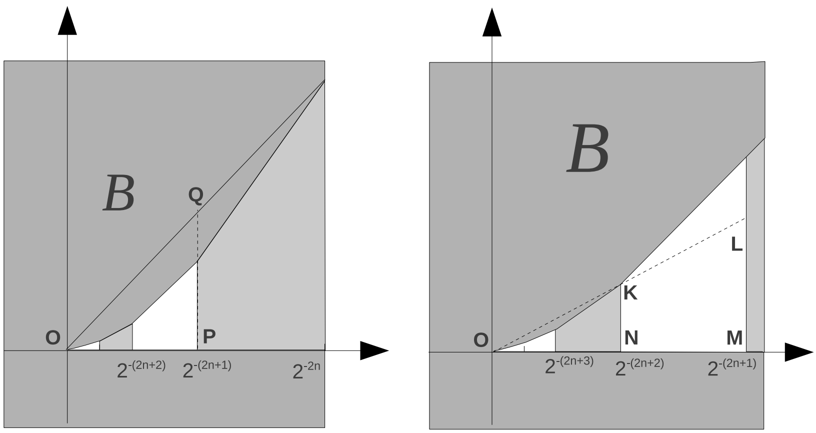
$$\frac{1}{\lambda}A = \Big[-\frac{1}{\lambda},\frac{1}{\lambda}\Big] \setminus \Big\{(x,y):\ x \in (0,1) \wedge y \in \Big(0,\frac{1}{\lambda}h(\lambda x)\Big)\Big\}.$$

Since h is convex and $h(0)=0$ we have that $\frac{1}{\lambda}h(\lambda x) < h(x)$. Hence $\frac{1}{\lambda}A \cap [-t,t]^2 \supset A \cap [-t,t]^2$ and at last $f_A(\lambda t) \geq f_A(t)$.

By virtue of the last observation and Proposition 1 we have that the function f_A is of bounded variation on $[0,1]$. Hence $(0,0)$ is the regular density point of A.

Now we shall show that $(0,0)$ is NOT the regular density point of B. First we estimate from below the value of the function f_B in point 2^{-2n}. In order to do this we estimate the measure of the complement of B laying on the left of the point $2^{-(2n+1)}$ by the area of the triangle $\triangle OPQ$. (see the picture)

$$f_B(2^{-2n}) > \frac{(2\cdot 2^{-2n})^2 - \frac{1}{8}\cdot(2^{-2n})^2\cdot\frac{1}{2n+1}}{(2\cdot 2^{-2n})^2} = 1 - \frac{1}{32\cdot(2n+1)}.$$



Now we shall estimate from above the value of the function f_B in the point $2^{-(2n+1)}$. In order to do this we estimate the measure of the complement of B

lying on the left of the point $2^{-(2n+1)}$ by the area of the trapezoid $\triangle\!\!\!\!\nabla\!\!\triangle KLMN$. (again see the picture)

$$
\begin{aligned}
f_B(2^{-(2n+1)}) &< \frac{(2\cdot 2^{-(2n+1)})^2 - \mu(\triangle\!\!\!\!\nabla\!\!\triangle KLMN)}{(2\cdot 2^{-(2n+1)})^2} = \frac{(2\cdot 2^{-(2n+1)})^2 - 3\mu(\triangle OKN)}{(2\cdot 2^{-(2n+1)})^2} \\
&= \frac{(2\cdot 2^{-(2n+1)})^2 - \frac{3}{2}\cdot(2^{-(2n+2)})^2\cdot\frac{1}{2n+3}}{(2\cdot 2^{-(2n+1)})^2} = 1 - \frac{3}{32\cdot(2n+3)}.
\end{aligned}
$$

Hence

$$
f_B(2^{-(2n+2)}) - f_B(2^{-(2n+1)}) > \frac{1}{16\cdot(2n+3)}.
$$

Since the last expression is a term of the divergent series, the total variation of the function f_B is unbounded.

Directly from the definition of an O'Malley point of the set A it follows that $\Phi_{OM}(A) \subset \Phi_{OM}(B)$ when $A \subset B$. Hence, by virtue of Theorem 1, $(0,0)$ is not an O'Malley point of A.

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Über Extrema mit Nebenbedingungen

Zusammenfassung. Zweck der vorliegenden Arbeit is es, eine gut handhabbare Methode zu zeigen, womit man die hinreichende Bedingung für die Existenz eines Extremums unter Nebenbedingungen behandeln kann. Das Resultat ist eigentlich nicht unbekannt, Einzelteile sind in mehreren Arbeiten wie etwa in [5], [10] oder in [16] enthalten. Es hat aber nicht Eingang in die neuere Lehrbuchliteratur gefunden (vgl. z. B. [1], [11] oder [13]) und ist nicht allgemein bekannt. Die Frage ist von einigem Interesse, da zum Beispiel zahlreiche Probleme in der angewandten Mathematik Extremwertaufgaben unter Nebenbedingungen sind.

1. Einleitung

In vielen Analysisbüchern und auch in vielen Analysisvorlesungen ist die Behandlung der hinreichenden Bedingung zweiter Ordnung für lokale Extrema unter Nebenbedingungen bezüglich Funktionen von mehreren Veränderlichen einfach vernachlässigt. An manchen Stellen dieser Werke gibt es vorsichtige Andeutungen darauf, daß zusätzliche Überlegungen notwendig sind, um zu entscheiden, ob ein und welche Art von Extremum vorliegt. In einigen Büchern wird die Überlegung über diese hinreichenden Bedingungen mit einer Aussage kurz abgeschlossen, nach der es hierfür kein einfaches allgemein andwendbares Verfahren gebe. In der Tat werden des öfteren andere Methoden als die hinreichende Bedingung zweiter Ordnung zur Rechtfertigung der Existenz eines Extremums herangezogen, wie etwa Kompaktheitsschluß oder die Tatsache, nach der es bei zahlreichen Abstandsaufgaben anschaulich klar ist, daß es eine Lösung gibt, und diese anschauliche Evidenz wird z. B. durch Ausnutzung der endlichdimensionaler Eigenschaft der gegebener Aufgabe untermauert. Es kommt aber oft vor, daß die Anzahl der Kandidaten für Extrema mehr als zwei ist und nicht nach globalen, sondern nach lokalen Extrema gesucht werden soll. In diesen Fällen ist das Kompaktheitsargument nicht immer ausreichend, da in diesen Stellen die zu optimierende Funktion recht verschiedene Werte annehmen kann. Unter anderem aus diesem Grunde heraus scheint die Kenntnis der hinreichenden Bedingung zweiter Ordnung vonnöten zu sein.

Diese Arbeit soll eine einfache und in vielen Aufgaben gut handhabbare Methode zur Anwendung der hinreichenden Bedingung zweiter Ordnung bekanntgeben,

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wobei vollständigkeitshalber und im Interesse der einheitlichen Behandlung auch die notwendige Bedingung (erster Ordnung) präsentiert wird. Um die Natur der Fragestellung zu verdeutlichen, werden mehrere nützliche Aufgaben gestellt und gelöst. Das Gemeinsame dieser Aufgaben ist das Auffinden einer Stelle, wo eine Funktion beschränkt auf eine Teilmenge ihres Definitionsbereiches ein (lokales) Minimum oder Maximum annimt. Dies wird präzisiert in der

Definition 1.1
Es sei Ω eine nichtleere offene Teilmenge des $\mathbb{R}^n$ und vorgelegt seien die Funktionen $f\colon \Omega \to \mathbb{R}$ (genannt auch Zielfunktion) und $\mathbf{g}\colon \Omega \to \mathbb{R}^m$, $1 \le m, n \in \mathbb{N}$. Wir sagen, f besitze in $\mathbf{c} \in \Omega$ ein lokales Extremum unter der Nebenbedingung $\mathbf{g} = \mathbf{0}$, wenn $\mathbf{c}$ zur Nebenbedingungsmenge

$$\{\mathbf{g} = \mathbf{0}\} := \{\mathbf{r} \in \Omega : \ \mathbf{g}(\mathbf{r}) = \mathbf{0}\} \neq \emptyset$$

gehört und eine Umgebung U von $\mathbf{c}$ vorhanden ist, so daß

$$f(\mathbf{r}) \le f(\mathbf{c}) \quad \text{bzw.} \quad f(\mathbf{r}) \ge f(\mathbf{c}), \qquad \mathbf{r} \in U \cap \{\mathbf{g} = \mathbf{0}\}$$

gilt.

2. Eine notwendige Bedingung

Die vektorielle Nebenbedingung $\mathbf{g} = \mathbf{0}$ zerfällt im Falle von $m < n$ nach Zerlegung in Komponenten in m skalare Nebenbedingungen

$$\begin{aligned} g_1(x_1, \ldots, x_{n-m}, x_{n-m+1}, \ldots, x_n) &= 0, \\ &\vdots \\ g_m(x_1, \ldots, x_{n-m}, x_{n-m+1}, \ldots, x_n) &= 0. \end{aligned}$$

Manchmal kann dieses System von m Gleichungen explizit nach m Veränderlichen, etwa nach $x_{n-m+1}, \ldots, x_n$ aufgelöst werden, so daß

$$x_{n-m+1} = \varphi_1(x_1, \ldots, x_{n-m}), \ldots, x_n = \varphi_m(x_1, \ldots, x_{n-m})$$

mit bekannten Funktionen $\varphi_1, \ldots, \varphi_m$ ist. In diesem Falle läuft die Aufgabe darauf hinaus, die „freien“lokalen Extrema der Funktion

$$\begin{aligned} &\Phi(x_1, \ldots, x_{n-m}) \\ &\quad := f(x_1, \ldots, x_{n-m}, \varphi_1(x_1, \ldots, x_{n-m}), \ldots, \varphi_m(x_1, \ldots, x_{n-m})) \\ &\qquad\qquad ((x_1, \ldots, x_{n-m}) \in \mathbb{R}^{n-m} : \ (x_1, \ldots, x_{n-m}, x_{n-m+1}, \ldots, x_n) \in \Omega) \end{aligned}$$

zu bestimmen. Dies wird benutzt bei der Lösung der

Aufgabe 2.1
Für eine Bewässerungsanlage soll ein Kanal mit gleichschenklig trapezförmigem Querschnitt aus drei gleich großen Betonfertigplatten der Breite b gebaut werden. Man bestimme die Anordnung der Platten, so daß möglichst viel Wasser transportiert werden kann!

Durch den Kanal kann möglichst viel Wasser fließen, wenn die Querschnittsfläche Q maximal wird. Je nachdem, in welchem Winkel α die Wände zur Horizontalen geneigt sind, ändern sich die Trapezhöhe h, die obere Breite β und mit ihnen die Querschnittsfläche Q. Für Q gilt nach der Flächenformel für ein Trapez

$$Q = \frac{\beta + b}{2} \cdot h \qquad \text{mit } h = b \cdot \sin(\alpha),\ \frac{\beta - b}{2} = b \cdot \cos(\alpha).$$

Die Zielfunktion und die Nebenbedingungsmenge sind also

$$f(\mathbf{r}) := \frac{\beta + b}{2} \cdot b \cdot \sin(\alpha),\ \{g = 0\} \qquad \text{mit } g(\mathbf{r}) := \frac{\beta - b}{2} - b \cdot \cos(\alpha)$$
$$(\mathbf{r} = (\alpha, \beta) \in \mathbb{R}^2 :\ 0 < \alpha < \pi,\ 0 < \beta < 3b).$$

Aus der Nebenbedingung errechnet man dann $\beta = \varphi(\alpha) := b \cdot (1 + 2\cos(\alpha))$. So sind die lokalen Extrema der Funktion

$$\Phi(\alpha) := f(\alpha, \varphi(\alpha)) = b^2 \cdot (1 + \cos(\alpha)) \cdot \sin(\alpha), \qquad \alpha \in (0, \pi)$$

aufzufinden. Wegen

$$\Phi'(\alpha) = b^2 \cdot (\cos^2(\alpha) - \sin^2(\alpha) + \cos(\alpha)) = b^2 \cdot (2\cos^2(\alpha) + \cos(\alpha) - 1),$$

und

$$\Phi''(\alpha) = -b^2 \cdot \sin(\alpha) \cdot (1 + 4\cos(\alpha)),$$

sowie

$$\Phi'\Big(\frac{\pi}{3}\Big) = 0 \quad \text{bzw.} \quad \Phi''\Big(\frac{\pi}{3}\Big) = -\frac{3\sqrt{3} \cdot b^2}{2} < 0$$

haben wir es mit einem Maximum zu tun. Der Kanal hat also den größten Durchfluß, wenn die Wände um $60°$ zur Horizontalen geneigt sind.

Im Falle $n = 2$, $m = 1$ könnte so eine Aufgabe zum Beispiel auch durch die Erfüllung der Nebenbedingung $g(x_1, x_2) = 0$ durch eine Parameterdarstellung

$$x_1 = \varphi_1(t),\ x_2 = \varphi_2(t), \quad t \in I \qquad \text{mit } \varphi_1, \varphi_2 \in \mathfrak{D}(I)$$

auf einem Intervall $I \subset \mathbb{R}$ gelöst und die Extremalstellen der Funktion F mit

$$F(t) := f(\varphi_1(t), \varphi_2(t)), \qquad t \in I$$

mit den üblichen Methoden bestimmt werden, vorausgesetzt f ist differenzierbar. Dieser Trick wird ausgenutzt bei der Lösung der

Aufgabe 2.2
Es sollen diejenigen Punkte einer durch die Gleichung

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (0 < a < b)$$

gegebenen Ellipse bestimmt werden, die von ihrem Mittelpunkt $(0, 0)$ maximalen bzw. minimalen Abstand haben.

Die Parametrisierung

$$x = \varphi_1(t) := a\cos(t), \quad y = \varphi_2(t) := b\sin(t), \qquad t \in [0, 2\pi]$$

der Ellipse führt zu der Funktion

$$F(t) := a^2\cos^2(t) + b^2\sin^2(t), \qquad t \in [0, 2\pi]$$

(statt des Abstandes wird – zur formalen Vereinfachung – das Quadrat des Abstandes verwendet: $f(x,y) := x^2 + y^2$, $(x,y) \in \mathbb{R}^2$; dabei werden dieselben Extremalstellen erhalten). Wegen

$$F'(t) = (b^2 - a^2)\sin(2t), \qquad t \in [0, 2\pi]$$

und $F'(s) = 0$, $s \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ sowie

$$F(0) = F(\pi) = a^2 < b^2 = F\left(\frac{\pi}{2}\right) = F\left(\frac{3\pi}{2}\right)$$

sind die gefundenen Punkte die Scheitelpunkte der Ellipse.

In der Praxis sind diese Methoden nur selten anwendbar: die explizite Auflösung ist in den meisten Fällen nicht möglich und auch die Bestimmung einer Parameterdarstellung ist oft ziemlich mühsam. Man denke nur an die Variante der Aufgabe 2.2, wo die gegebene Ellipse die Gleichung

$$ax^2 + 2bxy + cy^2 = 1 \qquad (a > 0,\ ac - b^2 > 0)$$

hat. Wie die Auflösung nach Variablen oder die Parameterbestimmung umgegangen werden kann, wird z. B. aus der folgenden Überlegung ersichtlich. Ist nämlich bei der Parameterbestimmung $\tau \in I$ derjenige Parameterwert, wofür $(\varphi_1(\tau), \varphi_2(\tau)) = \mathbf{c}$ gilt, so genügt er der Gleichung

$$0 = F'(t) = \partial_1 f(\varphi_1(t), \varphi_2(t)) \cdot \varphi_1'(t) + \partial_2 f(\varphi_1(t), \varphi_2(t)) \cdot \varphi_2'(t), \qquad t \in I.$$

Wegen $g(\varphi_1(t), \varphi_2(t)) = 0$, $t \in I$ gilt weiter

$$\partial_1 g(\varphi_1(t), \varphi_2(t)) \cdot \varphi_1'(t) + \partial_2 g(\varphi_1(t), \varphi_2(t)) \cdot \varphi_2'(t) = 0, \qquad t \in I.$$

Für $t = \tau$ sind also die Vektoren $f'(\mathbf{c})$ und $g'(\mathbf{c})$ linear abhängig, d. h. es gibt $\lambda \in \mathbb{R}$ derart, daß

$$\partial_1 f(\mathbf{c}) + \lambda\partial_1 g(\mathbf{c}) = 0, \quad \partial_2 f(\mathbf{c}) + \lambda\partial_2 g(\mathbf{c}) = 0 \quad \text{und} \quad g(\mathbf{c}) = 0$$

gelten.

Das hier skizzierte Verfahren wird für höhere Dimensionen im nächsten Satz formuliert.

Satz 2.3
Die folgenden drei Bedingungen seien erfüllt.

1. f *sei differenzierbar,* $\mathbf{g}$ *sei stetig differenzierbar.*

2. *f habe in $\mathbf{c} \in \Omega$ ein lokales Extremum unter der Nebenbedingung $\mathbf{g} = \mathbf{0}$.*

3. *Es gebe in $\mathbf{g}'(\mathbf{c})$ eine m-reihige Unterdeterminante, die nicht verschwindet, d. h. der Rang von $\mathbf{g}'(\mathbf{c})$ sei m (Rang- bzw. Regularitätsbedingung).*

Dann gibt es einen Vektor $\boldsymbol{\lambda} \in \mathbb{R}^m$ (Lagrange-Multiplikator), so daß mit $L := f + \langle \boldsymbol{\lambda}, \mathbf{g} \rangle$ (Lagrange-Funktion) gilt: $L'(\mathbf{c}) = \mathbf{0}$, d. h.

$$\partial_k f(\mathbf{c}) + \sum_{l=1}^{m} \lambda_l \partial_k g_l(\mathbf{c}) = 0, \qquad 1 \leq k \leq n.$$

Beweis. (vgl. [14]) Aus der Rangbedingung folgt, daß zwischen m und n die Relation $m \leq n$ bestehen muß. Im Falle von $m = n$ folgt aus dem Satz über inverse Funktionen, daß der Punkt $\mathbf{c}$ eine Umgebung hat, die gemeinsam mit der Nebenbedingungsmenge nur den Punkt $\mathbf{c}$ innehat. In diesem – hinsichtlich des Extremums – uninteressanten Fall, ist die Existenz eines $\boldsymbol{\lambda} \in \mathbb{R}^m$ wegen der dritten Bedingung offensichtlich. Des weiteren sei o. B. d. A. angenommen, daß $m < n$ gilt und die Matrix, die aus den letzten m Spalten von $\mathbf{g}'(\mathbf{c})$ gebildet wird, eine von Null verschiedene Determinante hat. (Notfalls benennt man die Variablen entsprechend um.) $\mathbf{g}'(\mathbf{c})$ läßt sich also in der Form $\mathbf{g}'(\mathbf{c}) = [G_1, G_2]$ schreiben, wobei $G_1 \in \mathbb{R}^{m \times (n-m)}$, $G_2 \in \mathbb{R}^{m \times m}$ mit

$$G_1 := \partial_1 \mathbf{g}(\mathbf{c}) := [\partial_j g_i(\mathbf{c})]_{i,j=1,1}^{m,n-m}, \qquad G_2 := \partial_2 \mathbf{g}(\mathbf{c}) := [\partial_j g_i(\mathbf{c})]_{i,j=1,n-m+1}^{m,n}$$

und $\det(\partial_2 \mathbf{g}(\mathbf{c})) \neq 0$. Nach dem Satz von der impliziten Funktion (angewandt auf $\mathbf{g}$) gibt es dann eine Umgebung U von $\mathbf{a} := (c_1, \ldots, c_{n-m}) \in \mathbb{R}^{n-m}$ und eine Umgebung V von $\mathbf{b} := (c_{n-m+1}, \ldots, c_n) \in \mathbb{R}^m$ mit $U \times V \subset \Omega$ und eine stetig differenzierbare Funktion $\boldsymbol{\varphi}\colon U \to V$ mit

$$\{\mathbf{g} = \mathbf{0}\} \cap (U \times V) = \{(\mathbf{r}, \boldsymbol{\varphi}(\mathbf{r})) :\ \mathbf{r} \in U\},$$

d. h.

$$\mathbf{g}(\mathbf{r}, \boldsymbol{\varphi}(\mathbf{r})) = \mathbf{0}, \qquad \mathbf{r} \in U$$

bzw.

$$\boldsymbol{\varphi}'(\mathbf{r})) = -[\partial_2 \mathbf{g}(\mathbf{r}, \boldsymbol{\varphi}(\mathbf{r})]^{-1} \cdot \partial_1 \mathbf{g}(\mathbf{r}, \boldsymbol{\varphi}(\mathbf{r})), \qquad \mathbf{r} \in U.$$

Da die Beschränkung von f auf $\{\mathbf{g} = \mathbf{0}\} \cap (U \times V)$ im Punkte $\mathbf{c} = (\mathbf{a}, \mathbf{b})$ $(\mathbf{b} = \boldsymbol{\varphi}(\mathbf{a}))$ ein lokales Extremum besitzt, hat auch die Funktion

$$\Phi(\mathbf{r}) := f(\mathbf{r}, \boldsymbol{\varphi}(\mathbf{r})), \qquad \mathbf{r} \in U$$

ein (freies) lokales Extremum in $\mathbf{c}$. Nach der notwendigen Bedingung für lokale Extrema gilt also

$$\mathbf{0} = \Phi'(\mathbf{c}) = \partial_1 f(\mathbf{a}, \mathbf{b}) + \partial_2 f(\mathbf{a}, \mathbf{b}) \cdot \boldsymbol{\varphi}'(\mathbf{a}),$$

woraus mit

$$\boldsymbol{\lambda} := -\partial_2 f(\mathbf{a}, \mathbf{b}) \cdot [\partial_2 \mathbf{g}(\mathbf{a}, \mathbf{b})]^{-1}$$

die Gleichheit

$$\mathbf{0} = \partial_1 f(\mathbf{a},\mathbf{b}) - \partial_2 f(\mathbf{a},\mathbf{b}) \cdot [\partial_2 \mathbf{g}(\mathbf{a},\mathbf{b})]^{-1} \cdot \partial_1 g(\mathbf{a},\mathbf{b}) = \partial_1 f(\mathbf{a},\mathbf{b}) + \boldsymbol{\lambda}\partial_1 \mathbf{g}(\mathbf{a},\mathbf{b}),$$

sowie (aus der Definition für $\boldsymbol{\lambda}$)

$$\partial_2 f(\mathbf{a},\mathbf{b}) + \boldsymbol{\lambda}\partial_2 \mathbf{g}(\mathbf{a},\mathbf{b}) = \mathbf{0}$$

folgt.

Um die kritische Stelle $\mathbf{c}$ zu bestimmen, muß also das System der $(m+n)$ Gleichungen

$$\begin{aligned} \partial_k f(\mathbf{r}) + \sum_{l=1}^{m} \lambda_l \partial_k g_l(\mathbf{r}) &= 0, \qquad 1 \le k \le n, \\ g_l(\mathbf{r}) &= 0, \qquad 1 \le l \le m \end{aligned} \tag{1}$$

für $\mathbf{r}$ und $\boldsymbol{\lambda}$ gelöst werden. Als Beispiel betrachten wir zwei Aufgaben.

Aufgabe 2.4
Es sei $A \in \mathbb{R}^{n\times n}$ eine symmetrische Matrix. Gesucht sind die Extrema der Funktion

$$f(\mathbf{r}) := \langle A\mathbf{r},\mathbf{r}\rangle = \sum_{i,j=1}^{n} a_{ij}x_i x_j \qquad (\mathbf{r} = (x_1,\ldots,x_n) \in \mathbb{R}^n : \ \|\mathbf{r}\|_2 = 1).$$

Da g die Form

$$g(\mathbf{r}) := \sum_{i=1}^{n} x_i^2 - 1, \qquad \mathbf{r} = (x_1,\ldots,x_n) \in \mathbb{R}^n$$

hat, ist die Rangbedingung $2\mathbf{c} \neq \mathbf{0}$, d. h. $\mathbf{c} \neq \mathbf{0}$ erfüllt, denn $\mathbf{0} \notin \{\mathbf{g} = \mathbf{0}\}$. Anhand (1) erhalten wir also das folgende Gleichungssystem:

$$\sum_{j=1}^{n} a_{kj}x_j + \sum_{i=1}^{n} a_{ki}x_i + 2\lambda x_k = 0, \quad 1 \le k \le n, \qquad \sum_{i=1}^{n} x_i^2 - 1 = 0.$$

$\mathbf{c}$ genügt der Gleichung (1), wenn

$$2\sum_{i=1}^{n} a_{ki}c_i + 2\lambda c_k = 0, \quad 1 \le k \le n \qquad \text{und} \qquad \sum_{i=1}^{n} c_i^2 = 1$$

gelten. Es gibt also ein $\mu \in \mathbb{R}$, so daß $A\mathbf{c} = \mu\mathbf{c}$ gilt, d. h. $\mathbf{c}$ ist ein normierter Eigenvektor der Matrix A zum Eigenwert $-\lambda$.

Aufgabe 2.5
Die lokalen Extrema der Funktion

$$f(x,y,z) := x + 2y + 3z \qquad ((x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 = 2, \ y + z = 1)$$

sollen bestimmt werden.

Die Rangbedingung ist erfüllt wenn

$$\mathrm{Rang}(\mathbf{g}'(\mathbf{c})) = \mathrm{Rang}\begin{pmatrix} 2c_1 & 2c_2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2$$

gilt. Anhand (1) hat das zu lösende Gleichungssystem die Form

$$1+2\lambda_1 x = 0, \quad 2+2\lambda_1 y+\lambda_2 = 0, \quad 3+\lambda_2 = 0, \quad x^2+y^2-2 = 0, \quad y+z-1 = 0.$$

Als Lösung haben wir zwei kritische Stellen: $\mathbf{c} \in \{(1,-1,2);(-1,1,0)\}$ (mit $\lambda_1 = -\frac{1}{2}$, $\lambda_1 = \frac{1}{2}$ bzw. $\lambda_2 = -3$), mit denen die Rangbedingung erfüllt ist.

3. Hinreichende Bedingungen

3.1. Kompaktheitsargumente

Ist die Nebenbedingungsmenge $\{\mathbf{g} = \mathbf{0}\}$ kompakt und die Zielfunktion f stetig, so ist die Existenz der absoluten Extrema gesichert, wie das in den Aufgaben 2.4 und 2.5 der Fall ist. Dort sind nämlich die Nebenbedingungsmengen die Einheitskugel und eine Ellipse (die sich als Schnitt eines Zylinders und einer Ebene ergibt), die beschränkt und abgeschlossen, somit kompakt sind. Da im ersten Falle $f(\mathbf{c}) = \langle A\mathbf{c}, \mathbf{c}\rangle = \mu$ gilt, wird das Maximum bzw. das Minimum bei einem Eigenvektor zum größten bzw. zum kleinsten Eigenwert angenommen. (Insbesondere folgt, daß eine reelle symmetrische Matrix mindestens einen reellen Eigenwert besitzt.) Im zweiten Falle entscheiden die Funktionswerte: $f(-1,1,0) = 1 < 5 = f(1,-1,2)$.

3.2. Abstandsaufgaben

Wenn es um Abstandsaufgaben geht, so erweist sich eine Behauptung, – die auf der Endlichdimensionalität des euklidischen Raumes $\mathbb{R}^d$ $(1 \le d \in \mathbb{N})$ beruht – als nützlich:

Satz 3.1
A sei eine kompakte und B eine abgeschlossene Teilmenge des $\mathbb{R}^d$; beide Mengen seien nicht leer. Dann gibt es in A einen Punkt $\mathbf{a}$ und in B einen Punkt $\mathbf{b}$ mit

$$\|\mathbf{a}-\mathbf{b}\| \le \|\mathbf{x}-\mathbf{y}\|, \qquad \mathbf{x} \in A,\ \mathbf{y} \in B,$$

wobei $\|\cdot\|$ irgendeine Norm auf $\mathbb{R}^d$ ist.

Statt des Beweises (siehe z. B. [11]), geben wir einen Fall an, wo dieser Satz vom Nutzen sein kann:

Aufgabe 3.2
Man bestimme denjenigen Punkt auf dem Schnitt der Ebenen $x+2y+z = 1$ und $2x-y-3z = 4$, der von der Koordinatenursprung den kleinsten (euklidischen) Abstand hat!

Die Zielfunktion und die Nebenbedingungsmenge sind also für $\mathbf{r} = (x,y,z) \in \mathbb{R}^3$

$$f(\mathbf{r}) := x^2+y^2+z^2,\ \{\mathbf{g} = \mathbf{0}\} \qquad \text{mit } \mathbf{g}(\mathbf{r}) := \begin{bmatrix} x+2y+z-1 \\ 2x-y-3z-4 \end{bmatrix}.$$

Die Rangbedingung ist überall erfüllt, denn

$$\operatorname{Rang}(\mathbf{g}'(\mathbf{r})) = \operatorname{Rang}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \end{bmatrix} = \operatorname{Rang}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \end{bmatrix} = 2, \qquad \mathbf{r} \in \mathbb{R}^3$$

gilt. Das Gleichungssystem (1) hat die Form

$$\begin{cases} 2x+\lambda_1+2\lambda_2=0, \\ 2y+2\lambda_1-\lambda_2=0, \\ 2z+\lambda_1-3\lambda_2=0, \\ x+2y+z-1=0, \\ 2x-y-3z-4=0. \end{cases}$$

Die einzige kritische Stelle $\mathbf{c}=(\frac{16}{15},\frac{1}{3}),-\frac{11}{15}$ mit $\lambda_1=-\frac{52}{75}$, $\lambda_2=-\frac{54}{75}$ von L steht also im Verdacht, die gesuchte Stelle zu sein. Da wegen Satz 3.1 eine solche Stelle wirklich vorhanden ist, muß $\mathbf{c}$ tatsächlich die Lösung der Aufgabe 3.2 sein.

3.3. Hinreichende Bedingungen zweiter Ordnung

Es ist offensichtlich, daß im Falle von Funktionen $f,\mathbf{g}\in\mathfrak{D}^2[\mathbf{c}]$ die Existenz eines lokalen Extremums unter Nebenbedingungen durch die Definitheit der Hesse-Matrix $L''(\mathbf{c})$ gesichert wird. Aus der Definitheit folgt nämlich, daß die Lagrange-Funktion in $\mathbf{c}$ ein lokales Extremum besitzt, also eine Umgebung $U\subset\Omega$ von $\mathbf{c}$ gibt, wofür

$$L(\mathbf{r})\leq L(\mathbf{c}) \quad \text{bzw.} \quad L(\mathbf{r})\geq L(\mathbf{c}), \qquad \mathbf{r}\in U$$

gilt. Aus $\mathbf{c}\in\{\mathbf{g}=\mathbf{0}\}$ folgt sofort die Ungleichung

$$f(\mathbf{r})=L(\mathbf{r})\leq L(\mathbf{c})=f(\mathbf{c}) \quad \text{bzw.} \quad f(\mathbf{r})=L(\mathbf{r})\geq L(\mathbf{c})=f(\mathbf{c}) \\ (\mathbf{r}\in U\cap\{\mathbf{g}=\mathbf{0}\}).$$

Dieses Resultat kann man z. B. bei der Lösung der folgenden Aufgabe ausnutzen.

Aufgabe 3.3
Man bestimme die Extrema der Funktion

$$f(\mathbf{r}):=x^2-2x+2y^2+z^2+z \quad (\mathbf{r}=(x,y,z)\in\mathbb{R}^3:\ x+y+z=1,\ 2x-y-z=5)!$$

Wie bei der Lösung der Aufgabe 3.2 kommt man hier mit der Kompaktheitsschluß ebenfalls nicht aus, denn die Nebenbedingunsmenge ist eine – als Schnitt von zwei Ebenen – ergebende Gerade, die unbeschränkt, somit nicht kompakt ist. Die Zielfunktion ist auch keine Abstandsfunktion, so scheint die Berechnung der zweiten Ableitung L'' in $\mathbf{c}$ sinnvoll zu sein. Die Lösung des Gleichungssystems (1) ist der einzige Vektor $\mathbf{c}=(2,0,-1)$ mit $\lambda_1=\lambda_2=-\frac{3}{2}$. Die Rangbedingung ist überall erfüllt. Die Hesse-Matrix $L''(\mathbf{c})=\operatorname{diag}\{2,4,2\}$ ist positiv definit, demzufolge hat f in $\mathbf{c}$ unter der Nebenbedingung $\{\mathbf{g}=\mathbf{0}\}$ ein lokales Minimum.

Oft kommt aber vor, daß $L''(\mathbf{c})$ nicht definit ist, wie das der Fall der folgenden Aufgabe zeigt.

Aufgabe 3.4
Man bestimme die lokalen Extrema der Funktion

$$f(\mathbf{r}):=xy+xz+xw+yz+yw+zw \quad (\mathbf{r}=(x,y,z,w)\in(\mathbb{R}^+)^4:\ xyzw=1)!$$

Die einzige kritische Stelle von L ist $\mathbf{c} := (1,1,1,1)$, mit $\lambda = 4$. Die Rangbedingung ist trivialerweise erfüllt, denn $g'(\mathbf{c}) = (1,1,1,1)$. Die Eigenwerte von $L''(\mathbf{c})$ sind -9 und 3 (dies letzte ist ein dreifacher Eigenwert), so ist $L''(\mathbf{c})$ nicht definit: indefinit.

In Fällen also, wo die obigen Hilfsmittel nicht ausreichen, könnte man eine hinreichende Bedingung zweiter Ordnung benutzen, die der eigentliche Gegenstand dieser Arbeit ist. Bevor wir aber die diesbezügliche Behauptung formulieren, wollen wir den folgenden Begriff bestimmen.

Definition 3.5
Es seien $0 < m, n \in \mathbb{N}$ mit $m < n$. Die symmetrische Matrix $A \in \mathbb{R}^{n\times n}$ bzw. die quadratische Form Q mit $Q(\mathbf{r}) := \langle A\mathbf{r}, \mathbf{r}\rangle$, $\mathbf{r} \in \mathbb{R}^n$ heißt positiv bzw. negativ definit bezüglich der Matrix $B \in \mathbb{R}^{m\times n}$ mit $\mathrm{Rang}(B) = m$, wenn Q positiv bzw. negativ definit auf dem Kernraum von B ist, d. h. aus $\mathbf{0} \neq \mathbf{r} \in \mathbb{R}^n$, $B\mathbf{r} = \mathbf{0}$ die Ungleichung $Q(\mathbf{r}) > 0$ bzw. $Q(\mathbf{r}) < 0$ folgt.

Beispiel 3.6
Im Fall $m = 1$, $n = 2$ habe die quadratische Form Q und die Matrix B die Gestalt

$$Q(\mathbf{r}) := ax^2 + 2bxy + cy^2, \quad \mathbf{r} = (x,y) \in \mathbb{R}^2, \qquad B := [d,\ e].$$

Der Vektor $\mathbf{r} = [x,y]^T$ gehört somit genau dann zum Kernraum von B, wenn $dx + ey = 0$ gilt. Die Forderung $\mathrm{Rang}(B) = 1$ heißt in diesem Falle, daß eine von den zwei Komponenten ungleich Null ist: $d^2 + e^2 > 0$. Ist z. B. $e \neq 0$, so bekommt man durch Einsetzen von $y = -\frac{dx}{e}$ in Q, daß

$$Q(\mathbf{r}) = ax^2 + 2bx\Big(-\frac{dx}{e}\Big) + c\Big(-\frac{dx}{e}\Big)^2 = \frac{(ae^2 - 2bde + cd^2)x^2}{e^2}, \quad \mathbf{r} = (x,y) \in \mathbb{R}^2.$$

Der Koeffizient von x^2 im Zähler lässt sich in der Form

$$ae^2 - 2bde + cd^2 = -\det\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & 0 \end{bmatrix} \tag{2}$$

schreiben. So ist z. B. die positive Definitheit von Q bezüglich B damit gleichwertig, daß die obige Determinante negativ ist.

Somit kann schon das Resultat über eine hinreichende Bedingung zweiter Ordnung für die Existenz eines Extremums unter Nebenbedingungen formuliert werden, das übrigens schon gegen Ende des 19. Jahrhunderts bekannt war.

Satz 3.7
Die Funktionen f bzw. $\mathbf{g}$ aus Satz 2.3 seien zweimal differenzierbar, ferner sei $\mathbf{c} \in \{\mathbf{g} = \mathbf{0}\}$, wofür die notwendige Bedingung erster Ordnung erfüllt ist, d. h. $\mathrm{Rang}(\mathbf{g}'(\mathbf{c})) = m$ und für den zugehörigen Lagrange-Multiplikator $\boldsymbol{\lambda} \in \mathbb{R}^m$ und für die Lagrange-Funktion $L := f + \langle \boldsymbol{\lambda}, \mathbf{g}\rangle$ gilt $L'(\mathbf{c}) = \mathbf{0}$. Ist die quadratische Form

$$Q^L_{\mathbf{c}}(\mathbf{r})\colon \mathbb{R}^n \to \mathbb{R}, \qquad Q^L_{\mathbf{c}}(\mathbf{r}) := \langle L''(\mathbf{c})\mathbf{r}, \mathbf{r}\rangle$$

bezüglich $\mathbf{g}'(\mathbf{c})$ positiv bzw. negativ definit, so hat f ein lokales Minimum bzw. Maximum in $\mathbf{c}$ unter der Nebenbedingung $\mathbf{g} = \mathbf{0}$.

Beweis. (vgl. [10]) Wir werden nur die Behauptung bezüglich Minimum beweisen, da offensichtliche Modifikationen der folgenden Argumentation zum Fall des Maximums führen.

Angenommen, hätte f kein lokales Minimum in $\mathbf{c}$ unter der Nebenbedingung $\mathbf{g}=\mathbf{0}$, so gäbe es eine Folge

$$\mathbf{r}_k \in \{\mathbf{g}=\mathbf{0}\}\backslash\{\mathbf{c}\},\quad k\in\mathbb{N} \qquad \text{mit } \lim(\|\mathbf{r}_k-\mathbf{c}\|)=0 \text{ und } f(\mathbf{r}_k)<f(\mathbf{c}),$$

wobei $\|\cdot\|$ die euklidische Norm auf $\mathbb{R}^n$ bezeichnet: $\|\cdot\|:=\|\cdot\|_2$. Da die Folge

$$\mathbf{h}_k := \frac{\mathbf{r}_k-\mathbf{c}}{\|\mathbf{r}_k-\mathbf{c}\|},\qquad k\in\mathbb{N}$$

beschränkt ist, hat sie eine konvergente Teilfolge $(\mathbf{h}_{\nu_k})$, $k\in\mathbb{N}$ mit

$$\lim(\mathbf{h}_{\nu_k}) =: \mathbf{h} \quad \text{und} \quad \|\mathbf{h}\|=1.$$

Wegen $\mathbf{g}(\mathbf{r}_{\nu_k})=\mathbf{g}(\mathbf{c})=\mathbf{0}$, $k\in\mathbb{N}$ folgt

$$\mathbf{0}=\lim\left(\frac{\mathbf{g}(\mathbf{r}_k)-\mathbf{g}(\mathbf{c})}{\|\mathbf{r}_k-\mathbf{c}\|}\right)=\mathbf{g}'(\mathbf{c})\mathbf{h} \quad \text{also} \quad Q_{\mathbf{c}}^L(\mathbf{h})>0, \tag{3}$$

denn laut Voraussetzung $Q_{\mathbf{c}}^L$ ist bezüglich $\mathbf{g}'(\mathbf{c})$ positiv definit. Da die Lagrange-Funktion L auf der Nebenbedingungsmenge $\{\mathbf{g}=\mathbf{0}\}$ mit f identisch ist, folgt

$$L(\mathbf{c})=f(\mathbf{c}) \quad \text{bzw.} \quad L(\mathbf{r}_k)=f(\mathbf{r}_k).$$

Die Taylorsche Formel mit Restglied nach Peano liefert dann für L:

$$\begin{aligned} f(\mathbf{c}) > f(\mathbf{r}_k) = L(\mathbf{r}_k) \\ &= L(\mathbf{c}) + \langle L'(\mathbf{c})(\mathbf{r}_k-\mathbf{c}), (\mathbf{r}_k-\mathbf{c})\rangle \\ &\quad + \frac{1}{2}\langle L''(\mathbf{c})(\mathbf{r}_k-\mathbf{c}), \mathbf{r}_k-\mathbf{c}\rangle + \eta(\mathbf{r}_k-\mathbf{c})\cdot\|(\mathbf{r}_k-\mathbf{c})\|^2 \\ &= f(\mathbf{c}) + 0 + \frac{1}{2}Q_{\mathbf{c}}^L(\mathbf{r}_k-\mathbf{c}) + \eta(\mathbf{r}_k-\mathbf{c})\|\mathbf{r}_k-\mathbf{c}\|^2 \end{aligned}$$

mit $\lim_{\mathbf{r}\to\mathbf{0}}\eta(\mathbf{r})=0$ (vgl. [15]) also

$$0 > Q_{\mathbf{c}}^L(\mathbf{r}_k-\mathbf{c}) + 2\eta(\mathbf{r}_k-\mathbf{c})\|\mathbf{r}_k-\mathbf{c}\|^2.$$

Dividiert man diese Ungleichung durch $\|\mathbf{r}_k-\mathbf{c}\|^2$ und bildet den Grenzübergang $k\to\infty$, so erhält man $0\geq Q_{\mathbf{c}}^L(\mathbf{h})$, was der Ungleichung in (3) widerspricht.

Als Beispiel wollen wir nochmal Aufgabe 2.5 behandeln. Dort ist die zweite Ableitung der Lagrange-Funktion semidefinit:

$$L''(\mathbf{c}) = \begin{bmatrix} 2\lambda_1 & 0 & 0 \\ 0 & 2\lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Die quadratische Form

$$Q_{\mathbf{c}}^{L}(\mathbf{r}) = \langle L''(\mathbf{c})\mathbf{r}, \mathbf{r}\rangle = 2\lambda_1(x^2+y^2), \qquad \mathbf{r} = (x,y,z) \in \mathbb{R}^3$$

ist aber bezüglich $\mathbf{g}'(\mathbf{c})$ definit, denn für alle $\mathbf{r} = (x,y,z) \in \mathbb{R}^3$ gilt

$$g'(\mathbf{c})\mathbf{r} = \begin{bmatrix} 2c_1 & 2c_2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2c_1x + 2c_2y \\ y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\iff \left(x = -\frac{c_2}{c_1}y \ \& \ y = -z\right),$$

also

$$g'(\mathbf{c})\mathbf{r} = \mathbf{0} \iff \mathbf{r} = (\xi, \xi, -\xi), \qquad \xi \in \mathbb{R},$$

d. h. für $\xi \neq 0$ gilt

$$Q_{\mathbf{c}}^{L}(\mathbf{r}) = \begin{cases} 4\lambda_1\xi^2 > 0\,, & \lambda_1 = \dfrac{1}{2}, \\ 4\lambda_1\xi^2 < 0\,, & \lambda_1 = -\dfrac{1}{2}. \end{cases}$$

So hat man in $(-1,1,0)$ mit einem Minimum bzw. in $(1,-1,2)$ mit einem Maximum zu tun.

Durch Einführung der sog. erweiterten Lagrange-Funktion

$$\widetilde{L}(\mathbf{r}, \boldsymbol{\lambda}) := f(\mathbf{r}) + \langle \boldsymbol{\lambda}, \mathbf{g}(\mathbf{r})\rangle, \qquad (\mathbf{r}, \boldsymbol{\lambda}) \in \Omega \times \mathbb{R}^m$$

hat die notwendige Bedingung erster Ordnung die Form $\widetilde{L}'(\mathbf{c}, \boldsymbol{\lambda}_{\mathbf{c}}) = \mathbf{0}$. Das Vektorpaar $(\mathbf{c}, \boldsymbol{\lambda}_{\mathbf{c}})$ genügt also dem Gleichungssystem (1). Ferner gilt für die zweite Ableitung

$$\widetilde{L}''(\mathbf{c}, \boldsymbol{\lambda}_{\mathbf{c}}) = \begin{bmatrix} L''(\mathbf{c}) & \mathbf{g}'(\mathbf{c})^T \\ \mathbf{g}'(\mathbf{c}) & \mathbf{O}_m \end{bmatrix},$$

wobei $\mathbf{O}_m$ die Nullmatrix mit m Spalten und m Zeilen bezeichnet. Die Matrix $\widetilde{L}''(\mathbf{c}, \boldsymbol{\lambda}_{\mathbf{c}})$ ist nicht definit, da sie mindestens eine Null in der Hauptdiagonale hat. Wir zeigen aber, daß sie eine entscheidende Rolle bei der Bestimmung der Definitheit $L''(\mathbf{c})$ bezüglich $\mathbf{g}'(\mathbf{c})$ spielt.

Durch Einführung der Matrizen

$$\mathbf{R}_1 := [x_1, \ldots, x_{n-m}]^T \quad \text{und} \quad \mathbf{R}_2 := [x_{n-m+1}, \ldots, x_n]^T$$

läßt sich die Bedingung $\mathbf{g}'(\mathbf{c})\mathbf{r} = \mathbf{0}$ in der Form $G_1\mathbf{R}_1 + G_2\mathbf{R}_2 = \mathbf{0}$ aufschreiben, wobei G_1 und G_2 die im Beweis des Satzes 2.3 eingeführten Matrizen sind. Unter Berücksichtigung der Annahme $\det(G_2) \neq 0$ folgt

$$\mathbf{R}_2 = -G_2^{-1} \cdot G_1 \cdot \mathbf{R}_1, \quad \text{d. h.} \quad \mathbf{r} = \begin{bmatrix} E_{n-m} \\ -G_2^{-1} \cdot G_1 \end{bmatrix} \cdot \mathbf{R}_1 =: M \cdot \mathbf{R}_1,$$

wobei E_{n-m} die Einheitsmatrix mit $(n-m)$ Spalten und $(n-m)$ Zeilen ist. So hat die quadratische Form $Q_{\mathbf{c}}^{L}$ die Gestalt

$$\langle L''(\mathbf{c})\mathbf{r}, \mathbf{r}\rangle = \mathbf{R}_1^T \cdot M^T \cdot L''(\mathbf{c}) \cdot M \cdot \mathbf{R}_1, \qquad \mathbf{r} \in \mathbb{R}^n.$$

Die positive bzw. negative Definitheit von $Q_{\mathbf{c}}^L$ bezüglich $\mathbf{g}'(\mathbf{c})$ ist also mit der (bedingungslosen) positiven bzw. negativen Definitheit der Matrix

$$\widehat{Q}_{\mathbf{c}}^L := M^T \cdot L''(\mathbf{c}) \cdot M \in \mathbb{R}^{(n-m)\times(n-m)} \tag{4}$$

gleichwertig.

Beispiel 3.8
Im Fall $m = 1, n = 2$ haben $L''(\mathbf{c})$, $g'(\mathbf{c})$ und somit M die Form

$$\begin{aligned} L''(\mathbf{c}) &= \begin{bmatrix} \partial_{11}f(\mathbf{c}) + \lambda\partial_{11}g(\mathbf{c}) & \partial_{12}f(\mathbf{c}) + \lambda\partial_{12}g(\mathbf{c}) \\ \partial_{21}f(\mathbf{c}) + \lambda\partial_{21}g(\mathbf{c}) & \partial_{22}f(\mathbf{c}) + \lambda\partial_{22}g(\mathbf{c}) \end{bmatrix}, \\ g'(\mathbf{c}) &= [\partial_1 g(\mathbf{c}), \partial_2 g(\mathbf{c})] \end{aligned}$$

und $M = \left[1, -\frac{\partial_1 g(\mathbf{c})}{\partial_2 g(\mathbf{c})}\right]^T$. Demzufolge gilt für $M^T \cdot L''(\mathbf{c}) \cdot M$ in diesem Fall

$$M^T \cdot L''(\mathbf{c}) \cdot M = \frac{ae^2 - 2bde + cd^2}{e^2}$$

mit

$$\begin{aligned} &a := \partial_{11}f(\mathbf{c}) + \lambda\partial_{11}g(\mathbf{c}), \quad b := \partial_{12}f(\mathbf{c}) + \lambda\partial_{12}g(\mathbf{c}) \\ &c := \partial_{22}f(\mathbf{c}) + \lambda\partial_{22}g(\mathbf{c}), \quad d := \partial_1 g(\mathbf{c}), \quad e := \partial_2 g(\mathbf{c}). \end{aligned}$$

Somit hat $M^T \cdot L''(\mathbf{c}) \cdot M$ wegen der Identität (2) die Form

$$M^T \cdot L''(\mathbf{c}) \cdot M = -\frac{1}{[\partial_2 g(\mathbf{c})]^2} \cdot D(\mathbf{c}),$$

wobei

$$D(\mathbf{c}) := \det \begin{bmatrix} \partial_{11}f(\mathbf{c}) + \lambda\partial_{11}g(\mathbf{c}) & \partial_{12}f(\mathbf{c}) + \lambda\partial_{12}g(\mathbf{c}) & \partial_1 g(\mathbf{c}) \\ \partial_{21}f(\mathbf{c}) + \lambda\partial_{21}g(\mathbf{c}) & \partial_{22}f(\mathbf{c}) + \lambda\partial_{22}g(\mathbf{c}) & \partial_2 g(\mathbf{c}) \\ \partial_1 g(\mathbf{c}) & \partial_2 g(\mathbf{c}) & 0 \end{bmatrix}. \tag{5}$$

So hat f ein lokales Maximum bzw. Minimum in $\mathbf{c}$ unter der Nebenbedingung $\mathbf{g} = \mathbf{0}$, wenn $D(\mathbf{c}) > 0$ bzw. $D(\mathbf{c}) < 0$ gilt.

Dieses Ergenbis könnte z. B. bei der Lösung der Aufgabe 2.1 nützlich sein. Dort hat nämlich die Lagrange-Funktion die Form

$$L(\alpha, \beta) = \frac{\beta + b}{2} \cdot b \cdot \sin(\alpha) + \lambda\Big(\frac{\beta - b}{2} - b \cdot \cos(\alpha)\Big), \qquad 0 < \alpha < \pi,\ 0 < \beta < 3b.$$

Die Rangbedingung ist überall erfüllt, denn $2g'(\alpha, \beta) \equiv (2b\sin(\alpha), 1)$, und die einzige kritische Stelle ist $\mathbf{c} = (\frac{\pi}{3}, 2b)$ mit $\lambda = \frac{-b\sqrt{3}}{2}$. Die zweite Ableitung der Lagrange-Funktion ist indefinit:

$$L''(\mathbf{c}) = \begin{bmatrix} \dfrac{-b^2\sqrt{3}}{2} & \dfrac{b}{4} \\ \dfrac{b}{4} & 0 \end{bmatrix}.$$

Durch die Berechnung der Determinante in (5) bekommt man

$$\det \begin{bmatrix} \frac{-b^2\sqrt{3}}{2} & \frac{b}{4} & \frac{b\sqrt{3}}{2} \\ \frac{b}{4} & 0 & \frac{1}{2} \\ \frac{b\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix} = \frac{b^2\sqrt{3}}{4} > 0,$$

so gibt es in der gefundenen Stelle ein Maximum.

Die Verallgemeinerung auf die Fälle $m, n \in \mathbb{N}$ mit $1 \le m < n$ wird formuliert im

Satz 3.9
Für $k \in \{1, \dots, m+n\}$ bezeichne H_k die Untermatrix der geänderten Hesse-Matrix

$$H := \begin{bmatrix} L''(\mathbf{c}) & \mathbf{g}'(\mathbf{c})^T \\ \mathbf{g}'(\mathbf{c}) & \mathbf{O}_m \end{bmatrix},$$

die aus den Einträgen der ersten k Zeilen und Spalten besteht, und die Voraussetzungen des Satzes 3.7 seien erfüllt. So hat f ein lokales Minimum bzw. Maximum in $\mathbf{c}$ unter der Nebenbedingung $\mathbf{g} = \mathbf{0}$, wenn die Ungleichungen

$$(-1)^m \det(H_k) > 0 \quad \textit{bzw.} \quad (-1)^{m+k} \det(H_k) > 0, \qquad k \in \{2m+1, \dots, m+n\}$$

gelten.

Beweis. Da die Matrix in (4) über die Form

$$\widehat{Q}^L_{\mathbf{c}} = \begin{bmatrix} E_{n-m} & (-G_2^{-1}G_1)^T \end{bmatrix} \cdot L''(\mathbf{c}) \cdot \begin{bmatrix} E_{n-m} \\ -G_2^{-1}G_1 \end{bmatrix}$$

verfügt, ist es zweckmäßig die Hesse-Matrix $L''(\mathbf{c})$ als Übermatrix der Blöcke L_{ij}, $i, j \in \{1; 2\}$ aufzufassen:

$$L''(\mathbf{c}) =: \begin{array}{rccl} & \overbrace{\hphantom{L_{11}}}^{n-m} & \overbrace{\hphantom{L_{12}}}^{m} & \\ n-m\{ & \boxed{L_{11}} & \boxed{L_{12}} & \}n-m \\ m\{ & \boxed{L_{21}} & \boxed{L_{22}} & \}m \\ & \underbrace{\hphantom{L_{21}}}_{n-m} & \underbrace{\hphantom{L_{22}}}_{m} & \end{array},$$

wobei $L_{21} = L_{12}^T$. So läßt sich die geänderte Hesse-Matrix in Blockgestalt

$$H = \begin{bmatrix} L_{11} & L_{12} & G_1^T \\ L_{21} & L_{22} & G_2^T \\ G_1 & G_2 & \mathbf{O}_m \end{bmatrix}$$

und $\widehat{Q}^L_{\mathbf{c}}$ in die Gestalt

$$\begin{aligned}\widehat{Q}^L_{\mathbf{c}} &= L_{11} - L_{12}(G_2^{-1}G_1) - (G_2^{-1}G_1)^T L_{21} + (G_2^{-1}G_1)^T L_{22}(G_2^{-1}G_1)\\ &= L_{11} - L_{12}(G_2^{-1}G_1) - G_1^T(G_2^{-1})^T L_{21} + G_1^T(G_2^T)^{-1} L_{22}(G_2^{-1}G_1)\\ &= L_{11} - L_{12}(G_2^{-1}G_1) - G_1^T(G_2^T)^{-1}\{L_{21} - L_{22}(G_2^{-1}G_1)\}\end{aligned}$$

schreiben. Laut Determinantensatz für Übermatrizen (vgl. [8]) gilt also

$$\begin{aligned}\det(\widehat{Q}^L_{\mathbf{c}}) &= \frac{1}{\det(G_2)}\cdot\det\begin{bmatrix} L_{11} - L_{12}(G_2^{-1}G_1) & G_1^T\\ L_{21} - L_{22}(G_2^{-1}G_1) & G_2^T\end{bmatrix}\\ &= \frac{1}{\det(G_2)}\cdot\frac{1}{\det(G_2)}\cdot\det\begin{bmatrix} L_{11} - L_{12}G_2^{-1}G_1 & G_1^T & L_{12}\\ L_{21} - L_{22}G_2^{-1}G_1 & G_2^T & L_{22}\\ \mathbf{O}_{n-m} & \mathbf{O}_m & G_2\end{bmatrix}\\ &= \frac{\det(A)}{(\det(G_2))^2}\end{aligned}$$

mit

$$A := \begin{bmatrix} L_{11} & G_1^T & L_{12}\\ L_{21} & G_2^T & L_{22}\\ G_1 & \mathbf{O}_m & G_2\end{bmatrix}.$$

So sind das $(\det(G_2))^2$-fache der Hauptminoren von $\widehat{Q}^L_{\mathbf{c}}$ der Ordnung k gleich den Hauptminoren von A der Ordnung $2m+k$, $k \in \{1,\dots,n-m\}$. $\widehat{Q}^L_{\mathbf{c}}$ ist also genau dann z. B. positiv definit, wenn diese Hauptminoren von A positiv sind. Diese Hauptminoren sind aber gleich dem $(-1)^m$-fachen der entsprechenden Hauptminoren von H, denn A bekommt man von H durch die Vertauschung der $(n-m+l)$-ten und der $(n+1+l)$-ten Spalten, $l \in \{1,\dots,m\}$.

Beispiel 3.10
Man rechnet leicht nach, daß die geränderte Hesse-Matrix im Falle der Aufgabe 3.4 die Form

$$H := \begin{bmatrix} 0 & -3 & -3 & -3 & 1\\ -3 & 0 & -3 & -3 & 1\\ -3 & -3 & 0 & -3 & 1\\ -3 & -3 & -3 & 0 & 1\\ 1 & 1 & 1 & 1 & 0\end{bmatrix} \tag{6}$$

hat. Für das Vorzeichen der entsprechenden Hauptminoren von H gilt wie folgt

$$(-1)^1\det(H_3) > 0, \quad (-1)^1\det(H_4) > 0 \quad \text{bzw.} \quad (-1)^1\det(H_5) > 0,$$

so haben wir mit einem lokalen Minimum unter Nebenbedingungen zu tun.

Das im Satz 3.9 formulierte Kriterium war zuerst in [12] zu lesen. Etwas einfachere Beweismethode findet sich in [16], woraus die Idee der obigen Herleitung stammt. Das Besondere an diesem Kriterium war, daß man durch die Berechnung der Hauptminoren viel einfacher vorgehen konnte als bei dem früher bekannten Kriterium von H. Hancock (vgl. [9]), nach der die positive bzw. negative Definitheit von $Q^L_{\mathbf{c}}$ begüglich $\mathbf{g}'(\mathbf{c})$ damit gleichwertig ist, daß die Koeffizienten $a_0, \ldots, a_{n-m}$ des Polynoms

$$\det \begin{bmatrix} L''(\mathbf{c}) - zE_n & \mathbf{g}'(\mathbf{c})^T \\ \mathbf{g}'(\mathbf{c}) & \mathbf{O}_m \end{bmatrix} = \sum_{k=0}^{n-m} a_k(-z)^{n-m-k}, \qquad z \in \mathbb{R} \tag{7}$$

das gleiche Vorzeichen haben bzw. alternieren.

Ein anderes Kriterium haben Chabriallac und Crouzeix bewiesen (vgl. [2]), nach der die positive bzw. negative Definitheit der Hesse-Matrix $L''(\mathbf{c})$ bezüglich $\mathbf{g}'(\mathbf{c})$ damit gleichwertig ist, daß die Trägheit der geränderten Hessematrix H (also das Tripel, bestehend aus der Anzahl der negativen, nullgleich bzw. positiven Eigenwerten) gleich $(m, 0, n)$ bzw. $(n, 0, m)$ ist. So sieht man wiederum, daß im Falle der Aufgabe 3.4 ein lokales Minimum vorliegt, indem man das Polynom (7) für die geränderte Hesse-Matrix H in (6) aufschreibt:

$$-4(-z)^3 - 36(-z)^2 - 108(-z) - 108, \qquad z \in \mathbb{R}.$$

Dies beweist auch die Trägheit der Matrix H: $(1, 0, 4)$, denn die Eigenwerte von H sind: $-4 - 2\sqrt{5}$, $-4 + 2\sqrt{5}$, 3, 3, 3.

Abgesehen davon, daß es immer die Frage der Beurteilung im Einzelfall ist, welche dieser Kriterien einfacher sind, zeichnet sich der Chabriallac-Crouzeix-Test dadurch aus, daß er gut auf einem Rechner implementierbar ist. Die Berechnung der Trägheit einer Matrix kann nämlich z. B. mit Hilfe des Rangreduktionsverfahrens von J. Egerváry (s. [3], [4]) durchgeführt werden. Eine gute Übersicht über die verschiedenen Rangreduktionsverfahren findet man z. B. in den Werken [6], [7].

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On Minkowski decomposition of Okounkov bodies on a Del Pezzo surface

Abstract. We show that on a blow up of $\mathbb{P}^2$ in 3 general points there exists a finite set of nef divisors $P_1, \ldots, P_s$ such that the Okounkov body $\Delta(D)$ of an arbitrary effective $\mathbb{R}$–divisor D on X is the Minkowski sum

$$\Delta(D) = \sum_{i=1}^{s} a_i \Delta(P_i) \tag{1}$$

with non-negative coefficients $a_i \in \mathbb{R}_{\geqslant 0}$.

1. Introduction

Okounkov bodies form a new and rapidly developing research area in algebraic geometry. They are convex bodies associated to algebraic varieties in a very general setting, and may be viewed as a vast generalization of toric geometry. The idea is to associate to a big divisor D on a variety X a convex body $\Delta(D)$ (the Okounkov body of D) in such a way that questions about the original variety and D can be answered from the geometry of this polytope.

A systematic development of the theory has been initiated in [8] and [5] and we refer to these articles for details and motivations. Here we recall the basic construction.

Let X be an irreducible projective variety of dimension n and

$$Y_\bullet : \; X = Y_0 \supset Y_1 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\}$$

be a flag of irreducible subvarieties of X such that $\operatorname{codim}_X(Y_i) = i$ and p is a smooth point of each Y_i for $i = 0, \ldots, n$.

Let D be a Cartier divisor on X. The flag $Y_\bullet$ defines an order n valuation-type mapping

$$\nu_{Y_\bullet} : H^0(X, kD) \to \mathbb{Z}^n \cup \{\infty\}$$

in the following way. Given a section $0 \neq s \in H^0(X, kD)$ we set

$$\nu_1 = (\nu_{Y_\bullet})_1(s) := \operatorname{ord}_{Y_1}(s).$$

AMS (2000) Subject Classification: 14C20, 14F05.

This determines a section

$$\widetilde{s} \in H^0(X, kD - \nu_1 Y_1),$$

which does not vanish identically along Y_1, and thus restricts to a non-zero section

$$s_1 \in H^0(Y_1, (kD - \nu_1 Y_1)|_{Y_1}).$$

We repeat the above construction for s_1 and so on. In this way we produce a valuation vector

$$\nu_{Y_\bullet}(s) = ((\nu_{Y_\bullet})_1(s), \ldots, (\nu_{Y_\bullet})_n(s)) \in \mathbb{Z}^n$$

and an element

$$(\nu_{Y_\bullet}(s), k) \in \Gamma_{Y_\bullet}(D) \subset \mathbb{Z}^{n+1}$$

in the *graded semigroup* of the linear series $|D|$. Let $S(D) \subset \mathbb{R}^n$ be the set of all normalized valuation vectors obtained as above, i.e.,

$$S(D) = \left\{ \frac{1}{k} \nu_{Y_\bullet}(s) : \; s \in H^0(X, kD),\; k = 1, 2, 3, \ldots \right\}.$$

Definition 1.1 (Okounkov body)
The *Okounkov body* $\Delta_{Y_\bullet}(D)$ associated to the divisor D is the closed convex hull of the set $S(D)$.

Note that the shape of the Okounkov body depends on the flag $Y_\bullet$. However some invariants, for example its volume, are independent of $Y_\bullet$. This is in fact the main result of [8].

Computing the Okounkov body explicitly is in general not an easy task. We address this question here for Del Pezzo surfaces. First, we need to recall some properties of Okounkov bodies on arbitrary surfaces.

2. Okounkov bodies on surfaces

A remarkable fact about divisors on arbitrary smooth surfaces is the existence of the Zariski decomposition. This fact goes back to Zariski [11]. We refer to [1] for a modern proof.

Theorem 2.1 (Zariski decomposition)
Let D be an effective divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{Q}$–divisors P_D and N_D such that

$$D = P_D + N_D$$

and

(i) *P_D is nef,*

(ii) *N_D is zero or has negative definite intersection matrix,*

(iii) $P_D \cdot C = 0$ *for all irreducible components* C *of* N.

Assume that p *is the smallest positive integer such that* pN_D *is a divisor defined by the section* n_D *of the line bundle* $\mathcal{O}_X(pN_D)$. *Then, multiplication by the section* $n_D^{\frac{k}{p}}$ *induces an isomorphism*

$$H^0(X, kP_D) \simeq H^0(X, kD) \tag{2}$$

for all k *divisible by* p.

We take a flag $Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 = \{p\}$ with the curve Y_1 not contained in the augmented base locus $B_+(D)$ (which in particular implies that Y_1 is not a component of N_D). Then because of (2) we have for an arbitrary section $s \in H^0(X, kD)$ with k divisible enough

$$s = t \cdot n_D^{\frac{k}{p}}$$

for some section $t \in H^0(X, kP_D)$ and

$$\nu_1(s) = \nu_1(t) + \nu_1\big(n_D^{\frac{k}{p}}\big) = \nu_1(t) + \frac{k}{p}\nu_1(n_D) = \nu_1(t). \tag{3}$$

Similarly, we have

$$\nu_2(s) = \nu_2(t) + \nu_2\big(n_D^{\frac{k}{p}}\big) = \nu_2(t) + k \cdot \frac{1}{p}\nu_2(n_D). \tag{4}$$

It follows that the Okounkov body of D is up to translation by $\frac{1}{p}\nu_2(n_D)$ equal to that of P_D.

Corollary 2.2
Let D *be an effective divisor on a smooth algebraic surface* X *with Zariski decomposition* $D = P_D + N_D$ *and let* $Y_\bullet$ *be a flag as above. Then*

$$\Delta(D) = \Delta(P_D) + (0, \mathrm{ord}_p(N_D)).$$

In the view of the above corollary, it is sufficient to know what Okounkov bodies of nef effective divisors are. It turns out that on Del Pezzo surfaces there are only finitely many building blocks. This is made precise in the next section.

3. Del Pezzo surfaces

Let r be a fixed integer $0 \leqslant r \leqslant 8$. We fix r points $p_1, \ldots, p_r$ in the projective plane $\mathbb{P}^2$ in general position. More precisely we assume that

a) no three of these points are collinear,

b) no six of them are on the same conic,

c) a cubic curve passing through 6 of them and singular in the seventh point, is not passing through the eighth.

Let $f_r\colon X_r \to \mathbb{P}^2$ be the blowing up of $P_1,\dots,P_r$ with exceptional divisors $E_1,\dots,E_r$. Under the above assumptions, X_r is a smooth Del Pezzo surface, i.e., the anticanonical divisor $-K_{X_r}$ is ample, see [4]. We denote the class of the pullback by f_r of a line in $\mathbb{P}^2$ by H.

From now on, we fix also the following flag. Let Y_1 be a line in $\mathbb{P}^2$ not passing through any of the points $P_1,\dots,P_r$ and let $p \in Y_1$ be a point not lying on the image under f_r in $\mathbb{P}^2$ of any of the (-1)–curves on X_r. This assumption, in view of (4) guarantees that

$$\Delta(D) = \Delta(P_D)$$

for an arbitrary big divisor D on X_r.

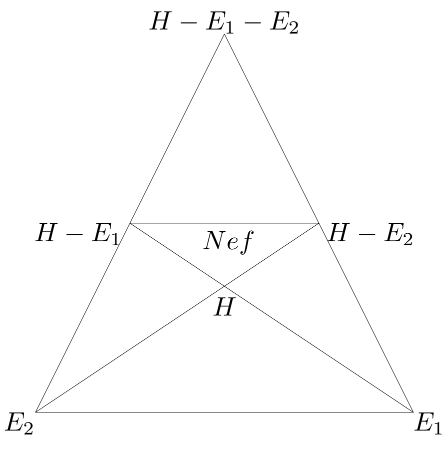
Del Pezzo surfaces are two-dimensional Fano varieties. It is well known from the Mori theory, see [3] and [10, Theorem 1.1.5], that the nef cone of a Fano variety is finitely generated.

For Del Pezzo surfaces it is easier to write down generators of the pseudo-effective cone than those of the nef cone. The effective cone is generated by classes of irreducible (-1)–curves on X_r for $r \geqslant 2$. For $r \leqslant 1$ one has to include also H in the set of generators.

One could naively expect, that in order to get the decomposition claimed in (1), one could take as the divisors P_i the generators of the nef cone. The following two simple examples show that this wouldn't work.

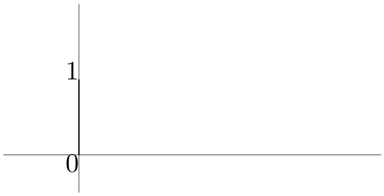
Example 3.1

Let X_2 be the blowup of $\mathbb{P}^2$ in two points. A slice of the effective cone of X_2 looks like in the following picture



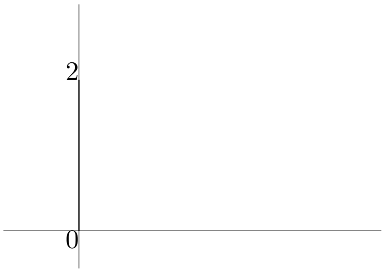
Picture 1. A slice of the effective cone of X_2.

We consider the generators $H - E_1$ and $H - E_2$ of the nef cone of X_2. The Okounkov bodies, constructed with respect to the flag, given in Section 2 coincide for both divisors. They are presented in the Picture 2.



Picture 2. Okounkov body of $H - E_i$ for $i = 1, 2$.

The Minkowski sum of two such segments is, again, a segment, presented in the next picture.

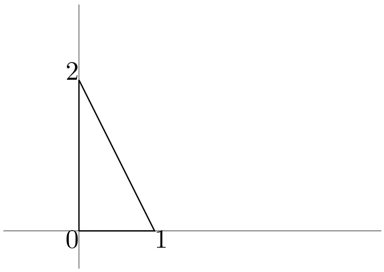


Picture 3. Minkowski sum of $\Delta(H - E_1)$ and $\Delta(H - E_2)$.

On the other hand

$$H - E_1 + H - E_2 = 2H - E_1 - E_2 = H + (H - E_1 - E_2)$$

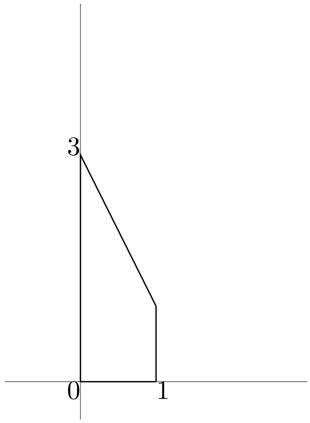
is a big (and nef) divisor, so its Okounkov body has in any case some positive volume. In fact, it is the triangle presented in the Picture 4.



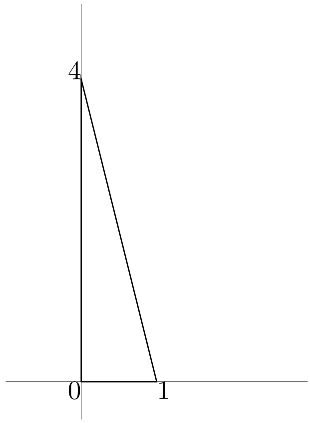
Picture 4. Okounkov body of $2H - E_1 - E_2$.

One might suspect that the reason for the bad behavior of the generators $H-E_1$ and $H - E_2$ is caused by them not being big. The next example shows that even for big and nef divisors the Okounkov bodies might not be additive.

Example 3.2
Now we look at X_3 with the flag fixed as explained in Section 2. We consider divisors $D_1 = 3H - 2E_1 - E_2$ and $4H - 2E_1 - 2E_2 - 2E_3$. They are both big and nef, with Okounkov bodies represented on Pictures 5 and 6, respectively.



Picture 5. Okounkov body of D_1.

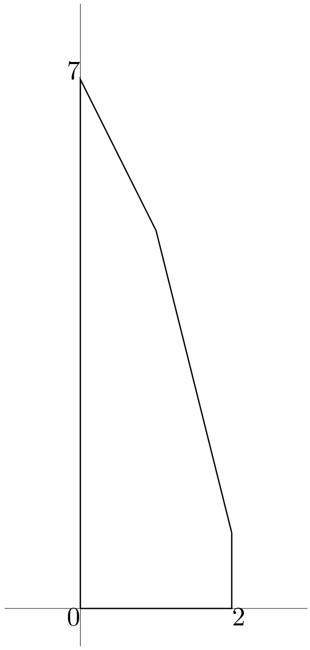


Picture 6. Okounkov body for D_2.

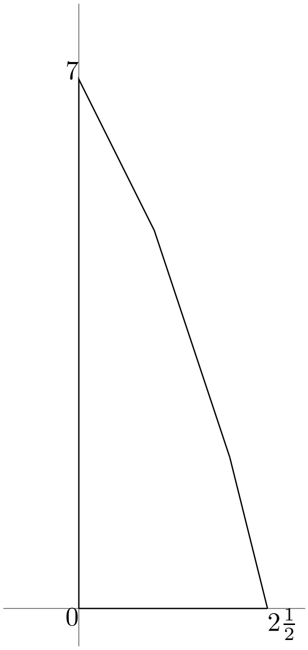
The Minkowski sum of $\Delta(D_1)$ and $\Delta(D_2)$ is presented in the Picture 7. On the other hand the Okounkov body of the sum

$$3H - 2E_1 - E_2 + 4H - 2E_1 - 2E_2 - 2E_3 = 7H - 4E_1 - 3E_2 - 2E_3$$

is presented in the Picture 8. The two figures do not agree.



Picture 7. Minkowski sum of $\Delta(D_1) + \Delta(D_2)$.



Picture 8. Okounkov body of $D_1 + D_2$.

These examples show that our main result stated in the next section is by no means obvious.

4. Minkowski decomposition

Theorem 4.1 (Minkowski decomposition on Del Pezzo surfaces)
Let X be a smooth Del Pezzo surface, i.e., $X = X_r$ for some r, or $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then there exists a finite set of nef divisors $P_1, \ldots, P_s$ such that for any big $\mathbb{R}$–divisor D we have

$$D = \sum_{i=0}^{s} a_i P_i + N_D \quad \textit{and} \quad \Delta(D) = \sum_{i=1}^{s} a_i \Delta(P_i)$$

with non-negative real numbers $a_i \in \mathbb{R}_{\geqslant 0}$.

Note that in the first equality there is the sum of divisors, whereas in the second equality the sum stands for the Minkowski sum of convex sets. We call the set $\{P_i\}$ the *Minkowski basis* of X_r, even though this is strictly speaking not a basis.

The complete proof of this theorem will appear in the forthcoming paper [9]. In this announcement we restrict our attention to the case $r = 3$ as this is already interesting enough.

Before we can proceed with the actual proof, we need to establish some notation. Following [2] we write

$$\mathrm{Null}(D) = \{C \subset X : \ C \text{ irreducible curve with } C \cdot D = 0\}$$

for the set of all irreducible curves orthogonal to a given $\mathbb{R}$–divisor D with respect to the intersection form on X.

We write

$$\mathrm{Null}^*(D) := \mathrm{Null}(D) \setminus \{E_1, \ldots, E_r\}$$

for the set $\mathrm{Null}(D)$ with $E_1, \ldots, E_r$ excluded.

The Neron-Severi group on X_r is generated by H and $E_1, \ldots, E_r$. We abbreviate

$$P(a; b_1, \ldots, b_r) := aH - b_1E_1 - \ldots - b_rE_r.$$

5. Proof of Theorem 4.1 for $r = 3$

We claim that as a set $\{P_i\}$ we can take the following divisors:

a) $P(1; 0, 0, 0)$,

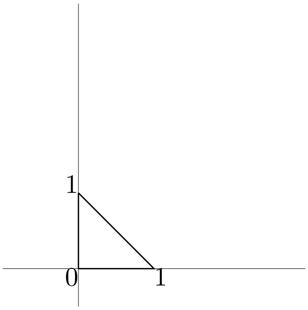
b) $P(1; 1, 0, 0)$, $P(1; 0, 1, 0)$, $P(1; 0, 0, 1)$,

c) $P(2; 1, 1, 0)$, $P(2; 1, 0, 1)$, $P(2; 0, 1, 1)$,

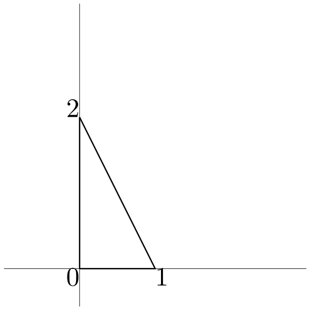
d) $P(2; 1, 1, 1)$,

e) $P(3; 2, 1, 1)$, $P(3; 1, 2, 1)$, $P(3; 1, 1, 2)$.

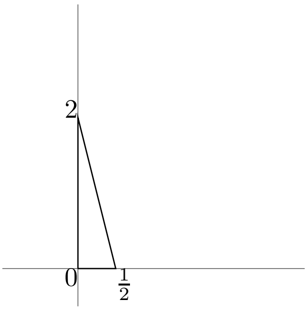
The divisors above are grouped in the obvious manner. The Okounkov bodies of divisors in each group are the same and they are depicted below. For the Okounkov bodies of the divisors of type b) see Picture 2 above.



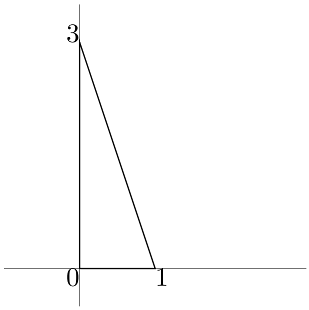
a) The Okounkov body of H



c) The Okounkov body of $2H - E_1 - E_2$



d) The Okounkov body of $2H - E_1 - E_2 - E_3$



e) The Okounkov body of $3H - 2E_1 - E_2 - E_3$

A nef divisor P can be written as a combination with non-negative coefficients of the divisors above (because the set contains the generators of the nef cone) but not in a unique way. It is in fact crucial for the Theorem to pick up the right decomposition. To this end we first list the space $\mathrm{Null}^*(\cdot)$ for each type of divisors in the Minkowski basis.

D	$\mathrm{Null}^*(D)$
H	$\emptyset$
$H - E_i$	$H - E_i$, $H - E_i - E_j$, $H - E_i - E_k$
$2H - E_i - E_j$	$H - E_i - E_j$
$2H - E_1 - E_2 - E_3$	$H - E_1 - E_2$, $H - E_1 - E_3$, $H - E_2 - E_3$
$3H - 2E_i - E_j - E_k$	$H - E_i - E_j$, $H - E_i - E_k$

The convention in this table is that i, j, k stay for mutually distinct indices.

In order to establish the theorem for arbitrary $\mathbb{Q}$–divisors, it is enough to work with the integral divisors, as the Okounkov bodies scale well. The claim for

$\mathbb{R}$–divisors follows then from the existence of the global Okounkov body, see [8, Theorem 4.5]. So we assume that P is an integral nef divisor on X_3. Next we compute the coefficients $\{a_i\}$ according to the following algorithm.

Let M be the divisor in the Minkowski basis given above with the property

$$\mathrm{Null}^*(M) = \mathrm{Null}^*(P).$$

Such an element exists and is unique. Indeed, it follows from the Index Theorem that $\mathrm{Null}^*(P)$ has a negative semi-definite intersection matrix. There are only finitely many such matrices possible on X_3 and each one of them appears in our list exactly once.

Then we set $P' := P - M$ and we claim that

$$\Delta(P) = \Delta(P') + \Delta(M). \tag{5}$$

Taking this for granted for a moment, we are finished with the proof of the Theorem, as we now apply our algorithm to P' and so on. This procedure terminates since we lower the absolute value of the coefficients of P in the basis $H, E_1, \ldots, E_r$ in every step.

The equality in (5) follows from observing that P and M lie on the same face (in the sense of convex geometry) of the nef cone of X_3. Moreover, subtracting M from P results in a divisor P' which either lies on the same face or on its boundary. Hence we can assume that

$$\mathrm{Null}^*(P) = \mathrm{Null}^*(M) = \{N_1, \ldots, N_s\}$$

and

$$\mathrm{Null}^*(P') = \{N_1, \ldots, N_s, N_{s+1}, \ldots, N_{s+t}\}.$$

Then

$$M = \mu_{Y_1}(M) \cdot Y_1 + \sum_{i=1}^{s} \alpha_i N_i \quad \text{and} \quad P' = \mu_{Y_1}(P') \cdot Y_1 + \sum_{j=1}^{s+t} \beta_j N_j,$$

where

$$\mu_{Y_1}(F) := \sup\{t \in \mathbb{R} : \ F - tY_1 \text{ is effective}\}.$$

Note that the exceptional divisors $E_1, \ldots, E_r$ do not appear in decompositions of nef divisors.

We claim that

$$\mu_{Y_1}(P' + M) = \mu_{Y_1}(P') + \mu_{Y_1}(M). \tag{6}$$

It is clear that we have the $\geqslant$ inequality in (6). Assume for the contrary that this inequality is sharp, i.e., $P = P' + M = \gamma \cdot Y_1 + R$ with some $\gamma > \mu_{Y_1}(P') + \mu_{Y_1}(M)$ and R, a pseudo-effective divisor. Comparing the two presentations of the sum $P' + M$ we have

$$(\gamma - \mu_{Y_1}(P') - \mu_{Y_1}(M)) \cdot Y_1 + R = \sum_{i=1}^{s} (\alpha_i + \beta_i) N_i + \sum_{i=s+1}^{s+t} \beta_i N_i.$$

In our case Y_1 is a big divisor, whereas the divisor on the right is not big and this contradiction shows (6).

Then we use the description of Okounkov bodies on surfaces from [8, Theorem 6.4]. For $P = P' + M$ we have

$$\Delta(P) = \{(x,y) \in \mathbb{R}^2 : \ 0 \leqslant x \leqslant \mu_{Y_1}(P) \text{ and } \alpha(x) \leqslant y \leqslant \beta(x)\}$$

with

$$\alpha(x) = \mathrm{ord}_p(N_{P-xY_1}) \quad \text{and} \quad \beta(x) = \mathrm{ord}_p(N_{P-xY_1}) + (Y_1 \cdot P_{P-xY_1}).$$

By our choice of the point p in the flag, $\alpha(x)$ is zero for all $0 \leqslant x \leqslant \mu_{Y_1}(D)$. The same is true for the α–functions for P' and M, so everything amounts to the computation of P_{P-xY_1}. We have

$$P_{P-xY_1} = \begin{cases} P' + P_{M-xY_1} & \text{for } 0 \leqslant x \leqslant \mu_{Y_1}(M), \\ P_{P'-(x-\mu_{Y_1}(M))Y_1} & \text{for } \mu_{Y_1}(M) \leqslant x \leqslant \mu_{Y_1}(P) = \mu_{Y_1}(P') + \mu_{Y_1}(M). \end{cases}$$

This shows the equality

$$\Delta(P' + M) = \Delta(P') + \Delta(M).$$

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Report of Meeting

14th International Conference on Functional Equations and Inequalities, Będlewo, September 11-17, 2011

Contents

The *Fourteenth International Conference on Functional Equations and Inequalities* was held from September 11 to 17, 2011 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Department of Mathematics of the Pedagogical University in Cracow* since 1984. For the fourth time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo.

This year's conference was dedicated to the memory of Professor Marek Kuczma, the founder of the *Polish School of Functional Equations and Inequalities*, who died 20 years ago.

The *Organizing Committee* of the 14th ICFEI consisted of Janusz Brzdęk as Chairman, Zbigniew Leśniak as Vice-Chairman, Anna Bahyrycz, Magdalena Piszczek, Paweł Solarz, Janina Wiercioch and Krzysztof Ciepliński, who also acted as Vice-Chairman and Scientific Secretary.

The *Scientific Committee* consisted of Professors: Dobiesław Brydak as Honorary Chairman, Janusz Brzdęk as Chairman, Nicole Brillouët-Belluot, Jacek Chmieliński, Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Zsolt Páles and Marek Cezary Zdun.

The 60 participants came from 12 countries: Australia, Austria, Denmark, France, Germany, Hungary, India, Iran, Israel, Russia, Serbia and Poland.

The conference was opened on Monday, September 12 by Professor Janusz Brzdęk – the Chairman of the Scientific and Organizing Committees, who welcomed the participants on behalf of the Organizing Committee and read a letter to them from Professor Władysław Błasiak, the Dean of the *Faculty of Mathe-*

matics, Physics and Technical Science of the Pedagogical University. The opening address was given by Professor Jacek Chmieliński, the Head of the Department of Mathematics. The opening ceremony was followed by the first scientific session chaired by Professor Roman Ger. Altogether, during 22 scientific sessions 3 lectures (given by Professors: Karol Baron – opening lecture, Stevo Stević and Boris Paneah) and 51 talks were delivered. They focused on functional equations in a single variable and in several variables, functional inequalities, stability theory, convexity, multifunctions, means and other topics. Several contributions have been made during special *Problems and Remarks* sessions.

On Tuesday, September 13, a picnic was organized. On the next day afternoon the participants visited Rogalin Palace with its gallery of paintings from the 19th and 20th centuries, and collection of horse-drawn vehicles. In the evening the piano recital was performed by Professor Marek Czerni. On Thursday, September 15, a banquet was held at the Palace in Będlewo. At the start of the banquet, George Gershwin's "Summertime" was performed by Professor Ewelina Mainka-Niemczyk.

The conference was closed on Friday, September 16 by Professor Janusz Brzdęk. He announced that Professors Ekaterina Shulman and László Székelyhidi had agreed to join the Scientific Committee. The 15th ICFEI will be organized in the south of Poland, in the spring of 2013.

The following part of the report contains the abstracts of the talks, the problems and remarks, and a list of the participants with their addresses.

Abstracts of Talks

Anna Bahyrycz *On the solutions of Wilson first generalization of d'Alembert's functional equation on some set*

Let A be a non-empty subset of an Abelian group. In the talk, under some simple additional assumptions on the set A, we deal with functions $f, g\colon A \to \mathbb{C}$ satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for $x, y \in A$ such that $x+y, x-y \in A$.

Karol Baron *Marek Kuczma*

The scientific output of Marek Kuczma consists of 179 papers (listed in [4]) published within the years 1958–1993 and three books [6, 7, 8] still used and quoted. Professor Marek Kuczma created and developed the theory of iterative functional equations but his name is also connected with important results on functional equations in several variables, in particular on Cauchy's equation and Jensen's inequality. In fact Marek Kuczma has founded a mathematical school: he supervised 13 Ph.D. dissertations, 10 his students have already their habilitation and 6 of them became full professors (cf. also [5]). In the talk I would like to present more information about this great teacher and, making also use of [1, 2, 3], some results of this outstanding mathematician.

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[2] B. Choczewski, *Papers of Marek Kuczma written in the last decade of his life*, Selected topics in functional equations and iteration theory (Graz, 1991), 7–16, Grazer Math. Ber. **316**, Karl-Franzens-Univ. Graz, Graz, 1992.

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[4] R. Ger, *Marek Kuczma, 1935-1991*, Aequationes Math. **44** (1992), 1–10.

[5] R. Ger, *Functional equations and inequalities* (Polish), *Half a century of mathematics in Upper Silesia* (Polish), 223–251, Pr. Nauk. Uniw. Śl. Katow. **2196**, Wydawn. Uniw. Śląskiego, Katowice, 2003.

[6] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne **46**, Państwowe Wydawnictwo Naukowe, Warszawa, 1968.

[7] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Pr. Nauk. Uniw. Śl. Katow. **489**, Uniwersytet Śląski, Katowice; Państwowe Wydawnictwo Naukowe, Warszawa, 1985 [Second edition: Edited and with a preface by Attila Gilányi, Birkhaüser Verlag, Basel, 2009].

[8] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications **32**, Cambridge University Press, Cambridge, 1990.

Janusz Brzdęk *Stability of linear equations of higher orders*
(joint work with **B. Xu** and **W. Zhang**)

We present some fixed point results, which provide a general method for investigations of the Hyers-Ulam stability of the linear operator equations of higher orders. In numerous cases, the Hyers-Ulam stability of such an equation is a consequence of a similar property of the corresponding first order equations. We describe some particular examples of applications for differential, integral, and functional equations.

Jacek Chmieliński *On approximate parallelogram identity in normed spaces*

Suppose that a norm in a real or complex space X approximately satisfies the parallelogram law, i.e.,

$$\big|\|x+y\|^2+\|x-y\|^2-2\|x\|^2-2\|y\|^2\big|\le\Phi(x,y),\qquad x,y\in X$$

with a given mapping $\Phi\colon X\to\mathbb{R}_+$. We show that for some control mappings Φ the above property yields that X is equivalent to an inner product space.

Jacek Chudziak *On continuous solutions of a composite functional equation*

Inspired by some problems concerning invariant utility functions we consider continuous solutions of the following functional equation

$$f(k(t)x+l(t))=a(t)f(x)+b(t).$$

Krzysztof Ciepliński *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*
(joint work with **J. Brzdęk**)

In the talk we present a fixed point theorem for complete non-Archimedean metric spaces and apply it to obtain the Hyers-Ulam stability of a quite wide class of functional equations in a single variable.

Marek Czerni *On a generalization of the problem of D. Brydak*

Let I be a real interval of the form $[0, a)$, where $0 < a \leq \infty$. Let $\psi\colon I \to \mathbb{R}$ be a continuous solution of the linear nonhomogeneous functional inequality

$$\psi[f(x)] \leq g(x)\psi(x) + h(x).$$

We assume the following hypotheses about given functions f, g and h:

(H_1) the function $f\colon I \to \mathbb{R}$ is continuous and strictly increasing. Moreover, $0 < f(x) < x$ for $x \in I^\star = I \setminus \{0\}$,

(H_2) the function $g\colon I \to \mathbb{R}$ is continuous and $g(x) > 0$ for $x \in I^\star$,

(H_3) the function $h\colon I \to \mathbb{R}$ is continuous and $h(0) = 0$,

(H_4) the functional sequence $G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]$ converges to zero almost uniformly in $I^\star$,

(H_5) the functional sequence $\varphi_n^\star(x) = \sum_{i=0}^{n-1} \frac{h[f^i(x)]}{G_{i+1}(x)}$ converges almost uniformly in $I^\star$.

In the talk we give partial answer to the following question: does there always exist a continuous solution $\varphi\colon I \to \mathbb{R}$ of the functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + h(x)$$

or

$$\varphi[f(x)] = g(x)\varphi(x)$$

such that the finite limit $\lim_{x\to 0^+} \frac{\psi(x)}{\varphi(x)}$ exists?

At the 3rd International Symposium on Functional Equations and Inequalities in Noszvaj (Hungary) in September 1986 D. Brydak put the similar problem for linear homogeneous inequality (see [1]). This problem was solved in [2].

[1] *Report on the Third International Symposium on Functional Equations and Inequalities. Abstracts from the symposium held in Noszvaj, September 21-27, 1986*, Publ. Math. Debrecen **38** (1991), 1–38.

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Włodzimierz Fechner *Functional equations with exotic addition*

S. Northshield in [1] introduced the term exotic addition for two types of operations he dealt with. One of them was the following "sine-type" addition on the real line

$$x \oplus y := xf(y) + yf(x)$$

with $f\colon \mathbb{R} \to \mathbb{R}$ enjoying some regularity properties. The name "sine-type" addition which appears in [1] is justified by the fact that if one takes mapping $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = \sqrt{1-x^2}$ for $x \in [-1,1]$, then in this particular case the sine function acts as a homomorphism between the real line with ordinary addition and the interval $[-1,1]$ with "exotic" addition $\oplus$. In other words, we have

$$\sin(x+y) = \sin(x) \oplus \sin(y), \qquad x, y \in \mathbb{R}.$$

Northshield provided several conditions equivalent to the associativity of $\oplus$ (see Theorem 4 and Corollary 1 in [1]). From these results it follows that the associativity seems to be a fairly restrictive assumption since it implies a particular form of the mapping f (given in an implicit form involving solutions of some ODE's).

We will deal with operation $\oplus$ without assuming its associativity and for mapping f defined on an arbitrary interval. We solve "exotic" modifications of some functional equations, including the equation of derivations, with ordinary addition replaced by $\oplus$.

[1] S. Northshield, *On two types of exotic addition*, Aequationes Math. **77** (2009), 1–23.

Żywilla Fechner *On some integral generalizations of trigonometric functional equations*

Let $(G,+)$ be a locally compact Abelian group, $\mathcal{B}(G)$ the space of all Borel subsets of G and $\mu\colon \mathcal{B}(G) \to \mathbb{C}$ a bounded regular measure. The following equation

$$\int_G \{f(x+y-s) + f(x-y+s)\}\, d\mu(s) = f(x)f(y), \qquad x, y \in G,$$

where $f\colon G \to \mathbb{C}$ is essentially bounded, was introduced and solved by Z. Gajda in [3]. We are going to present some other possible generalizations of this functional equation.

[1] Ż. Fechner, *A generalization of Gajda's equation*, J. Math. Anal. Appl. **354** (2009), 584–593.

[2] Ż. Fechner, *A note on a modification of Gajda's equation*, Aequationes Math. **82** (2011), 135–141.

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Roman Ger *On a subsequent problem of Roger Cuculière*

In the May 2011 issue of *The American Mathematical Monthly* (**118**, Problems and Solutions, p. 464) the following problem was proposed by Roger Cuculière:

Let E be a real normed vector space of dimension at least 2. Let f be a mapping from E into E, bounded on the unit sphere $\{x \in E : \|x\| = 1\}$, such that whenever

x and y are in E, $f(x + f(y)) = f(x) + y$. Prove that f is a continuous, linear involution on E. (Problem **11578**).

We shall present the general solution of the functional equation in question (in much more general setting) from which the proof spoken of will be obtained as a corollary.

Dorota Głazowska *Uniformly bounded composition operators in the space of functions of bounded φ-variation with weight in the sense of Riesz*
(joint work with **J. Matkowski**)

We prove that if a uniformly bounded (or equidistantly bouned) Nemytskij operator maps a space of functions of bounded φ-variation with weight function in the sense of Riesz into another space of the same type and its generator function is continuous with respect to the second variable, then this generator function is an affine function in the second variable.

Moshe Goldberg *Submultiplicativity and stability of sup norms on homotonic algebras*

An algebra $\mathcal{A}$ of real or complex valued functions defined on a set S shall be called *homotonic* if $\mathcal{A}$ is closed under forming of absolute values, and if for all f and g in $\mathcal{A}$, the product $f \times g$ satisfies $|f \times g| \leq |f| \times |g|$. Our purpose in this talk is to offer several examples of homotonic algebras and provide a simple inequality which characterizes submultiplicativity and strong stability for weighted sup norms on such algebras.

Niyati Gurudwan *Strong convergence theorem for finite family of m-accretive operators in Banach spaces*
(joint work with **B.K. Sharma**)

The purpose of this presentation is to propose a composite iterative scheme for approximating a common solution of a finite family of m-accretive (nonlinear) operators in a strictly convex Banach space having a uniformly Gateaux differentiable norm. As a consequence, the strong convergence of the scheme for a common fixed point of a finite family of pseudocontractive mappings is also obtained. The results presented herein improve and extend the corresponding results of Kim and Xu, Qin and Su, Xu, and Zegeye and Shahzad (see [1, 2, 3, 4] and the references given there) to a finite family of operators in a strictly convex Banach space.

[1] T.-H. Kim, H.-K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[2] X. Qin, Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl. **329** (2007), 415–424.

[3] H.-K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314** (2006), 631–643.

[4] H. Zegeye, N. Shahzad, *Strong convergence theorems for a common zero for a finite family of m-accretive mappings*, Nonlinear Anal. **66** (2007), 1161–1169.

Attila Házy *On (α,β,a,b)-convex functions*

Bernstein and Doetsch (see [1]) proved that the local upper boundedness of a Jensen-convex function yields its local boundedness and continuity as well on the whole domain, which implies the convexity of the function.

In this talk we present some Bernstein-Doetsch type results for (α,β,a,b)-convex functions, which were intoduced by Maksa and Páles (see [4]) in the following way:

Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open (α,β)-convex (that is, $\alpha(t)x+\beta(t)y \in D$ whenever $x,y \in D$ and $t \in [0,1]$) set, and $\alpha,\beta,a,b\colon [0,1] \to \mathbb{R}$ be given functions. The function f is called (α,β,a,b)-convex function if

$$f(\alpha(t)x+\beta(t)y) \leq a(t)f(x)+b(t)f(y), \qquad x,y \in D,\ t \in [0,1]$$

holds. To avoid the trivialities and the unimportant cases, we suppose that there exists an element t_0 such that $\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0$.

[1] F. Bernstein, G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Ann. **76** (1915), 514–526.

[2] P. Burai, A. Házy, *On approximately h-convex functions*, J. Convex Anal. **18** (2011), 447–454.

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[4] Gy. Maksa, Zs. Páles, *The equality case in some recent convexity inequalities*, Opuscula Math. **31** (2011), 269–277.

[5] A. Házy, *Bernstein-Doetsch type results for h-convex functions*, Math. Inequal. Appl. **14** (2011), 499–508.

[6] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

Eliza Jabłońska *On the pexiderized Gołąb-Schinzel equation*

Let X be a linear space over a commutative field $\mathbb{K}$. We characterize a general solution $f,g,h,k\colon X \to \mathbb{K}$ of the pexiderized Gołąb-Schinzel equation

$$f(x+g(x)y) = h(x)k(y),$$

as well as, in the case $\mathbb{K}=\mathbb{R}$, continuous on rays solutions of the equation.

Justyna Jarczyk *On some equality problem connected with conjugate means* (joint work with **J. Dascăl**)

Let $I \subset \mathbb{R}$ be an open interval and $p,q \in (0,1)$. We present some partial results on solutions (φ,ψ) of the functional equation

$$\varphi^{-1}\Big(\varphi(x)+\varphi(y)-\frac{1}{2}(\varphi(px+(1-p)y)+\varphi(qx+(1-q)y))\Big) = \psi^{-1}\Big(\frac{\psi(x)+\psi(y)}{2}\Big),$$

where $\varphi,\psi\colon I \to \mathbb{R}$ are four times continuously differentiable functions.

Witold Jarczyk *Note on an equation occurring in a problem of Nicole Brillouët-Belluot*
(joint work with **J. Morawiec**)

We study the functional equation

$$f(x)f^{-1}(x) = x^2$$

imposing no continuity assumptions on its bijective solutions defined on an interval. All the continuous bijections of an interval were determined in [2] when solving a problem posed by N. Brillouët-Belluot (see [1]).

[1] N. Brillouët-Belluot, *Problem posed during the Forty-nine International Symposium on Functional Equations, June 19–26, 2011, Graz-Mariatrost, Austria.*

[2] J. Morawiec, *On a problem of Nicole Brillouët-Belluot,* Aequationes Math. (in print).

Vyacheslav Kalnitsky *Solution of the Kuczma equation*

In the works of M. Kuczma and J. Sándor (see [1, 2]) was noted that all monotone convex (vertex) solutions of a special class of Stamate-type equation

$$\frac{f(x)-f(y)}{x-y} = \varphi(h(x), h(y)),$$

where $\varphi(u,v) = u+v, uv, \frac{1}{u+v}$, are arcs of cone sections. Assuming $h, \varphi \in C$, $\varphi(u,u)$ being invertible, the above equation is reducible to the form

$$\frac{f(x)-f(y)}{x-y} = K(f'(x), f'(y)), \tag{1}$$

where K is necessarily generalized mean. In this form Kuczma's result can be formulated in the following way: all solutions for arithmetical, geometrical and harmonical means are arcs of vertical paraboles, vertical hyperbols and horizontal parabols correspondently.

If for a mean K there is the function f solving equation (1), then we call it *solvable mean.* The simple criteria of solvability is proven: the nondegenerate mean $K \in C^1([\alpha] \times [\beta])$ is solvable if and only if for any four numbers $a, b, c, d \in [\alpha, \beta]$,

$$\begin{vmatrix} K(a,b) & K(a,b) & K(c,d) \\ K(c,a) & K(a,d) & K(c,d) \\ K(c,b) & K(b,d) & K(c,d) \end{vmatrix} = \begin{vmatrix} K(a,d) & K(b,d) & K(b,d) \\ K(a,d) & K(a,b) & K(c,b) \\ K(c,a) & K(c,a) & K(c,b) \end{vmatrix}.$$

Equation (1) has relations with different disciplines such as economics, operational research and low-dimensional geometry (see [3, 4, 5]).

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[3] E. Akerman, V. Kalnitsky, *The purpose function in the problem of the effective distribution of resurces*, Works of Int. Sci. School "MA SRQ 2001", 18-22 June 2001, St.P., Russia.

[4] V. Kalnitsky, *The reconstruction operator of two-argument function by its section*, Proc. of Int. Sci. Conf. "Lobachevsky readings-2001", Kazan.

[5] V. Kalnitsky, *Arithmetic properties of the Lagrange equation solutions*, Proc. of ICM'2002, August 20-28, Beijing, China.

Tomasz Kochanek *Steinhaus' lattice points problem for Banach spaces*

A classical property of the Euclidean plane, which goes back to H. Steinhaus, asserts that for any $n \in \mathbb{N}$ one may find a circle surrounding exactly n lattice points. P. Zwoleński generalized this result to the setting of Hilbert spaces replacing the set of lattice points by a *quasi-finite* set, i.e., a countably infinite set such that every ball contains only finitely many of its points.

We extend his result by giving a geometrical characterization (involving only the shape of the unit ball) of what we shall call a *Steinhaus property* of a given Banach space X:

(S) for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for any $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centered at y with $|A \cap B| = n$.

It turns out that every strictly convex Banach space shares this property, but in any dimension greater than 2 property (S) is weaker than strict convexity (e.g., as we will see, the space $L_1(0,1)$ satisfies (S), nonetheless it is not strictly convex). We will give some positive and negative examples for property (S) and discuss its connection with the existence of an equivalent strictly convex norm.

Barbara Koclęga-Kulpa *On a functional equation connected to Hermite quadrature rule*
(joint work with **T. Szostok**)

In the talk we deal with the functional equation

$$F(y) - F(x) = (y-x)\Big[\alpha f(x) + \beta f\Big(\frac{x+y}{2}\Big) + \alpha f(y)\Big] + (y-x)^2[g(y) - g(x)],$$

which is connected to Hermite quadrature rule. It is easy to note that particular cases of this equation generalize many well-known functional equations connected to quadrature rules and mean value theorems. Thus the set of solutions is too complicated to be described completely and therefore we prove that (under some assumptions) all solutions of the above equation have to be polynomials.

We obtain the aforementioned result using a lemma proved by M. Sablik, however this lemma works only in case $\beta \neq 0$. Taking $\beta = 0$, we obtain the following equation

$$F(y) - F(x) = (y-x)[f(x) + f(y)] + (y-x)^2[g(y) - g(x)],$$

which will also be solved in the talk.

Zygfryd Kominek *On pexiderized Jensen-Hosszú functional equation on the unit interval*

We solve the functional equation of the form

$$2f\left(\frac{x+y}{2}\right) = g(x+y-xy) + h(xy)$$

in the class of real functions defined on the unit interval $[0,1]$. We prove that it is not stable, but if two functions from the triple $\{f,g,h\}$ coincide the analogue equation is stable in the Hyers-Ulam sense.

Dawid Kotrys *Hermite-Hadamard inequality for convex stochastic processes*

In 1980 K. Nikodem introduced convex stochastic processes and investigated their regularity properties. In 1992 A. Skowroński obtained some further results on convex stochastic processes which generalize some known properties of convex functions. The aim of this talk is to extend the classical Hermite-Hadamard inequality to convex stochastic processes.

Grażyna Łydzińska *On iterative roots of some multifunctions with a unique set-value point*

In [1] and [2] the authors considered the problem of the existence of square iterative roots of multifunctions with exactly one set-value point. In this talk we present a generalization of some results from these papers.

[1] L. Li, J. Jarczyk, W. Jarczyk, W. Zhang, *Iterative roots of mappings with a unique set-value point*, Publ. Math. Debrecen **75** (2009), 203–220.

[2] W. Jarczyk, W. Zhang, *Also set-valued functions do not like iterative roots*, Elem. Math. **62** (2007), 73–80.

Ewelina Mainka-Niemczyk *Set-valued sine families*

Let K be a convex cone in a normed linear space X and let $F_t\colon K \to n(X)$, $E_t\colon K \to n(K)$ for $t \geq 0$. A family $\{E_t : t \geq 0\}$ is called a *sine family associated with family* $\{F_t : t \geq 0\}$ if

$$E_{t+s}(x) = E_{t-s}(x) + 2F_t(E_s(x)), \qquad 0 \leq s \leq t,\ x \in K.$$

Our primary objective in the talk is to show some properties of sine families, such as continuity and correlation with cosine families. Moreover, an integral representation of sine families is given.

Judit Makó *Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities*
(joint work with **Zs. Páles**)

In this talk, the connection between the functional inequalities

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \alpha_J(x-y), \qquad x,y \in D$$

and

$$\int_0^1 f(tx+(1-t)y)\rho(t)\,dt \le \lambda f(x)+(1-\lambda)f(y)+\alpha_H(x-y), \qquad x,y\in D$$

is investigated, where D is a convex subset of a linear space, $f\colon D \to \mathbb{R}$, $\alpha_H, \alpha_J\colon D-D \to \mathbb{R}$ are even functions, $\lambda \in [0,1]$, and $\rho\colon [0,1] \to \mathbb{R}_+$ is an integrable nonnegative function with $\int_0^1 \rho(t)\,dt = 1$.

Bartosz Micherda *On some inequalities of Hermite-Hadamard-Fejér type for (k,h)-convex functions*
(joint work with **T. Rajba**)

Let k, h be two given real functions defined on the interval $(0,1)$, and choose a nonempty set $D \subset \mathbb{R}$. Then a function $f\colon D \to \mathbb{R}$ will be called (k,h)-*convex* if, for all $x, y \in D$ and $t \in (0,1)$, $k(t)x + k(1-t)y \in D$ and

$$f(k(t)x+k(1-t)y) \le h(t)f(x)+h(1-t)f(y). \tag{1}$$

Condition (1), for conveniently chosen mappings k and h, produces various families of well-known functions, e.g. s-Orlicz convex functions, h-convex functions, subadditive functions and starshaped functions.

In our talk we present two new inequalities of Hermite-Hadamard-Fejér type for (k,h)-convex functions, and we apply them to some special kinds of mappings. This extends results given e.g. in [1] and [2].

[1] M. Bombardelli, S. Varošanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58** (2009), 1869–1877.

[2] S.S. Dragomir, S. Fitzpatrick, *Hadamard's inequality for s-convex functions in the first sense and applications*, Demonstratio Math. **31** (1998), 633–642.

[3] B. Micherda, T. Rajba, *On some Hermite-Hadamard-Fejér inequalities for (k,h)-convex functions* (preprint).

Krzysztof Misztal *Midconvexity for finite sets*
(joint work with **Jacek Tabor** and **Józef Tabor**)

Motivated by increasing role of computers we introduce two definitions of midconvexity for a finite subset X of $\mathbb{R}^N$:

Definition 1
We say that $W \subset X$ is *X-midconvex* if

$$\frac{v+w}{2} \in X \implies \frac{v+w}{2} \in W, \qquad v,w \in W.$$

Definition 2
We say that $W \subset X$ is *function X-midconvex* if there exists a function $f\colon X \to \mathbb{R}_+$ such that

$$\frac{x_1+x_2}{2} \in X \implies f\Big(\frac{x_1+x_2}{2}\Big) \le \frac{f(x_1)+f(x_2)}{2}, \qquad x_1,x_2 \in X$$

and $W = f^{-1}(0)$.

The properties of such notions are investigated, and the analogues of some classical results are shown. In particular we show that, for the second definition, an analogue of the theorem stating that compact convex set in $\mathbb{R}^N$ is a convex hull of its extremal points is valid.

Janusz Morawiec *On a problem of Nicole Brillouët-Belluot*

We solve the problem posed by Nicole Brillouët-Belluot during the 49th International Symposium on Functional Equations determining all continuous bijections $f\colon I \to I$ satisfying

$$f(x)f^{-1}(x) = x^2, \qquad x \in I,$$

where I is an arbitrary subinterval of the real line.

Marek Niezgoda *Schur-convexity and similar separability of vectors*

Let G be a compact group acting on an inner product space $(V, \langle\cdot,\cdot\rangle)$. A vector $y \in V$ is said to be *G-majorized* by a vector $x \in V$, written as $y \prec_G x$, if y lies in the the convex hull of the orbit $\{gx : \ g \in G\}$. A function $F\colon V \to \mathbb{R}$ is called *G-increasing* if for $x, y \in V$, $y \prec_G x$ implies $F(y) \le F(x)$.

A group G acting on V is said to be a *reflection group*, if G is the closure of a subgroup of the orthogonal group $O(V)$ generated by a set of reflections in the form $S_r x = x - 2\langle x, r\rangle r$ for $x \in V$, where $r \in V$, $\|r\| = 1$.

A differential characterization of G-increasing functions, due to Eaton and Perlman, is as follows.

Let G be a reflection group acting on V with $\dim V < \infty$. Assume $F\colon V \to \mathbb{R}$ is a G-invariant function, i.e., $F(gx) = F(x)$ for $x \in V$ and $g \in G$. If F is differentiable on V, then a necessary and sufficient condition that F be G-increasing on V is

$$\langle x, r\rangle \cdot \langle \nabla F(x), r\rangle \ge 0$$

for $x \in V$ and $r \in V$ such that $S_r \in G$, where $\nabla F(x)$ stands for the gradient of F at x.

In the particular situation when G is the permutation group $\mathbb{P}_n$ acting on $V = \mathbb{R}^n$, the preorder $\prec_G$ reduces to the classical majorization $\prec$ on $\mathbb{R}^n$. In this case, the G-increasing functions are called *Schur-convex* functions.

A differential characterization of Schur-convex functions is included in the following Schur-Ostrowski's Theorem.

If $F\colon \mathbb{R}^n \to \mathbb{R}$ is a symmetric differentiable function, then a necessary and sufficient condition that F be a Schur-convex function on $\mathbb{R}^n$ is

$$(x_i - x_j)\Big(\frac{\partial F}{\partial x_i}(x) - \frac{\partial F}{\partial x_j}(x)\Big) \ge 0, \qquad x \in \mathbb{R}^n,\ i, j = 1, 2, \ldots, n.$$

The aim of this talk is to present some extensions of the sufficiency part of Schur-Ostrowski and Eaton-Perlman's Theorems from majorized vectors to similarly separable vectors. A generalized Schur-Ostrowski's condition is introduced.

The obtained results are applied for cone orderings and group-induced cone orderings.

Agata Nowak *On a generalization of the Gołąb-Schinzel equation*

Inspired by a problem posed by J. Matkowski in [1] we investigate the equation

$$f\big(p(x,y)(xf(y)+y)+(1-p(x,y))(yf(x)+x)\big)\big) = f(x)f(y), \qquad x,y\in\mathbb{R},$$

where functions $f\colon\mathbb{R}\to\mathbb{R}$, $p\colon\mathbb{R}^2\to\mathbb{R}$ are assumed to be continuous.

[1] J. Matkowski, *A generalization of the Gołąb-Schinzel functional equation*, Aequationes Math. **80** (2010), 181–192.

Andrzej Olbryś *On some derivatives and (s,t)-convex functions*

In our talk we consider some kinds of derivatives and investigate their connections with (s,t)-convexity.

Zsolt Páles *On the generalization of the lower Hermite-Hadamard inequality and Korovkin type theorems*

We investigate functions that satisfy an approximate or strenghtened version of the lower Hermite-Hadamard inequality. Under certain assumptions we deduce that they are also approximately or strongly convex in an appropriate sense. The approach involves certain Korovkin type approximation theorems.

Boris Paneah *On the general theory of multidimensional functional operators: new problems and new approaches*

At first, using the frameworks of the classical triad: where we are? who we are? where do we go? we discuss the notion "General theory of MFO". The most meaningful part here is undoubtedly nonphilosophical second part. Up to recently the only solvability of the Ulam–stability problem for new functional operators traditionally considered as an advance in the general theory of MFO. But now such approach can not be treated as progressive and, moreover, it is harmful one.

The following problem is deep, very interesting, and plays an important role in applications:

given an MFO $\mathcal{P}$*, to describe asymptotic behavior of solutions to nonhomogeneous* MFE

$$\mathcal{P}F = H_\varepsilon(x), \qquad x\in D\subset\mathbb{R}^n$$

with $H_\varepsilon(x)=\mathcal{O}(\varepsilon)$*, as* $\varepsilon\to 0$.

In his book Ulam guessed the answer for the Cauchy operator, and Hyers verified this. The answer is:

$$F(t)=\lambda t+\mathcal{O}(\varepsilon),$$

where λ is an arbitrary real number. Thus, the function $\varphi(t)=\lambda t$ is the main term of the asymptotic of the solution to the equation

$$\mathcal{P}F = H_\varepsilon, \quad \text{as } H_\varepsilon\to 0 \text{ for } \varepsilon\to 0.$$

The excellent question, excellent answer, and ... no mystical stability.

In 2006 it was established something cardinally new in searching asymptotic behavior of the same function F. Namely, to identify the main term of the asymptotic we do not need to know the function $\mathcal{P}F$ on the full domain D. It suffices to know it only at the points of a curve Γ (Γ-asymptotic). This result generates very actual problem. Given an operator $\mathcal{P}$ to describe a set of submanifolds Γ for which the Γ-asymptotic problem

$$\mathcal{P}F\,|_{\Gamma}= H_\varepsilon(x),\ x \in D,\ H_\varepsilon(x) = \mathcal{O}(\varepsilon) \implies F = \psi + \mathcal{O}(\varepsilon)$$

is solvable.

It is not difficult to formulate a series important technical problems leading to the significant extension of the class MFO of operators $\mathcal{P}$ generating for some Γ a solvable Γ-asymptotic problem.

Example: $a = b + c$.

The following problems are formulated in a general form for the first time, although some particular cases have been considered earlier by the speaker.

Inverse problem. To describe a class of MFO in a domain D such that any operator from this class can be uniquely determined by its asymptotic behavior.

It is a surprising result reminding the famous inverse problem in the spectral theory of the differential operators. The possibility to reconstruct an MFO operator using only the asymptotic behavior of the solution to the equation $\mathcal{P}F = H_\varepsilon$ must find many important applications.

Uniqueness problem. Given an MFO operator $\mathcal{P}$ in a domain $D \subset \mathbb{R}^n$, to find a curve $\Gamma \subset D$ such that

$$\text{if } \mathcal{P}F\,|_{\Gamma}= 0,\ \text{then } \mathcal{P}F = 0 \text{ in } D.$$

It is an analogue of the famous uniqueness theorem in the theory of holomorphic functions.

Magdalena Piszczek *The properties of functional inclusions and Hyers-Ulam stability*

We show the properties of some inclusions, especially we prove that a set-valued function satisfying these inclusions admits, in appropriate conditions, a unique selection. As a consequence we obtain a result on the Hyers-Ulam stability of the functional equation

$$\Psi \circ f \circ a = f,$$

where $\Psi\colon Y \to Y$, $f\colon K \to Y$, $a\colon K \to K$, K is a nonempty set and Y is a complete metric space.

Wolfgang Prager *On a system of inhomogeneous linear functional equations* (joint work with **J. Schwaiger**)

Given $a, A \in \mathbb{R}$ with $aA \neq 0$, and an additive function $\phi\colon \mathbb{R} \to \mathbb{R}$, we give the possible additive solution(s) of the equation

$$\alpha(ax) - A\alpha(x) = \phi(x). \tag{1}$$

Imposing the same assumptions as above on b, B, ψ, we consider (1) together with

$$\alpha(bx) - B\alpha(x) = \psi(x) \tag{2}$$

and investigate solvability of the system (1), (2) within the set of additive functions. Finally, the role of system (1), (2) in our efforts to find necessary and sufficient conditions for solvability of the inhomogeneous general linear functional equation will be discussed.

Ludwig Reich *Reversible power series and generalized Abel equations*
(joint work with **P. Kahlig**)

An invertible formal power series F with complex coefficients is called *reversible* if there exists an invertible series T such that

$$F^{-1} = T^{-1} \circ F \circ T \tag{1}$$

holds. If, in particular, we can choose $T(X) = \eta X$ in (1), then (1) yields the generalized Legendre-Gudermann equation

$$F^{-1}(X) = \frac{1}{\eta} F(\eta X) \tag{2}$$

for F.

We are interested here in solutions F with $F(X) = X + \dots,\ F(X) \neq X$. We characterize the values of η for which (2) has such a solution, then we construct the set of all solutions of (2) using ideas of J. Haneczok and we discuss the following connections with (generalized) Abel equations.

Theorem
(i) $F(X) = X + \dots,\ F(X) \neq X$, is reversible if and only if there exists a non-constant Laurent series and a Möbius transformation L such that

$$V(F(X)) = L(V(X))$$

holds.
(ii) A formal series F as in (i) is a solution of (2) if and only if there exists a Laurent series V and a constant $C \in \mathbb{C} \setminus \{0\}$ such that

$$V(\eta X) = -V(X),$$
$$V(F(X)) = V(X) + C$$

holds.

This theorem can also be used to construct reversible series.

Maciej Sablik *Functional equations characterizing future life-time*

In our talk we will present a source of functional equations appearing in actuarial mathematics. The analytic form of future life-time has been an essential concept for actuaries since it makes calculations easier. Another facilitation of calculations is used in the procedure of “group insurance” where many lives are replaced by one artificial, usually aggregated. The method leads to some functional equations which in turn characterize models of de Moivre, Gompertz, Makeham and Weibull among others.

Jens Schwaiger *On the construction of functional equations with prescribed solutions of a certain type*

In the literature one may find certain functional equations such that their general solution is a homogeneous polynomial of degree n where $n \leq 5$. One example is the equation

$$f(kx+y)+f(ky-y)=k^2(f(x+y)+f(x-y))+2k^2(k^2-1)f(x)-2(k^2-1)f(y).$$

This equation was considered in [1] for $k \in \mathbb{N}$, $k \geq 2$. One can motivate the special form of the coefficients not only for this equation but for much more general cases. For example the following holds true.

Theorem
Let $n \in \mathbb{N}$ and let $\rho_1, \rho_2, \ldots, \rho_n$ be rationals different from 0 such that the squares ρ_i^2 are different in pairs. Then for any given rational number k a certain linear system of equations has a unique solution $(\alpha_0, \alpha_1, \ldots, \alpha_n)$. If, moreover, V, W are non-trivial rational vector spaces, then all generalized homogeneous polynomials $f\colon V \to W$ of degree $2n$ satisfy

$$f(kx+y)+f(kx-y)+\sum_{i=1}^{n-1}\alpha_i(f(x+\rho_i y)+f(x-\rho_i y))+\alpha_0 f(x)+\alpha_n f(y)=0,\ x,y \in V.$$

Provided that k is not a zero of a certain polynomial with rational coefficients this equation does not have other solutions.

[1] M. Eshaghi Gordji, Ch. Park, M.B. Savadkouhi, *The stability of a quartic type functional equation with the fixed point alternative*, Fixed Point Theory **11** (2010), 265–272.

Ekaterina Shulman *Subadditive set-functions on groups and applications to functional equations*

Let G be a group and Ω be an arbitrary set. A map $F\colon G \to 2^\Omega$ is called *subadditive* if $F(gh) \subset F(g) \cup F(h)$ for all $g, h \in G$. Let us denote by $|M|$ the number of elements of a subset $M \subset \Omega$. It will be shown that

$$\Big|\bigcup_{g\in G} F(g)\Big| \leq 4 \sup_{g\in G} |F(g)|.$$

We also establish the extensions of this inequality to maps with values in measurable subsets of a measure space and to maps with values in subspaces of a linear space. We apply this technique to the functional equation

$$f(g_1 g_2 \ldots g_n) = \sum_E \sum_{j=1}^{N_E} u_j^E v_j^E, \tag{1}$$

where E runs through all proper non-empty subsets of $\{1, 2, \ldots, n\}$, $N_E \in \mathbb{N}$ and for each E, the functions u_j^E only depend on variables g_i with $i \in E$, while the v_j^E only depend on g_i with $i \notin E$. Namely, we prove that any bounded continuous function f on G satisfying (1), for an $n > 2$, is a matrix element of a continuous finite-dimensional representation of G. Earlier this was known only for topologically finitely generated G.

Dhiraj Kumar Singh *On three sum form functional equations*
(joint work with **P. Nath**)

The general solutions of three sum form functional equations, without imposing any regularity conditions on any of the mappings appearing in these equations, have been obtained.

Barbara Sobek *Wilson's functional equation on a restricted domain*

Assume that X is a real or complex linear topological space and D is a non-empty, open and connected subset of $X \times X$. Let

$$\begin{aligned} D_+ &:= \{x + y :\ (x, y) \in D\}, \\ D_- &:= \{x - y :\ (x, y) \in D\}, \\ D_1 &:= \{x :\ (x, y) \in D \text{ for a } y \in X\} \end{aligned}$$

and

$$D_2 := \{y :\ (x, y) \in D \text{ for an } x \in X\}.$$

We study the equation

$$f(x + y) + g(x - y) = h(x)k(y), \qquad (x, y) \in D,$$

where $f\colon D_+ \to \mathbb{C}$, $g\colon D_- \to \mathbb{C}$, $h\colon D_1 \to \mathbb{C}$ and $k\colon D_2 \to \mathbb{C}$ are unknown functions. We investigate the problem of existence and uniqueness of extensions of the solutions and determine the general solution of this equation. Some results concerning conditional d'Alembert's equation are also presented.

Przemysław Spurek *Strict numerical verification of optimality condition for approximately midconvex functions*
(joint work with **Jacek Tabor**)

Let X be a normed space and V be a convex subset of X. Let $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$. A function $f\colon V \to \mathbb{R}$ is called *α-midconvex* if

$$f\Big(\frac{x+y}{2}\Big) - \frac{f(x) + f(y)}{2} \le \alpha(\|x - y\|), \qquad x, y \in V.$$

It can be shown that every continuous α-midconvex function satisfies the following estimation:

$$f(tx + (1-t)y) - tf(x) - (1-t)f(y) \le \sum_{k=0}^{\infty} \frac{1}{2^k} \alpha(d(2^{kt}\|x - y\|)), \qquad t \in [0, 1],$$

where $d(t) := 2\mathrm{dist}(t, \mathbb{Z})$ for $t \in [0, 1]$.

An important problem lies in verifying for which functions α the above estimation is optimal. The conjecture of Zs. Páles that this is the case for functions of type $\alpha(r) = r^p$ for $p \in (0,1)$, was proved by J. Makó and Zs. Páles in [1].

In this paper we present a computer assisted method to verifying optimality of this estimation in the class of piecewise linear functions α.

[1] J. Makó, Zs. Páles, *Approximate convexity of Takagi type function*, J. Math. Anal. Appl. **369** (2010), 545–554.

Henrik Stetkær *Levi-Civitá's functional equation*

Let G be a group. Levi-Civitá's functional equation

$$f(xy) = \sum_{l=1}^{N} g_l(x)h_l(y), \qquad x, y \in G, \tag{1}$$

where $f, g_1, \dots, g_N, h_1, \dots, h_N \colon G \to \mathbb{C}$ are unknown, has been thoroughly studied on abelian groups by Székelyhidi in [1]. But little is known about its solutions on non-abelian groups, even for N as small as 2.

We shall briefly discuss the general structure of the solutions of (1) on any group G, before we concentrate on a choice example, viz.

$$f(xy) = f(x)h(y) + f(y), \qquad x, y \in G,$$

$f, h \colon G \to \mathbb{C}$ being the unknown functions. It turns out that h is multiplicative on any group G, if $f \neq 0$. We will prove that f is central on all nilpotent groups, and give an example of a non-central f on the $(ax+b)$-group. We conclude that the solution formulas for f are the same on nilpotent groups as on abelian, and that a new phenomenon occurs on the $(ax+b)$-group.

[1] L. Székelyhidi, *Convolution type functional equations on topological abelian groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

Stevo Stević *On some nonlinear recurrences*

Studying nonlinear difference equations and systems has attracted considerable attention in the last few decades. Usually such equations cannot be solved so that the behavior of their solutions is investigated by various analytic methods. Here we present several classes of difference equations and systems whose solutions can be explicitly found.

First we present some classical methods for solving the nonhomogeneous linear difference equation of the first order

$$x_{n+1} = p_n x_n + q_n, \qquad n \in \mathbb{N}_0,$$

where $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are arbitrary real sequences and $x_0 \in \mathbb{R}$. Then we emphasize the role of the equation by giving numerous applications of it, for example in: getting Cauchy-Binet formula, solving nonhomogeneous second order difference equation with constant coefficients, solving some homogeneous second order difference equations with nonconstant coefficients, solving Beverton-Holt difference equation and studying its periodic solutions etc.

Further we show that the system of difference equations

$$u_{n+1} = \frac{w_n}{1+s_n}, \qquad v_{n+1} = \frac{t_n}{1+r_n},$$

where w_n, s_n, t_n and r_n are some of the sequences u_n or v_n, with $u_0, v_0 \in \mathbb{R}$, can be solved in many cases.

Finally we show that the system of difference equations

$$x_{n+1} = \frac{ax_{n-1}}{by_n x_{n-1} + c}, \quad y_{n+1} = \frac{\alpha y_{n-1}}{\beta x_n y_{n-1} + \gamma}, \qquad n \in \mathbb{N}_0,$$

where parameters $a, b, c, \alpha, \beta, \gamma$ and initial values x_{-1}, x_0, y_{-1}, y_0 are real numbers, can be also solved.

[1] S. Stević, *More on a rational recurrence relation*, Appl. Math. E-Notes **4** (2004), 80–85.

[2] S. Stević, *On a system of difference equations*, Appl. Math. Comput. (to appear).

[3] S. Stević, *On some solvable systems of difference equations*, Appl. Math. Comput. (to appear).

László Székelyhidi *Polynomial functions on Abelian groups*

Polynomial functions on Abelian groups play a basic role in the theory of functional equations and spectral analysis. In this paper we investigate the ring-structure of polynomial functions on topological Abelian groups. We show that polynomial functions form a Noetherian ring if and only if the linear space of continuous homomorphisms of the group into the additive group of complex numbers is finite dimensional. In the case of discrete Abelian groups this is equivalent to the presence of spectral synthesis.

Jacek Tabor *New approach to entropy*
(joint work with **M. Śmieja**)

The classical approach to entropy lies in the division of the given measure space into pairwise disjoint sets. We show that we can equivalently use a partition of the measure into measures with not necessarily disjoint supports.

The basic role in our proof plays the classical Hardy-Littlewood-Polya Theorem.

Józef Tabor *Uniform convexity*
(joint work with **Jacek Tabor**)

We present some convenient tools to compute the modulus of uniform convexity of a given convex function $f\colon I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$.

We first show that if f' is convex or concave then the modulus of uniform convexity of f equals to the Bergman distance at a respective endpoint of I. Then, in our main result, we give an estimation from below the modulus of uniform convexity of f by applying moduli of uniform convexities of f restricted to intervals J, K such that I is the union of J and K.

Jointly the two above mentioned results allow to estimate the moduli of uniform convexity for a large class of convex functions.

Jörg Tomaschek *On the solvability of generalized Dhombres functional equations*

The generalized Dhombres functional equation in the complex domain was introduced in [1] and is given by

$$f(zf(z)) = \varphi(f(z)), \tag{1}$$

where f is an unknown function and φ is a known one. In [2] it is shown that (1) is equivalent to the transformed generalized Dhombres functional equation

$$g(w_0 z + zg(z)) = \tilde{\varphi}(g(z)).$$

We discuss solutions f of (1) with $f(0) = w_0$, where w_0 is a root of unity of order $l \geq 2$, and we characterize those equations (1) which have non-trivial solutions. After that an example where the given function $\tilde{\varphi}$ is a Möbius transformation is computed.

[1] L. Reich, J. Smítal, M. Štefánková, *Local analytic solutions of the generalized Dhombres functional equation I*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **214** (2005), 3–25.

[2] L. Reich, J. Smítal, M. Štefánková, *Local analytic solutions of the generalized Dhombres functional equation II*, J. Math. Anal. Appl. **355** (2009), 821–829.

Hamid Vaezi *Fuzzy approximation of an additive functional equation*

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of the functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big)$$

in fuzzy Banach spaces. Some applications of our results to the stability of the above equation in the case when f is a mapping from a normed space to a Banach space will also be presented.

Szymon Wąsowicz *Spline approximation method in higher-order convexity business*

It is well-known that a continuous convex function $f\colon [a,b] \to \mathbb{R}$ can be uniformly approximated on $[a,b]$ by convex polygonal functions. This property allows us to give easy proofs of many linear inequalities involving (continuous) convex functions, among others of the celebrated Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Our considerations will be based on the observation that the left hand side inequality of (1) gives the better estimate of the integral mean value from the right one

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad (2)$$

whenever f is convex on $[a,b]$.

The similar approximation property holds for convex functions of higher order. Namely, every continuous n-convex function $f\colon [a,b]\to\mathbb{R}$ can be uniformly approximated on $[a,b]$ by n-convex spline functions of order n (Bojanic, Roulier, 1974). In the talk we will show an aplication of this result to prove some counterparts of (2) for convex functions of higher order.

Alfred Witkowski *Interpolations of Schwab-Borchardt mean*

For positive numbers x, y the pair of sequences

$$x_{n+1} = \frac{x_n+y_n}{2}, \quad y_{n+1} = \sqrt{y_n\frac{x_n+y_n}{2}}, \quad x_0 = x, \quad y_0 = y \qquad (1)$$

converges to a common limit called the Schwab-Borchardt mean

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\arccos\frac{x}{y}}, & x<y,\\ \dfrac{\sqrt{x^2-y^2}}{\operatorname{arccosh}\frac{x}{y}}, & y<x,\\ x, & x=y. \end{cases}$$

Algorithm (1) was known to Gauss but has been rediscovered by Borchradt and named after him.

Two means introduced by Seiffert

$$P(x,y) = \begin{cases} \dfrac{x-y}{2\arcsin\frac{x-y}{x+y}}, & x\neq y,\\ x, & x=y, \end{cases}$$

$$T(x,y) = \begin{cases} \dfrac{x-y}{2\arctan\frac{x-y}{x+y}}, & x\neq y,\\ x, & x=y \end{cases}$$

are of great interest for many mathematicians. Neuman and Sándor proved that both are particular cases of the Schwab-Borchardt means, namely

$$P(x,y) = SB\Big(\sqrt{xy}, \frac{x+y}{2}\Big) \quad \text{and} \quad T(x,y) = SB\bigg(\frac{x+y}{2}, \sqrt{\frac{x^2+y^2}{2}}\bigg).$$

Interesting inequalities between P, T, arithmetic, geometric, logarithmic, identric and power means were obtained by many authors using analytic approach or properties of the Schwab-Borchardt algorithm.

In this talk we use geometric properties of the "upper" part of SB to generalize those results and to obtain some new estimates. In particular we show some new interpolations of the Seiffert means.

David Yost *Pseudolinear functions, Banach spaces and polyhedra*

We begin with F-spaces (which are not necessarily locally convex) and a class of mappings between them defined by a certain functional inequality. We briefly describe how their study led us to some results about Minkowski decomposability of finite-dimensional convex sets.

Marek C. Zdun *On some applications of Kuczma's ideas to Schröder's equation in multidimensional case*

Let $U \subset \mathbb{R}^N$ be a neighbourhood of the origin, a function $F\colon U \to U$ be of class C^2 and $0 \in \operatorname{Int} U$ be an attractive fixed point of F. We consider a problem when a regular solution φ of Schröder's equation

$$\varphi(F(x)) = S\varphi(x),$$

where $S = \mathrm{d}F(0)$, is given by

$$\varphi(x) = \lim_{n\to\infty} S^{-n} F^n(x).$$

We give some sufficient conditions for truthfulness of this formula as well as some conditions which imply its falsehood.

Marek Żołdak *Approximately convex functions on Abelian topological groups*

Let $(G,+)$ be an Abelian topological group and let $\alpha\colon G \to \mathbb{R}_+$ be an even function. A function $f\colon D \to \mathbb{R}$, where D is a subset of G, is called *α-convex* if

$$f(z) \le \frac{f(x)+f(y)}{2} + \alpha(x-y)$$

for all $x, y, z \in D$ such that $x + y = 2z$.

Our main result is that if $\alpha(0) = 0$, α is continuous at zero, D is open and connected, f is α-convex and locally bounded above at a point, then f is locally uniformly continuous. The same is true if we replace the assumption that f is locally bounded above at a point by assumption that f is Haar measurable or Baire measurable.

Problems and Remarks

1. Remark.

We consider the Sierpiński Carpet $\mathcal{L}$ (defined in [1]), which is the ICFEI-Logo-Set. For convenience, we work with the set shifted and sized according to the requirement

$$\operatorname{conv}\{\mathcal{L}\} = \Big[-\frac{1}{2}, \frac{1}{2}\Big]^2.$$

From the symmetry and invariance properties of this set we obtain the following covering of $\mathcal{L}$ by its eight subsets (the *self-similarity equation*)

$$\mathcal{L} = \frac{1}{3}\mathcal{A} + \frac{1}{3}\mathcal{L} = \bigcup_{a\in\mathcal{A}} \Big(\frac{1}{3}a + \frac{1}{3}\mathcal{L}\Big), \tag{1}$$

where $\mathcal{A} = \{-1,0,1\}^2 \setminus \{(0,0)\}$. Accordingly, we seek a suitably invariant Borel probability measure $\mu_{\mathcal{L}}$ concentrated on $\mathcal{L}$. In terms of the random variables

- L is a 2-dimensional random variable (r.v.) with the probability distribution (p.d.) $P_L = \mu_{\mathcal{L}}$,
- A is an independent 2-dimensional r.v. with the classical p.d. $P_A = P_{\mathcal{A}}^{\text{class}}$ on the set $\mathcal{A}$,

the expected invariance of $\mu_{\mathcal{L}}$ is expressed as follows (see (1))

$$P_L = \frac{1}{8}\sum_{a\in\mathcal{A}} P_{\frac{1}{3}a+\frac{1}{3}L} = P_{\frac{1}{3}A+\frac{1}{3}L} = P_{\frac{1}{3}A} * P_{\frac{1}{3}L}, \tag{2}$$

where $*$ stands for the convolution of measures. Now, property (2) is equivalent to the following *Poincaré equation* for the characteristic function of L

$$\varphi_L(t) = \varphi_{\frac{1}{3}A}(t)\cdot\varphi_{\frac{1}{3}L}(t) = \varphi_A\Big(\frac{t}{3}\Big)\cdot\varphi_L\Big(\frac{t}{3}\Big). \tag{3}$$

Thus, by iteration procedure, with the use of continuity at 0 only ($\lim_{x\to 0}\varphi_L(x) = 1$), we arrive at the unique solution of (3)

$$\varphi_L(t) = \prod_{n=1}^{\infty}\varphi_A\Big(\frac{t}{3^n}\Big) = \varphi_S(t),$$

where almost surely $S = \sum_{n=1}^{\infty}\frac{A_n}{3^n}$ and the random vectors A_n for $n\in\mathbb{N}$ are independent, all with the same classical p.d. on $\mathcal{A}$. Since the intersection of $\mathcal{L}$ and the open square $(-\frac{1}{6},\frac{1}{6})^2$ is empty, and the same property has the set of values of the series S, with the use of the symmetry of $\mathcal{L}$ we obtain the following well-known result

Corollary
The Borel measure $\mu_{\mathcal{L}} = P_L$ concentrated on $\mathcal{L}$ and satisfying (2) exists and is unique. Moreover, all possible values of the infinite sums of vectors $\sum_{n=1}^{\infty}\frac{a_n}{3^n}$ with $a_n\in\mathcal{A} = \{-1,0,1\}^2\setminus\{(0,0)\}$ form a set of full measure $\mu_{\mathcal{L}}$.

[1] W. Sierpiński, *On a curve which contains the image of any curve* (Russian), Mat. Sb. **30** (1916), 267–287.

Joachim Domsta

2. Remark.

Let d denote the distance function from integer numbers defined by

$$d(x) := \inf\{|x-k|,\ k\in\mathbb{Z}\}.$$

Then one can see that d is nonnegative and even. It is not difficult to prove that

d is also subadditive. It easily follows from these properties that

$$|d(x) - d(y)| \leq d(x - y), \qquad x, y \in \mathbb{R}. \tag{1}$$

As an application of these properties we have the following result.

THEOREM
Let $a, b, a_1, b_1, \ldots, a_k, b_k$ be positive numbers such that the rectangle of sides a, b is the union of rectangles of sides a_i, b_i $(i = 1, \ldots, k)$ such that these rectangles have no interior points in common. Then

$$d(a)d(b) \leq \sum_{i=1}^{k} d(a_i)d(b_i). \tag{2}$$

Proof. Assume that $I = [0, a] \times [0, b]$ and $I_i = [x_i, x_i + a_i] \times [y_i, y_i + b_i]$ for some points $(x_i, y_i) \in I$. Let

$$f(x, y) := d'(x)d'(y), \qquad (x, y) \in I,$$

and compute the integral of f over I in two ways.

First, using Fubini's Theorem,

$$\begin{aligned} \int_I f &= \int_0^a \Big(\int_0^b f(x, y)\, dy \Big)\, dx = \int_0^a d'(x)\, dx \cdot \int_0^b d'(y)\, dy \\ &= (d(a) - d(0)) \cdot (d(b) - d(0)) = d(a) \cdot d(b). \end{aligned}$$

Secondly, also using the additivity of the integral and (1),

$$\begin{aligned} \int_I f &= \sum_{i=1}^{k} \int_{I_i} f = \sum_{i=1}^{k} \int_{x_i}^{x_i + a_i} \Big(\int_{y_i}^{y_i + b_i} f(x, y)\, dy \Big)\, dx \\ &= \sum_{i=1}^{k} \int_{x_i}^{x_i + a_i} d'(x)\, dx \cdot \int_{y_i}^{y_i + b_i} d'(y)\, dy \\ &= \sum_{i=1}^{k} (d(x_i + a_i) - d(x_i)) \cdot (d(y_i + b_i) - d(y_i)) \\ &\leq \sum_{i=1}^{k} |d(x_i + a_i) - d(x_i)| \cdot |d(y_i + b_i) - d(y_i)| \\ &\leq \sum_{i=1}^{k} d(a_i) \cdot d(b_i). \end{aligned}$$

Thus, the proof is complete.

The following consequence is a well-known result.

Corollary

Let $a, b, a_1, b_1, \ldots, a_k, b_k$ be positive numbers such that the rectangle of sides a, b can be decomposed as the (almost disjoint) union of rectangles of sides a_i, b_i ($i = 1, \ldots, k$). Assume that, for all i, either a_i or b_i is an integer number. Then a or b is also an integer.

Proof. If a_i or b_i is an integer then $d(a_i) \cdot d(b_i) = 0$ for all i. Thus, by (2), we get $d(a) \cdot d(b) = 0$. Therefore, $d(a) = 0$ or $d(b) = 0$ holds. This shows that a or b is an integer.

Zsolt Páles

3. Problem.

Let $g\colon \mathbb{C} \to \mathbb{C}$ be an entire function, and assume that

$$g(z) = \frac{1}{2} g\Big(\frac{z}{2}\Big) + \frac{1}{2} g\Big(\frac{z+1}{2}\Big), \qquad z \in \mathbb{C}$$

holds. Then F. Schottky and G. Herglotz showed that g is constant.

What is known about entire solutions g of functional equations of the form

$$p_0(z) g(z) = \sum_{j=1}^{N} p_j(z) g(\alpha_j z + \beta_j) + R(z), \qquad z \in \mathbb{C},$$

where $p_0, \ldots, p_N$ are slowly growing entire functions (e.g. polynomials), R is a given entire function and α_j, β_j satisfy appopriate conditions?

Ludwig Reich

4. Problems.

1. Let V denote a translation invariant linear subspace in the space of complex polynomials in k variables. Suppose that $(p_n)_{n \in \mathbb{N}}$ is a sequence in V which converges pointwise to the polynomial p. Does it follow that p is in V? In the case $k = 1$ the answer is "yes". (This problem has been presented at the 49th ISFE, Graz-Mariatrost, 2011.)

2. Does there exist a strictly descending infinite chain of translation invariant linear spaces of complex polynomials in k variables? In the case $k = 1$ the answer is "no".

László Székelyhidi

5. Remark.

The classical Hermite-Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \overset{(1)}{\leqslant} \frac{1}{b-a} \int_a^b f(x)\, dx \overset{(2)}{\leqslant} \frac{f(a)+f(b)}{2}$$

holds for all convex functions $f\colon [a, b] \to \mathbb{R}$. It is well-known that inequality (1) gives the better estimate of the integral mean value than inequality (2). After

Sz. Wąsowicz's talk M. Goldberg asked the speaker the question what about the multivariate case. Below we give the negative answer.

If $S \subset \mathbb{R}^n$ is a simplex with vertices $p_0, p_1, \dots, p_n$, then the following Hermite-Hadamard type inequality holds:

$$f\Big(\frac{1}{n+1}\sum_{i=0}^{n} p_i\Big) \leqslant \frac{1}{\mathrm{vol(S)}}\int_S f(\mathbf{x})\,d\mathbf{x} \leqslant \frac{1}{n+1}\sum_{i=0}^{n} f(p_i),$$

whenever $f\colon S \to \mathbb{R}$ is a convex function (cf. [1, 2]).

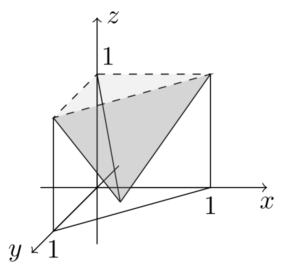
Now let $S = \mathrm{conv}\big\{(0,0),(0,1),(1,0)\big\}$ be the unit simplex in $\mathbb{R}^2$. Then the above inequalities have the form

$$f\Big(\frac{1}{3},\frac{1}{3}\Big) \overset{(3)}{\leqslant} 2\iint_S f(x,y)\,dx\,dy \overset{(4)}{\leqslant} \frac{f(0,0)+f(0,1)+f(1,0)}{3}.$$

For the convex function $f(x,y) = x^2$ we obtain

$$\frac{1}{9} \leqslant \frac{1}{6} \leqslant \frac{1}{3}$$

which means that inequality (3) estimates the integral mean value better than (4). Take now another convex function, whose graph is the surface of a pyramid shown in the picture below.



Then it is easy to observe that for this function inequalities (3) and (4) have the form

$$0 \leqslant 2\Big(\frac{1}{2} - \frac{1}{6}\Big) = \frac{2}{3} \leqslant 1$$

and in this case (4) estimates the integral mean value better than (3).

[1] M. Bessenyei, *The Hermite-Hadamard inequality on simplices*, Amer. Math. Monthly **115** (2008), 339–345.

[2] Sz. Wąsowicz, *Hermite-Hadamard-type inequalities in the approximate integration*, Math. Inequal. Appl. **11** (2008), 693–700.

Szymon Wąsowicz and Alfred Witkowski

6. Problem.

Let us say that a function $f\colon X \to Y$ between normed spaces has the property $\mathcal{P}_n$ (for a fixed $n \in \mathbb{N}$) if it is homogenous and satisfies the functional inequality

$$\Big\| f\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} f(x_i) \Big\| \le K \sum_{i=1}^{n} \|x_i\|, \qquad x_1, \dots, x_n \in X.$$

The classic property quasilinearity is simply $\mathcal{P}_2$. It is clear that $\mathcal{P}_{n+1} \implies \mathcal{P}_n$ for all n. What about the converse? If we denote by K_n the best constant for which f has $\mathcal{P}_n$, a short calculation shows that $K_{n+1} \le K_n + K_2$. Can this estimate be improved? For the "worst" example (which has $X = \ell_1$ and $Y = \mathbb{R}$) it is only known that $K_n \ge \log n$. (When the domain X is a so-called $\mathcal{K}$-space, $\mathcal{P}_2$ already implies $\mathcal{P}_n$ for all n with a common value for K. This class includes all super-reflexive spaces, all quotients of $\mathcal{L}_\infty$ spaces, in particular all classical spaces except ℓ_1.) Dropping the homogeneity requirement leads to a very different problem, which may also be interesting.

David Yost

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