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The research on the strong Markov property

Abstract. Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$. In this paper, we give the definitions of σ-algebras prior to α and post-α and discuss their properties. At the same time, we prove that the strong Markov property holds for an arbitrary Markov process, that is, we prove that the Markov property is equivalent to the strong Markov property.

1. Introduction

Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$. So for every $A \in \mathcal{E}$, $\{\omega :\ x_t(\omega) \in A\} \in \mathcal{F}$, where $(E, \mathcal{E})$ is an abstract space and t is the time parameter. The points of E are denoted as $x, y, \dots$. The sets of $\mathcal{E}$ are denoted as $A, B, \dots$. For convenience, suppose that $\mathcal{E}$ contains all sets of simple points of E, that is, $\{x\} \in \mathcal{E}$ for every $x \in E$.

Throughout this paper, suppose that $X(t,\omega)$ is non-interruptive Markov process unless mentioned. Otherwise, we may enlarge the state space E to $\tilde{E} = E \cup \{d\}$ by joining a single point d with $d \notin E$ into E, and change $X(t,\omega)$ into non-interruptive process $\tilde{X}(t,\omega)$ on $\tilde{E}$. It does not affect all conclusions in this paper.

Let $\alpha(\omega)$ be a random variable which might be ∞. In order to show that x_α is well defined when $\alpha = \infty$, choose a random variable $\beta(\omega)$ valued in $(E, \mathcal{E})$, and define $x_\infty(\omega) \triangleq \beta(\omega)$. Then $x_{\alpha(\omega)}(\omega)$ is well defined for all $\omega \in \Omega$. Now, define $\mathcal{F}(x_\alpha)$ as

$$\mathcal{F}(x_\alpha) \triangleq \{\{\omega :\ (x_{\alpha(\omega)}(\omega), \alpha(\omega)) \in A\} :\ A \in \mathcal{E} \times \mathcal{B}([0,\infty])\}, \tag{1.1}$$

where $\mathcal{B}([0,\infty])$ is a *Borel* σ-algebra generated by $[0,\infty]$. $\{(x,s)\}$ is an atom of $\mathcal{E} \times \mathcal{B}([0,\infty])$ for every $x \in E$, $s \in [0,\infty]$, namely, $\{(x,s)\} \in \mathcal{E} \times \mathcal{B}([0,\infty])$, and does not contain any proper subsets of $\mathcal{E} \times \mathcal{B}([0,\infty])$. It follows that $\{x_{\alpha(\omega)}(\omega) = x\} \cap \{\alpha(\omega) = s\}$ is an atom of $\mathcal{F}(x_\alpha)$. Since $\alpha(\omega)\colon \Omega \to \bar{\mathbb{R}}^+ \triangleq [0,\infty]$ is a mapping from Ω to $\bar{\mathbb{R}}^+$, and $x_t(\omega)\colon \Omega \to E$ is also a mapping from Ω to E for every fixed

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$t \geq 0$, it follows that $x_{\alpha(\omega)}(\omega)$ is a mapping from Ω to $E \times \bar{\mathbb{R}}^+$. Note that $\mathcal{E} \times \mathcal{B}([0,\infty])$ is a σ-algebra, therefore $\mathcal{F}(x_\alpha)$ is a σ-algebra by [2, Property 2.2.2].

Definition 1.1
$\mathcal{F}(x_\alpha)$ is called the σ-algebra generated by $x_{\alpha(\omega)}(\omega)$.

The core of the Markov process is the Markov property which is the base of theoretic and applied research on Markov process. But we often need a stronger property: "the strong Markov property". We know that "present" in the explanation of the Markov property is a fixed time t which has nothing to do with ω. But in many problems, "present" is required to be a random time $\alpha(\omega)$ which may take different values according to different ω, such as hitting time. Let $\eta_A(\omega)$ be the hitting time of $A \in \mathcal{E}$. Whether $X(t,\omega)$ satisfies Markov property at time $\eta_A(\omega)$. Note that $\eta_A(\omega)$ depends on ω. So the strong Markov property is distinct from the Markov property.

More precisely, this problem is explained as follows: Let $X(t,\omega) \triangleq \{x_t(\omega);\ t \geq 0\}$ be a Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on $(E, \mathcal{E})$, that is, for any *Borel* subset B of $(-\infty, \infty)$, we have

$$\{x:\ f(x) \in B\} \in \mathcal{E}. \tag{1.2}$$

Let $\alpha(\omega)$ be a random variable. Does the following equality

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|\mathcal{F}(x_\alpha)], \quad P_{\Omega_\alpha}\text{–a.e.}$$

hold? Here $\mathcal{N}_\alpha^+$ is a σ-algebra prior to α generated by $X(t,\omega)$, which is defined in Section 2.1; $\Omega_\alpha = \{\omega:\ \alpha(\omega) < \infty\}$; $\mathcal{F}(\cdot)$ denotes the smallest σ-algebra on Ω generated by all sets of bracket.

In order to prove (1.2), many scholars made great efforts, and obtained many fine results. The first one who thought (1.2) should be seriously proven is Doob (1945). To make (1.2) hold, what should we do?

(1) What restricted conditions should $\alpha(\omega)$ have?

(2) How to define the σ-algebra prior to $\alpha(\omega)$ so that it includes the special case $\alpha(\omega) \equiv$ constant?

(3) How to define the function $f(x)$ so that $f(x_t)$ is a random variable and $E[f(x_t)|\mathcal{F}(x_\alpha)]$ is $\mathcal{N}_\alpha^+$-measurable?

The questions above were mentioned in [1, P106].

2. σ-algebra prior to $\alpha(\omega)$ and its properties

2.1. The definition of the σ-algebra prior to $\alpha(\omega)$

Recall the σ-algebras $\mathcal{N}_T \triangleq \mathcal{F}(x_s(\omega);\ s < T)$ and $\mathcal{N}_T^+ \triangleq \mathcal{F}(x_s(\omega);\ s \le T)$, generated by the trajectory of $X(t,\omega)$ prior to T, are defined by

$$\mathcal{N}_T \triangleq \mathcal{F}(x_s(\omega);\ s < T) \triangleq \mathcal{F}\Big(\bigcup_{s<T} x_s^{-1}(\mathcal{E})\Big) \tag{2.1}$$

and

$$\mathcal{N}_T^+ \triangleq \mathcal{F}(x_s(\omega);\ s \le T) \triangleq \mathcal{F}\Big(\bigcup_{s\le T} x_s^{-1}(\mathcal{E})\Big), \tag{2.2}$$

respectively. In particular, taking $T = \infty$, we have

$$\mathcal{N}_\infty = \mathcal{F}\Big(\bigcup_{s<\infty} x_s^{-1}(\mathcal{E})\Big) \qquad \text{and} \qquad \mathcal{N}_\infty^+ = \mathcal{F}\Big(\bigcup_{s\le\infty} x_s^{-1}(\mathcal{E})\Big).$$

Here $x_s^{-1}(\mathcal{E}) \triangleq \{\{x_s(\omega) \in B\} :\ B \in \mathcal{E}\}$. Intuitively, $\mathcal{F}(x_s(\omega);\ s < T)$ or $\mathcal{F}(x_s(\omega);\ s \le T)$ is the σ-algebra generated by the stochastic process prior to T of $X(t,\omega)$, that is, generated by the two stochastic precesses $(x_s(\omega);\ s < T)$ and $(x_s(\omega);\ s \le T)$, respectively. Of course, here T is a constant that has nothing to do with ω. How to define the $\mathcal{N}_{\alpha(\omega)}$ and $\mathcal{N}_{\alpha(\omega)}^+$ if α is a random variable? Similarly to the way of defining $\mathcal{N}_T$ and $\mathcal{N}_T^+$, they are defined as follows: Let $y_t(\omega) = x_t(\omega)$ if $t < \alpha(\omega)$; $\bar{y}_t(\omega) = x_t(\omega)$ if $t \le \alpha(\omega)$. Put $Y(t,\omega) = \{y_t(\omega);\ t \ge 0\}$ and $\bar{Y}(t,\omega) \triangleq \{\bar{y}_t(\omega);\ t \ge 0\}$. Then they satisfy:

$$Y(t,\omega) = (X(t,\omega);\ t < \alpha(\omega))$$

and

$$\bar{Y}(t,\omega) = (X(t,\omega);\ t \le \alpha(\omega)).$$

That is, $\{y_t(\omega) \in B\} = \{x_t(\omega) \in B,\ t < \alpha(\omega)\}$ and $\{\bar{y}_t(\omega) \in B\} = \{x_t(\omega) \in B,\ t \le \alpha(\omega)\}$ for any $t \ge 0$ and $B \in \mathcal{E}$, where when $t = \infty$, $\{x_t(\omega) \in B,\ t < \alpha(\omega)\} = \emptyset$ and $\{x_t(\omega) \in B,\ t \le \alpha(\omega)\} = \{\beta(\omega) \in B,\ \alpha(\omega) = \infty\}$, respectively. By the definition of a stochastic process, $Y(t,\omega)$ and $\bar{Y}(t,\omega)$ are two stochastic processes prior to $\alpha(\omega)$ of $x(t,\omega)$, that is, the two processes end at time $t < \alpha(\omega)$ and $t \le \alpha(\omega)$, respectively.

From (2.1), (2.2) it follows that the σ-algebras prior to α of $X(t,\omega)$ are defined by

$$\mathcal{N}_\alpha \triangleq \mathcal{F}(y_t(\omega);\ t < \infty) \triangleq \mathcal{F}\Big(\bigcup_{t<\infty} y_t^{-1}(\mathcal{E})\Big) \tag{2.3}$$

and

$$\mathcal{N}_\alpha^+ \triangleq \mathcal{F}(\bar{y}_t(\omega);\ t \le \infty) \triangleq \mathcal{F}\Big(\bigcup_{t\le\infty} \bar{y}_t^{-1}(\mathcal{E})\Big), \tag{2.4}$$

respectively. When $t = \infty$, $\bar{y}_t^{-1}(\mathcal{E})$ is defined by $\bar{y}_t^{-1}(\mathcal{E}) \triangleq \{\{\omega :\ \beta(\omega) \in B,\ \alpha(\omega) = \infty\} :\ B \in \mathcal{E}\}$. Obviously, if $\alpha(\omega) \equiv T$ (constant), $\mathcal{N}_\alpha = \mathcal{N}_T$ and $\mathcal{N}_\alpha^+ = \mathcal{N}_T^+$.

Definition 2.1
$\mathcal{N}_\alpha$ and $\mathcal{N}_\alpha^+$ defined by (2.3) and (2.4), respectively, are called σ-algebras prior to α of $X(t,\omega)$.

2.2. The properties of σ-algebra prior to $\alpha(\omega)$

We now discuss the properties of $\mathcal{N}_\alpha$ and $\mathcal{N}_\alpha^+$, which are the foundations of studying the strong Markov property.

Theorem 2.2

$$\mathcal{F}(\alpha) \subseteq \mathcal{N}_\alpha; \qquad \mathcal{F}(\alpha) \subseteq \mathcal{N}_\alpha^+.$$

Proof. The proofs of both statements are similar, we only prove the first relation. Since $\{x_s(\omega) \in E\} = \Omega$, we have

$$\{\alpha(\omega) > s\} = \{x_s(\omega) \in E,\ \alpha(\omega) > s\} = \{y_s(\omega) \in E\} \in \mathcal{N}_\alpha.$$

It is well known that $\mathcal{F}(\alpha) = \mathcal{F}(\alpha(\omega) > s;\ s \geq 0)$. Hence, the theorem is valid.

Theorem 2.3
Let

$$\begin{aligned} \Pi &= \{\{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha > s\} : \\ &\qquad n \geq 1;\ t_1 \leq \ldots \leq t_n \leq s;\ A_1, \dots, A_n \in \mathcal{E}\}; \\ \Pi^+ &= \{\{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha \geq s\} : \\ &\qquad n \geq 1;\ t_1 \leq \ldots \leq t_n \leq s \leq \infty;\ A_1, \dots, A_n \in \mathcal{E}\}, \end{aligned}$$

where for $s = \infty$, $\{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha > s\} = \emptyset$ *and* $\{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha \geq s\} = \{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha = \infty\}$. *Then*

$$\mathcal{F}(\Pi) = \mathcal{N}_\alpha \qquad \text{and} \qquad \mathcal{F}(\Pi^+) = \mathcal{N}_\alpha^+.$$

Proof. $\{x_{t_1} \in A_1, \dots, x_{t_n} \in A_n,\ \alpha > s\} = \{y_{t_1} \in A_1, \dots, y_{t_n} \in A_n,\ y_s \in E\} \in \mathcal{N}_\alpha$, hence, $\mathcal{F}(\Pi) \subseteq \mathcal{N}_\alpha$. Again, for every $t \geq 0$ and $A \in \mathcal{E}$, obviously, $\{y_t \in A\} = \{x_t \in A,\ \alpha > t\} \in \Pi$. Therefore, $\mathcal{N}_\alpha = \mathcal{F}(\bigcup_{t<\infty} y_t^{-1}(\mathcal{E})) \subseteq \mathcal{F}(\Pi)$, from which and above it follows that $\mathcal{F}(\Pi) = \mathcal{N}_\alpha$. Similarly as above we obtain $\mathcal{F}(\Pi^+) = \mathcal{N}_\alpha^+$.

Theorem 2.4
Let $\alpha(\omega)$ *be a nonnegative random variable. Then*

$$\mathcal{N}_\alpha \subseteq \mathcal{N}_\alpha^+.$$

Proof. For any $t_1 \leq t_2 \leq \ldots \leq t_m \leq s < t$, from $\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m,\ \alpha \geq t\} \in \mathcal{N}_\alpha^+$ we get

$$\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m,\ \alpha > s\} = \lim_{t \downarrow s} \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\} \in \mathcal{N}_\alpha^+.$$

Here $\lim_{t\downarrow s}\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\}$ is defined by $\bigcup_{n=1}^\infty \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha \geq a_n\}$ for an arbitrary sequence of number $\{a_n\}_{n\geq 1}$,

$a_n \downarrow s$ as $n \uparrow \infty$. We easily verify that $\lim_{t\downarrow s}\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \geq t\}$ has nothing to do with the chosen $\{a_n\}_{n\geq 1}$. Hence, by Theorem 2.3, the theorem is proven.

THEOREM 2.5
Let $\alpha(\omega)$ be a stopping time with respect to $\mathcal{N}_t^+$, that is, $\{\alpha \leq t\} \in \mathcal{N}_t^+$ for every $t \geq 0$. Then $A \cap \{\alpha \leq t\} \in \mathcal{N}_t^+$ and $A \cap \{\alpha < t\} \in \mathcal{N}_t^+$ for every $A \in \mathcal{N}_\alpha^+$.

Proof. Suppose that A has the following shape

$$A = \{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n,\, \alpha \geq s\}$$

for any $n \geq 1$ and $t_1 \leq \ldots \leq t_n \leq s$ and $A_1, \ldots, A_n \in \mathcal{E}$. Obviously, $A \cap \{\alpha \leq t\} = \{x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \cap \{s \leq \alpha \leq t\} \in \mathcal{N}_t^+$. So, by λ-π-system method and Theorem 2.3, the first assertion is obtained. Again, $A \cap \{\alpha < t\} = \lim_{u\uparrow t} A \cap \{\alpha \leq u\} \in \mathcal{N}_t^+$, which is the other assertion.

3. σ-algebra post-$\alpha(\omega)$ and its properties

3.1. The definition of the σ-algebra post-$\alpha(\omega)$

Let $w_t(\omega) = x_t(\omega)$ if $\alpha(\omega) < t$ and $\bar{w}_t(\omega) = x_t(\omega)$ if $\alpha(\omega) \leq t$. Set $W(t,\omega) \triangleq \{w_t(\omega);\ t \geq 0\} = (X(t,\omega); \alpha(\omega) < t)$. $\bar{W}(t,\omega) \triangleq \{\bar{w}_t(\omega);\ t \geq 0\} = (X(t,\omega); \alpha(\omega) \leq t)$. That is, $\{w_t(\omega) \in B\} = \{x_t(\omega) \in B,\ \alpha(\omega) < t\}$ and $\{\bar{w}_t(\omega) \in B\} = \{x_t(\omega) \in B,\ \alpha(\omega) \leq t\}$ for any $t \geq 0$ and $B \in \mathcal{E}$. Here for $t = \infty$, $\{x_t(\omega) \in B,\ \alpha(\omega) < t\} = \{\beta(\omega) \in B,\ \alpha(\omega) < \infty\}$ and $\{x_t(\omega) \in B,\ \alpha(\omega) \leq t\} = \{\beta(\omega) \in B\}$, respectively. We adjoin a point Δ with $\Delta \notin E$ to E to expand E into $\hat{E} = E \cup \{\Delta\}$, and set $\hat{\mathcal{E}} \triangleq \mathcal{F}(\mathcal{E},\, \{\Delta\})$. Let

$$\tilde{w}_t(\omega) \triangleq \begin{cases} w_t(\omega), & t > \alpha(\omega), \\ \Delta, & t \leq \alpha(\omega) \end{cases} = \begin{cases} x_t(\omega), & t > \alpha(\omega), \\ \Delta, & t \leq \alpha(\omega); \end{cases}$$

$$\tilde{\bar{w}}_t(\omega) \triangleq \begin{cases} \bar{w}_t(\omega), & t \geq \alpha(\omega), \\ \Delta, & t < \alpha(\omega) \end{cases} = \begin{cases} x_t(\omega), & t \geq \alpha(\omega), \\ \Delta, & t < \alpha(\omega). \end{cases}$$

Then $\tilde{W}(t,\omega) \triangleq \{\tilde{w}_t(\omega);\ t \geq 0\}$ and $\tilde{\bar{W}}(t,\omega) \triangleq \{\tilde{\bar{w}}_t(\omega);\ t \geq 0\}$ are changed into non-interruptive processes on $(\hat{E}, \hat{\mathcal{E}})$, respectively. The state Δ is the starting point of $\tilde{W}(t,\omega)$ and $\tilde{\bar{W}}(t,\omega)$, that is, for all $\omega \in \Omega$, $\tilde{W}(t,\omega)$ and $\tilde{\bar{W}}(t,\omega)$ start from state Δ, and stay time at Δ is $\alpha(\omega)$, then the ω enter into E to move according to the primary trajectory. The σ-algebras post-α ${}_\alpha\mathcal{N}$ and ${}_\alpha\mathcal{N}^+$ are defined by

$$ {}_\alpha\mathcal{N} \triangleq \mathcal{F}(\tilde{w}_t(\omega);\, t \leq \infty) \triangleq \mathcal{F}\left(\bigcup_{t\leq\infty} \tilde{w}_t^{-1}(\mathcal{E})\right) \tag{3.1}$$

and

$$ {}_\alpha\mathcal{N}^+ \triangleq \mathcal{F}(\tilde{\bar{w}}_t(\omega);\, t \leq \infty) \triangleq \mathcal{F}\left(\bigcup_{t\leq\infty} \tilde{\bar{w}}_t^{-1}(\mathcal{E})\right), \tag{3.2}$$

respectively. Here when $t = \infty$, $\tilde{w}_t^{-1}(\mathcal{E})$ and $\tilde{\bar{w}}_t^{-1}(\mathcal{E})$ are defined by $\tilde{w}_t^{-1}(\mathcal{E}) \triangleq$

$\{\{\beta \in B,\, \alpha < \infty\} :\ B \in \mathcal{E}\}$ and $\tilde{\bar{w}}_t^{-1}(\mathcal{E}) \triangleq \{\{\beta \in B\} :\ B \in \mathcal{E}\}$, respectively. By the definition of $\mathcal{F}(\cdot)$ on Ω, obviously,

$$_\alpha\mathcal{N} = \mathcal{F}\Big(\bigcup_{t\le\infty} w_t^{-1}(\hat{\mathcal{E}})\Big); \qquad _\alpha\mathcal{N}^+ = \mathcal{F}\Big(\bigcup_{t\le\infty} \bar{w}_t^{-1}(\hat{\mathcal{E}})\Big).$$

Definition 3.1
$_\alpha\mathcal{N}$ and $_\alpha\mathcal{N}^+$ defined by (3.1) and (3.2) are called σ-algebras post-α of $X(t,\omega)$, respectively.

Intuitively, $_\alpha\mathcal{N}$ or $_\alpha\mathcal{N}^+$ is the σ-algebra generated by the stochastic process post-α of $X(t,\omega)$.

3.2. The properties of the σ-algebra post-$\alpha(\omega)$

Similarly to the proof of Theorem 2.2 we obtain the following theorem.

Theorem 3.2

$$\mathcal{F}(\alpha) \subseteq {}_\alpha\mathcal{N}; \qquad \mathcal{F}(\alpha) \subseteq {}_\alpha\mathcal{N}^+.$$

Theorem 3.3

$$\mathcal{F}(\alpha) \subseteq \mathcal{F}(x_\alpha).$$

Proof. Since $\{x_{\alpha(\omega)}(\omega) \in E\} = \Omega$, from (1.1), it follows that $\{\alpha \in B\} \in \mathcal{F}(x_\alpha)$.

Theorem 3.4
Let

$$\Gamma = \{\{\alpha < s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} :\ n \ge 1,\, s \le t_1 \le \dots \le t_n,\, A_1, \dots, A_n \in \mathcal{E}\}$$

and

$$\Gamma^+ = \{\{\alpha \le s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} :\ n \ge 1,\, s \le t_1 \le \dots \le t_n,\, A_1, \dots, A_n \in \mathcal{E}\},$$

where when $s = \infty$, $\{\alpha < s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} = \{\alpha < \infty,\, \beta \in A_1, \dots, \beta \in A_n\}$ *and* $\{\alpha \le s,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} = \{\beta \in A_1, \dots, \beta \in A_n\}$. *Then*

$$\mathcal{F}(\Gamma) = {}_\alpha\mathcal{N} \qquad \textit{and} \qquad \mathcal{F}(\Gamma^+) = {}_\alpha\mathcal{N}^+.$$

Proof. The proof is analogous to the proof of Theorem 2.3.

Theorem 3.5

$$_\alpha\mathcal{N} \subseteq {}_\alpha\mathcal{N}^+.$$

Proof. By Theorem 3.4, $\{\alpha \le u,\, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n\} \in {}_\alpha\mathcal{N}^+$. So

$$\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} = \lim_{u\uparrow s}\{\alpha \le u,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \in {}_\alpha\mathcal{N}^+,$$

where $\lim_{u\uparrow s}\{\alpha \le u,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\}$ is defined by $\bigcup_{i=1}^{\infty}\{\alpha \le a_i,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\}$ for any sequence of number $\{a_n\}_{n\ge 1}$, $a_n \uparrow s$ as $n \uparrow \infty$. When $s = \infty$, $\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} = \{\alpha < \infty,\, \beta \in A_1, \ldots, \beta \in A_n\} \in {}_\alpha\mathcal{N}^+$ from the definition of ${}_\alpha\mathcal{N}^+$ and Theorem 3.2. So $\{\alpha < s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} \in {}_\alpha\mathcal{N}^+$ for every $s \le \infty$. By Theorem 3.4 the proof is accomplished.

Theorem 3.6
Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in measurable space $(E, \mathcal{E})$. Then

$$\mathcal{F}(x_\alpha) \subseteq \mathcal{N}_\alpha^+; \qquad \mathcal{F}(x_\alpha) \subseteq {}_\alpha\mathcal{N}^+.$$

Proof. For any $A \in \mathcal{E}$, obviously,

$$\begin{aligned}\{\omega : x_{\alpha(\omega)}(\omega) \in A\} &= \{\omega : x_{\alpha(\omega)}(\omega) \in A\} \cap \{\omega : \alpha(\omega) \le \infty\} \\ &= \bigcup_{s<\infty}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \\ &\quad + \{\omega : \beta(\omega) \in A\} \cap \{\omega : \alpha(\omega) = \infty\}.\end{aligned} \tag{3.3}$$

By Theorem 2.3 and Theorem 2.4, for every $s \ge 0$,

$$\begin{aligned}&\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\} \\ &\quad = \{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) \ge s\} \\ &\qquad - \{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) > s\} \in \mathcal{N}_\alpha^+.\end{aligned} \tag{3.4}$$

Now we prove

$$\bigcup_{s<\infty}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \in \mathcal{N}_\alpha^+ \tag{3.5}$$

by virtue of transfinite induction. Suppose that $\preceq$ is well ordering on $[0,\infty)$ with the first element a_0. By (3.4),

$$\begin{aligned}&\bigcup_{s\preceq a_0}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \\ &\quad = \{\omega : x_{a_0}(\omega) \in A\} \cap \{\omega : \alpha(\omega) = a_0\} \in \mathcal{N}_\alpha^+.\end{aligned}$$

Suppose that

$$\bigcup_{s\preceq a}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \in \mathcal{N}_\alpha^+$$

for any a with $a \prec T$. Chosen an increasing sequence $\{a_i :\ i \ge 1,\, a_i \prec T\}$ satisfying that for any given number $t \prec T$, there exists a_i such that $t \preceq a_i$. So

$$\begin{aligned}&\bigcup_{s\prec T}(\{\omega : x_s(\omega) \in A\} \cap \{\omega : \alpha(\omega) = s\}) \\ &\quad = \bigcup_{i=1}^{\infty}\left[\bigcup_{s\preceq a_i}(\{\omega : x_s \in A\} \cap \{\omega : \alpha = s\})\right] \in \mathcal{N}_\alpha^+.\end{aligned}$$

Hence,

$$\begin{aligned}&\bigcup_{s\preceq T}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\&\qquad=\bigcup_{s\prec T}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\&\qquad\quad+\{\omega:\ x_T(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=T\}\in\mathcal{N}_\alpha^+.\end{aligned}\tag{3.6}$$

Transfinite induction implies that (3.6) holds for any $T\in[0,\infty)$. Again, take an increasing sequence $\{T_i:\ i\geq 1\}$ satisfying that for any given number $s\in[0,\infty)$, there exists a T_i such that $s\preceq T_i$. So

$$\begin{aligned}&\bigcup_{s<\infty}(\{\omega:\ x_s(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=s\})\\&\qquad=\bigcup_{i=1}^{\infty}\Big[\bigcup_{s\preceq T_i}(\{\omega:\ x_s\in A\}\cap\{\omega:\ \alpha=s\})\Big]\in\mathcal{N}_\alpha^+.\end{aligned}$$

Again, $\{\omega:\ \beta(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=\infty\}\in\mathcal{N}_\alpha^+$ from the definition of $\mathcal{N}_\alpha^+$. Hence, by (3.3), $\{\omega:\ x_{\alpha(\omega)}(\omega)\in A\}\in\mathcal{N}_\alpha^+$. By Theorem 2.2,

$$\{\omega:\ (x_\alpha(\omega),\alpha(\omega))\in A\times B\}=\{x_{\alpha(\omega)}(\omega)\in A\}\cap\{\alpha(\omega)\in B\}\in\mathcal{N}_\alpha^+$$

for every $B\in\mathcal{B}([0,\infty])$. Note that $\mathcal{E}\times\mathcal{B}([0,\infty])=\mathcal{F}(A\times B;\ A\in\mathcal{E},\ B\in\mathcal{B}([0,\infty])$, and $\{A\times B;\ A\in\mathcal{E},\ B\in\mathcal{B}([0,\infty])\}$ is a π-system. So, by λ-π-system method, it follows that $\mathcal{F}(x_\alpha)\subseteq\mathcal{N}_\alpha^+$. Again, $\{\omega:\ \beta(\omega)\in A\}\cap\{\omega:\ \alpha(\omega)=\infty\}\in{}_\alpha\mathcal{N}^+$ from the definition of ${}_\alpha\mathcal{N}^+$ and Theorem 3.2. Similarly to the proof of the fact $\mathcal{F}(x_\alpha)\subseteq\mathcal{N}_\alpha^+$, we get $\mathcal{F}(x_\alpha)\subseteq{}_\alpha\mathcal{N}^+$.

4. The strong Markov property

Suppose that Θ_s denotes the shift operator, that is

$$\Theta_s(f(t_1+s,\ldots,t_n+s))=f(t_1,\ldots,t_n)$$

for any natural number n and function $f(t_1,\ldots,t_n)$ of n-variables defined on n-dimensional real number space $\mathbb{R}^n$. Generally, if $s=s(\omega)$ is a function of ω, then, for $|s(\omega)|<\infty$, $\Theta_{s(\omega)}$ denotes the shift as follows:

$$\begin{aligned}\Theta_{s(\omega)}(f(t_1,\ldots,t_n))&=f(t_1-s(\omega),\ldots,t_n-s(\omega))\\&=\sum_{-\infty<u<\infty}f(t_1-u,\ldots,t_n-u)\mathcal{X}_{\{s(\omega)=u\}}.\end{aligned}$$

More generally, if $f=f(t_1,\ldots,t_n,x_{s(\omega)}(\omega))$ is also a function of $x_{s(\omega)}(\omega)$, then, for $|s(\omega)|<\infty$, $\Theta_{s(\omega)}$ denotes the shift as follows:

$$\begin{aligned}&\Theta_{s(\omega)}(f(t_1,\ldots,t_n,x_{s(\omega)}(\omega)))\\&\quad=f(t_1-s(\omega),\ldots,t_n-s(\omega),x_{s(\omega)}(\omega))\\&\quad=\sum_{-\infty<u<\infty}\sum_{x\in E}f(t_1-u,\ldots,t_n-u,x)\mathcal{X}_{\{s(\omega)=u,x_{s(\omega)}(\omega)=x\}}(\omega),\end{aligned}$$

where $\mathcal{X}_{\{s(\omega)=u,x_{s(\omega)}(\omega)=x\}}(\omega)$ is an indicator relative to $\{s(\omega) = u,\, x_{s(\omega)}(\omega) = x\}$, that is, $X_{\{s(\omega)=u,\, x_{s(\omega)}(\omega)=x\}}(\omega) = 1$ if $\omega \in \{s(\omega) = u,\, x_{s(\omega)}(\omega) = x\}$ and $X_{\{s(\omega)=u,\, x_{s(\omega)}(\omega)=x\}}(\omega) = 0$ otherwise.

Definition 4.1
$X(t, \omega)$ is called a homogeneous Markov process if

$$p(s, t + s; x, A) = p(0, t; x, A),$$

where $p(s, t; x, A)$ is the transition probability function of $X(t, \omega)$.

Let $\omega_0 \in \{x_s = x\}$. For an arbitrary $\mathcal{E}$-measurable bounded real-valued function $f(x)$, by [2, Theorem 5.2.5], $E[f(x_{t+s})|x_s](\omega_0)(\overset{\triangle}{=} E[f(x_{t+s})|\mathcal{F}(x_s)](\omega_0))$ may be denoted by $K(s, t + s; x, f(x_{t+s}))$. So $E[f(x_{t+s})|x_s](\omega)$ may be denoted as $\sum_{x\in E} K(s, t + s; x, f(x_{t+s}))\mathcal{X}_{\{x_s=x\}}(\omega) = K(s, t + s; x_s(\omega), f(x_{t+s}))$.

Definition 4.1'
$X(t, \omega)$ is called a homogeneous Markov process if for an arbitrary $\mathcal{E}$-measurable bounded real-valued function $f(x)$, such that

$$E[f(x_{t+s})|x_s] = \Theta_s E[f(x_{t+s})|x_s] = K(0, t; x_s(\omega), f(x_t)) \overset{\triangle}{=} E_{x_s}[f(x_t)]. \quad (4.1)$$

Let $f = \mathcal{X}_A(x), A \in \mathcal{F}$. By Markov property, (4.1) holds if and only if

$$\begin{aligned} E[\mathcal{X}_A(x_{t+s})|x_s](\omega) = p(s, t + s; x_s(\omega), A) &= \sum_{x\in E} p(s, t + s; x, A)\mathcal{X}_{\{x_s=x\}}(\omega) \\ &= \sum_{x\in E} p(0, t; x, A)\mathcal{X}_{\{x_s=x\}}(\omega) \\ &= p(0, t; x_s(\omega), A), \quad P_{\mathcal{F}(x_s)}\text{–a.e..} \end{aligned}$$

So, by $\mathcal{L}$-system method (Appendix B, Theorem B.5), it follows that the two definitions are equivalent.

Lemma 4.2
$f(x_t)$ is $x_t^{-1}(\mathcal{E})$-measurable real-valued function if and only if $f(x)$ is a $\mathcal{E}$-measurable real-valued function defined on a measurable space $(E, \mathcal{E})$. So $f(x_t)$ is a random variable if $f(x)$ is a $\mathcal{E}$-measurable real-valued function defined on a measurable space $(E, \mathcal{E})$.

Proof. Let $g(\omega) \overset{\triangle}{=} x_t(\omega)$. Then $g(\omega)$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ for any fixed $t \geq 0$. If we rewrite $f(x_t(\omega))(\omega) \overset{\triangle}{=} f \circ g(\omega)$, from [2, Theorem 2.2.13] $f(x_t(\omega))$ is a $x_t^{-1}(\mathcal{E})$-measurable (so it is also $\mathcal{F}$-measurable) mapping from Ω to $\bar{\mathbb{R}}(\overset{\triangle}{=} \mathbb{R} \cup \{\infty\})$ if and only if there exists a $\mathcal{E}$-measurable real-valued function $f(x)$ such that $f(x_t(\omega)) = f \circ g(\omega)$.

Lemma 4.3
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$. Put $z_s(\omega) \triangleq E[f(x_{t+s})|x_s] = K(s, t+s; x_s(\omega), f(x_{t+s}))$. Set

$$H_s = \{z_s(\omega)\}, \qquad H = \bigcup_{s\leq\infty} H_s.$$

Let $\mathcal{B}(H)$ denote the σ-algebra generated by all Borel subsets in H. Then:

(1) *$Z(s,\omega) \triangleq \{z_s(\omega) : \ s \geq 0\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(H, \mathcal{B}(H))$.*

(2) *$Z(s,\omega)$ is a martingale relative to σ-algebra filtration $\{\mathcal{N}_s^+; \ 0 \leq s \leq t\}$.*

Proof. (1) First, we prove $z_s(\omega)$ is a random variable for any fixed s. $z_s(\omega)\colon \Omega \to \bar{\mathbb{R}} \triangleq \{\infty\} \cup \mathbb{R}$ is a $\mathcal{F}(x_s)$-measurable real-valued function (here assume without loss of generality that the mathematical expectation may only value $+\infty$) by the definition of conditional mathematical expectation, namely, for every *Borel* subset A of $\bar{\mathbb{R}}$,

$$\{\omega : \ z_s(\omega) \in A\} \in x_s^{-1}(\mathcal{E}) \subseteq \mathcal{F}. \tag{4.2}$$

Let $\mathcal{B}(\bar{\mathbb{R}})$ be the *Borel* σ-algebra generated by $\mathbb{R} \cup \{\infty\}$. Then $\mathcal{B}(H) \subseteq \mathcal{B}(\bar{\mathbb{R}})$, from which and (4.2) it follows that

$$\{\omega : \ z_s(\omega) \in A\} \in \mathcal{F} \tag{4.3}$$

for every $A \in \mathcal{B}(H)$. From (4.3) it follows $z_s(\omega)$ is a random variable valued in a measurable space $(H, \mathcal{B}(H))$ for every fixed $s \geq 0$. Therefore, $Z(s,\omega)$ is a stochastic process valued in a measurable space $(H, \mathcal{B}(H))$ from the definition of stochastic process.

(2) Since $Z(s,\omega) = E[f(x_t)|\mathcal{N}_s^+](\omega)$ by Markov property, for any $s \leq u$,

$$E[Z(u,\omega)|\mathcal{N}_s^+] = E\{E[f(x_t)|\mathcal{N}_u^+]|\mathcal{N}_s^+\} = E[f(x_t)|\mathcal{N}_s^+] = Z(s,\omega), \quad P_{\mathcal{N}_s^+}\text{–a.e.},$$

from which it follows (2) is valid.

Note. $z_s(\omega)$ is also regarded as a composite mapping with $x_s(\omega)$ as intermediate variable and ω as independent variable.

Lemma 4.4
Let $\alpha(\omega)$ be an arbitrary nonnegative random variable. Set

$$\alpha^{(n)}(\omega) = \sum_{k=1}^{n2^n} \frac{k}{2^n} \mathcal{X}_{\{\frac{k-1}{2^n} < \alpha \leq \frac{k}{2^n}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha > n\}}(\omega);$$

$$\alpha_-^{(n)}(\omega) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathcal{X}_{\{\frac{k-1}{2^n} < \alpha \leq \frac{k}{2^n}\}}(\omega) + n\mathcal{X}_{\{\alpha > n\}}(\omega).$$

Then

(1) $\alpha^{(n)}(\omega) \downarrow \alpha(\omega)$, $\alpha_-^{(n)}(\omega) \uparrow \alpha(\omega)$ *as* $n \uparrow \infty$,

(2) $\mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha^{(n+1)})$, $\mathcal{F}(\alpha_-^{(n)}) \subseteq \mathcal{F}(\alpha_-^{(n+1)})$ *for every* $n \geq 1$,

(3) $\mathcal{F}(\alpha) = \mathcal{F}(\alpha^{(\infty)}) = \mathcal{F}(\alpha_-^{(\infty)})$.

Here $\mathcal{F}(\alpha^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\alpha^{(n)}))$; $\mathcal{F}(\alpha_-^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\alpha_-^{(n)}))$.

For the convenience of representation, $[0, \frac{1}{2^n}]$ and $\mathcal{X}_{\{\frac{0}{2^n} \leq \alpha \leq \frac{1}{2^n}\}}$ are marked by $(0, \frac{1}{2^n}]$ and $\mathcal{X}_{\{\frac{0}{2^n} < \alpha \leq \frac{1}{2^n}\}}$, respectively throughout this paper.

Proof. (1) By the property of construction of measurable function it follows (1).

(2) Set $A_k^{(n)} = \{\frac{k-1}{2^n} < \alpha \leq \frac{k}{2^n}\}$ for every $k = 1, 2, \ldots, n2^n$, $A_{n2^n+1}^{(n)} = \{\alpha > n\}$. Obviously,

$$\mathcal{F}(\alpha^{(n)}) = \mathcal{F}\big(A_k^{(n)};\ 1 \leq k \leq n2^n + 1\big). \tag{4.4}$$

For $1 \leq k \leq n2^n$,

$$A_k^{(n)} = \Big\{\frac{2(k-1)}{2^{n+1}} < \alpha \leq \frac{2k-1}{2^{n+1}}\Big\} + \Big\{\frac{2k-1}{2^{n+1}} < \alpha \leq \frac{2k}{2^{n+1}}\Big\} \in \mathcal{F}(\alpha^{(n+1)}), \tag{4.5}$$

$$\{\alpha > n\} = \sum_{k=n2^{n+1}+1}^{(n+1)2^{n+1}+1} A_k^{(n+1)} \in \mathcal{F}(\alpha^{(n+1)}). \tag{4.6}$$

The first assertion of (2) follows from (4.4)–(4.6). Similarly we get the second assertion of (2).

(3) Obviously, $\mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha)$ for every $n \geq 1$, hence, $\mathcal{F}(\alpha^{(\infty)}) \subseteq \mathcal{F}(\alpha)$. Next we prove $\mathcal{F}(\alpha) \subseteq \mathcal{F}(\alpha^{(\infty)})$. It is well known that $\mathcal{F}(\alpha) = \mathcal{F}(\{\alpha \geq s\}; s \geq 0)$. Hence, it is sufficient to prove $\{\alpha \geq s\} \in \mathcal{F}(\alpha^{(\infty)})$ for any $s \geq 0$. Set $a_n = \min(s - \frac{k}{2^n};\ s - \frac{k}{2^n} \geq 0,\ 1 \leq k \leq n2^n)$ and $K_n = s - a_n$, Obviously, $(K_n, \infty) \downarrow [s, \infty)$ as $n \uparrow \infty$, where $(K_n, \infty) \downarrow [s, \infty)$ is defined by $\bigcap_{n=1}^{\infty}(K_n, \infty) = [s, \infty)$. Hence $\{\alpha \geq s\} = \bigcap_{n=1}^{\infty}\{\alpha > K_n\} \in \mathcal{F}(\alpha^{(\infty)})$ since $\{\alpha > K_n\} \in \mathcal{F}(\alpha^{(n)}) \subseteq \mathcal{F}(\alpha^{(\infty)})$. In the same manner as above it follows the rest part of (3).

The intuitive idea of Lemma 4.4 is that: the interval $[0, \infty)$ is partitioned into $n2^n + 1$ many pairwise disjoint little intervals $[0, \frac{1}{2^n}], (\frac{1}{2^n}, \frac{2}{2^n}], \ldots, (\frac{n2^n-1}{2^n}, n]$, (n, ∞). We then construct two simple function $\alpha^{(n)}(\omega)$ and $\alpha_-^{(n)}(\omega)$, whose values in every little interval are taken the maximum and the infimum values of $\alpha(\omega)$ in the corresponding little interval. (But $\alpha^{(n)}(\omega)$ take value $n+1$ in little interval (n, ∞)), respectively. We have the same conclusion as Lemma 4.4 if $[0, n] = \bigcup_{k=1}^{n2^n}(a_{k-1}^{(n)}, a_k^{(n)}]$ is an arbitrary partition of $[0, n]$ into a sequence of pairwise disjoint little intervals. This partition method is given a token "$B(2^{(n)})$", called it partition method "$B(2^{(n)})$". Let $d^{(n)} = \max_{1 \leq k \leq n2^n} (a_k^{(n)} - a_{k-1}^{(n)})$. $d^{(n)}$ is called the distance of $B(2^{(n)})$.

LEMMA 4.4'
Let $\alpha(\omega)$ be a nonnegative random variable. For every $n \geq 1$, $[0, n]$ is partitioned into $n2^n$ many pairwise disjoint little intervals $[0, a_1^{(n)}], (a_1^{(n)}, a_2^{(n)}], \ldots, (a_{n2^n-1}^{(n)}, n] \triangleq (a_{n2^n-1}^{(n)}, a_{n2^n}^{(n)}]$, and these partitions satisfy the following conditions:

(a) *For every $n \geq 1$, every such a little interval of partition method "$B(2^{(n)})$" is equal to the sum of such two disjoint little intervals of partition method $B(2^{(n+1)})$.*

(b) $\lim_{n\to\infty} d^{(n)} = 0$.

Let

$$\bar{\alpha}^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_k^{(n)} \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha > n\}}(\omega);$$

$$\bar{\alpha}_-^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_{k-1}^{(n)} \mathcal{X}_{\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}}(\omega) + n\mathcal{X}_{\{\alpha > n\}}(\omega).$$

Then

(1) $\bar{\alpha}^{(n)}(\omega) \downarrow \alpha(\omega)$, $\bar{\alpha}_-^{(n)}(\omega) \uparrow \alpha(\omega)$ *as* $n \uparrow \infty$,

(2) $\mathcal{F}(\bar{\alpha}^{(n)}) \subseteq \mathcal{F}(\bar{\alpha}^{(n+1)})$, $\mathcal{F}(\bar{\alpha}_-^{(n)}) \subseteq \mathcal{F}(\bar{\alpha}_-^{(n+1)})$ *for every* $n \geq 1$,

(3) $\mathcal{F}(\alpha) = \mathcal{F}(\bar{\alpha}^{(\infty)}) = \mathcal{F}(\bar{\alpha}_-^{(\infty)})$.

Here $\mathcal{F}(\bar{\alpha}^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\bar{\alpha}^{(n)}))$; $\mathcal{F}(\bar{\alpha}_-^{(\infty)}) \triangleq \mathcal{F}(\bigcup_{n=1}^{\infty} \mathcal{F}(\bar{\alpha}_-^{(n)}))$.

LEMMA 4.5
Let $\alpha(\omega)$ be a nonnegative random variable. Then

(1) $\mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$ *for every* $n \geq 1$,

(2) $\mathcal{N}_{\bar{\alpha}_-^{(n)}} \subseteq \mathcal{N}_{\bar{\alpha}_-^{(n+1)}}$ *for every* $n \geq 1$,

(3) $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}} = \mathcal{N}_{\bar{\alpha}_-^{(\infty)}}$.

Proof. (1) Let $\Pi_n = \{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} : m \geq 1;\ t_1 \leq \ldots \leq t_m \leq s;\ A_1, \ldots, A_m \in \mathcal{E}\}$ for every $n = 1, 2, \ldots$. By Theorem 2.3 it follows that $\mathcal{N}_{\alpha_-^{(n)}} = \mathcal{F}(\Pi_n)$ for every $n = 1, 2, \ldots$. Suppose, without loss of generality, that $s \in (\frac{k-1}{2^n}, \frac{k}{2^n}]$. Then s must lie in either the interval $(\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$ or the interval $[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}]$.

If $s \in (\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$, then

$$\{\alpha_-^{(n)} > s\} = \Big\{\alpha_-^{(n)} \geq \frac{k}{2^n}\Big\} = \Big\{\alpha > \frac{k}{2^n}\Big\},$$

$$\{\alpha_-^{(n+1)} > s\} = \Big\{\alpha_-^{(n+1)} \geq \frac{2k-1}{2^{n+1}}\Big\} = \Big\{\alpha > \frac{2k-1}{2^{n+1}}\Big\}.$$

Hence,

$$\{\alpha_-^{(n+1)} > s\} = \{\alpha_-^{(n)} > s\} + \Big\{\alpha_-^{(n+1)} = \frac{2k-1}{2^{n+1}}\Big\},$$

from which it follows that

$$\begin{aligned}\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} &\cap \{\alpha_-^{(n)} > s\} \\ &= \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n+1)} > s\} \\ &\quad - \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \Big\{\alpha_-^{(n+1)} = \frac{2k-1}{2^{n+1}}\Big\} \\ &= \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n+1)} > s\} \\ &\quad - \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \Big\{s < \alpha_-^{(n+1)} \le \frac{2k-1}{2^{n+1}}\Big\}.\end{aligned} \tag{4.7}$$

By Theorem 2.2 and Theorem 2.3 as $t_m \le s < \frac{2k-1}{2^{n+1}}$ it follows

$$\begin{aligned}\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} &\cap \Big\{s < \alpha_-^{(n+1)} \le \frac{2k-1}{2^{n+1}}\Big\} \\ &= \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n+1)} > s\} \cap \Big\{\alpha_-^{(n+1)} \le \frac{2k-1}{2^{n+1}}\Big\} \\ &\in \mathcal{N}_{\alpha_-^{(n+1)}},\end{aligned}$$

this and (4.7) yield

$$\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}} \tag{4.8}$$

for every $s \in (\frac{2(k-1)}{2^{n+1}}, \frac{2k-1}{2^{n+1}})$.

If $s \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})$, obviously,

$$\{\alpha_-^{(n+1)} > s\} = \Big\{\alpha_-^{(n+1)} \ge \frac{2k}{2^{n+1}}\Big\} = \Big\{\alpha > \frac{k}{2^n}\Big\} = \Big\{\alpha_-^{(n)} \ge \frac{k}{2^n}\Big\} = \{\alpha_-^{(n)} > s\},$$

from which it follows that

$$\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}} \tag{4.9}$$

for every $s \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})$.

If $s = \frac{k}{2^n}$, an analogous treatment of (4.8) implies

$$\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > s\} \in \mathcal{N}_{\alpha_-^{(n+1)}}, \tag{4.10}$$

from (4.8)–(4.10) it follows that $\Pi_n \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$. Hence, by Theorem 2.3, we get $\mathcal{N}_{\alpha_-^{(n)}} = \mathcal{F}(\Pi_n) \subseteq \mathcal{N}_{\alpha_-^{(n+1)}}$ for every $n \ge 1$.

(2) By an analogous treatment of (1) we complete the proof of (2).

(3) Obviously, $\mathcal{N}_{\alpha_-^{(\infty)}} \subseteq \mathcal{N}_\alpha$ by $\mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_\alpha$ for every n. Next we prove $\mathcal{N}_\alpha \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$. Set $a_n = \min(\frac{k}{2^n} - s : \frac{k}{2^n} - s \geq 0,\, 1 \leq k \leq n2^n)$; $K_n = s + a_n$, from which it follows $(K_n, \infty) \uparrow (s, \infty)$ as $n \uparrow \infty$, where $(K_n, \infty) \uparrow (s, \infty)$ is defined by $\bigcup_{n=1}^{\infty}(K_n, \infty) = (s, \infty)$. Therefore, by $\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha_-^{(n)} > K_n\} \in \mathcal{N}_{\alpha_-^{(n)}} \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$ and $\{\alpha_-^{(n)} > K_n\} \uparrow$ as $n \uparrow$, it follows that

$$\begin{aligned}&\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m,\, \alpha > s\}\\&\quad = \lim_{n\uparrow\infty}\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m,\, \alpha_-^{(n)} > K_n\} \in \mathcal{N}_{\alpha_-^{(\infty)}},\end{aligned}$$

this and Theorem 2.3 gives $\mathcal{N}_\alpha \subseteq \mathcal{N}_{\alpha_-^{(\infty)}}$. Finally, $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}}$. In the same manner one can prove $\mathcal{N}_\alpha = \mathcal{N}_{\alpha_-^{(\infty)}}$, $\mathcal{N}_\alpha = \mathcal{N}_{\bar{\alpha}_-^{(\infty)}}$.

Lemma 4.6

Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$, $\alpha(\omega)$ be a stopping time, that is, $\{\alpha \leq t\} \in \mathcal{N}_t^+$ for every $t \geq 0$. Put

$$Z(s,\omega) \triangleq E[f(x_t)|x_s](\omega).$$

Suppose that $\bar{Z}(s,\omega)$ is a $\mathcal{N}_{s+}^+(\triangleq \bigcap_{u>s}\mathcal{N}_u^+)$-adaptive process which is uniquely determined by $Z(s,\omega)$ according to [7, Theorem 3.5]. Then

$$E[f(x_t)|x_\alpha] = \bar{Z}(\alpha(\omega),\omega), \quad P_{\{\alpha\leq t\}}\text{–a.e.}$$

that is,

$$\mathcal{X}_{\{\alpha\leq t\}}E[f(x_t)|x_\alpha] = \mathcal{X}_{\{\alpha\leq t\}}\bar{Z}(\alpha(\omega),\omega), \quad P_{\mathcal{F}(x_\alpha)}\text{–a.e..}$$

Proof. Take

$$\bar{\alpha}^{(n)}(\omega) = \sum_{k=1}^{n2^n} a_k^{(n)}\mathcal{X}_{\{a_{k-1}^{(n)}<\alpha\leq a_k^{(n)}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha(\omega)>n\}}(\omega),$$

and the corresponding to partition of $[0,n] = \sum_{k=1}^{n2^n}(a_{k-1}^{(n)}, a_k^{(n)}]$ satisfies that t is a partition point when $n > t$. So there exists K_n with $1 \leq K_n \leq n2^n$ such that $\{\alpha \leq t\} = \sum_{k=1}^{K_n}\{a_{k-1}^{(n)} < \alpha \leq a_k^{(n)}\}$. Take $\{a_k^{(n)} : n \geq 1, 1 \leq k \leq n2^n\} \subseteq D$, where D is defined as that in [7, Theorem 3.5], for every $A \in \mathcal{E}$,

$$\begin{aligned}&\int\limits_{x_\alpha^{-1}(A)\{\alpha\leq t\}} E[f(x_t)|x_\alpha]\,P(\mathrm{d}\omega)\\&\quad = \int\limits_{x_\alpha^{-1}(A)\{\alpha\leq t\}} f(x_t)\,P(\mathrm{d}\omega)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K_n} \int\limits_{x_\alpha^{-1}(A)\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}} E\big[f(x_t)|\mathcal{N}^+_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega)\\
&= \sum_{k=1}^{K_n} \int\limits_{x_\alpha^{-1}(A)\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}} E\big[f(x_t)|x_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \sum_{k=1}^{K_n} Z\big(a_k^{(n)},\omega\big)\mathcal{X}_{\{a_{k-1}^{(n)}<\alpha\le a_k^{(n)}\}}(\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \sum_{k=1}^{K_n} Z\big(a_k^{(n)},\omega\big)\mathcal{X}_{\{\bar\alpha^{(n)}=a_k^{(n)}\}}(\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} Z(\bar\alpha^{(n)},\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \lim_{n\uparrow\infty} Z(\bar\alpha^{(n)},\omega)\, P(\mathrm{d}\omega)\\
&= \int\limits_{x_\alpha^{-1}(A)\{\alpha\le t\}} \bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega),
\end{aligned}
$$

where $x_\alpha^{-1}(A) = \{\omega :\ x_\alpha \in A\}$. The first equality follows from the definition of conditional expectation; the second equality follows from $x_\alpha^{-1}(A)\cap\{a_{k-1}^{(n)} < \alpha \le a_k^{(n)}\} \in \mathcal{F}(x_\alpha) \subseteq \mathcal{N}_\alpha^+$ and Theorem 2.5; the third equality follows from Markov property; the seventh equality follows from dominated convergence theorem; the last equality follows from [7, Theorem 3.5]. Similarly to the above proof we obtain

$$
\begin{aligned}
&\int\limits_{x_\alpha^{-1}(A)\{\alpha\le u\}} \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega)\\
&\qquad = \int\limits_{x_\alpha^{-1}(A)\{\alpha\le u\}} \mathcal{X}_{\{\alpha\le t\}} \bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega)
\end{aligned}
$$

for every $u \ge 0$, from which and λ-π-system method it follows

$$
\int\limits_{x_\alpha^{-1}(A)\{\alpha\in B\}} \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) = \int\limits_{x_\alpha^{-1}(A)\{\alpha\in B\}} \mathcal{X}_{\{\alpha\le t\}} \bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega)
$$

for every $B \in \mathcal{B}([0,\infty])$, Note that $\mathcal{F}(x_\alpha) = \mathcal{F}(x_\alpha^{-1}(A)\{\alpha \in B\};\ A \in \mathcal{E},\ B \in \mathcal{B}([0,\infty]))$. From which and λ-π-system method it follows, for every $C \in \mathcal{F}(x_\alpha)$,

$$
\int\limits_C \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) = \int\limits_C \mathcal{X}_{\{\alpha\le t\}} \bar Z(\alpha(\omega),\omega)\, P(\mathrm{d}\omega). \tag{4.11}
$$

Since $E[f(x_t)|x_{a_k^{(n)}}]$ is a measurable function with $x_{a_k^{(n)}}(\omega)$ as intermediate variable and ω as independent variable, that is, there exists a function $K(a_k^{(n)}, t; x, f(x_t))$ on $(E, \mathcal{E})$ such that $E[f(x_t)|x_{a_k^{(n)}}] = K(a_k^{(n)}, t; x_{a_k^{(n)}}, f(x_t))$. So, by [2, Theorem 2.2.13], we have that $E[f(x_t)|x_{a_k^{(n)}}]$ is both $\mathcal{E}$-measurable (in this case, $E[f(x_t)|x_{a_k^{(n)}}]$ is regarded as defined on space $(E, \mathcal{E})$) and $\mathcal{F}(x_{a_k^{(n)}})$-measurable (in this case, $E[f(x_t)|x_{a_k^{(n)}}]$ is regarded as defined on space $(\Omega, \mathcal{F}(x_{a_k^{(n)}}))$). Rewrite $Z(x, a_k^{(n)}) \triangleq K(a_k^{(n)}, t; x, f(x_t))$. Let $Z^{(n)}(x, s) = \sum_{k=1}^{K_n} Z(x, a_k^{(n)})\mathcal{X}_{\{a_{k-1}^{(n)} < s \leq a_k^{(n)}\}}$. Hence $Z^{(n)}(x, s)$ is $\mathcal{E} \times \mathcal{B}([0, \infty])$-measurable (see [9, Section 2.6, Problem 8]). So $\lim_{n\uparrow\infty} Z^{(n)}(x, s)$ is also $\mathcal{E} \times \mathcal{B}([0, \infty])$-measurable. Since $\{a_k^{(n)} : n \geq 1, 1 \leq k \leq n2^n\} \subseteq D$, and $\bar{Z}(s, \omega)$ is right continuous and $\bar{Z}(\bar{\alpha}^{(n)}(\omega), \omega) = \sum_{k=1}^{K_n} Z(x, a_k^{(n)})\mathcal{X}_{\{a_{k-1}^{(n)} < \alpha(\omega) \leq a_k^{(n)}\}}$ by [7, Theorem 3.5], then

$$\bar{Z}(\alpha(\omega), \omega) = \lim_{n\uparrow\infty} \bar{Z}(\bar{\alpha}^{(n)}(\omega), \omega) = \lim_{n\uparrow\infty} Z^{(n)}(x, \alpha(\omega)).$$

Thus $\bar{Z}(\alpha(\omega), \omega)$ is $\mathcal{F}(x_\alpha)$-measurable by [2, Theorem 2.2.13]. By Radon–Nikodym Theorem and (4.11) the proof of theorem is comleted.

Remark 4.7
We easily verify $\alpha^{(n)}(\omega)$ and $\bar{\alpha}^{(n)}(\omega)$ are stopping times if $\alpha(\omega)$ is a stopping time. In fact, for any $t \geq 0$, if $\frac{1}{2^n} \leq t < n + 1$, letting $a_n = \max(a_k^{(n)} : t \geq \frac{k}{2^n}, 1 \leq k \leq n2^n)$, then $\{\alpha^{(n)} \leq t\} = \{\alpha^{(n)} \leq a_n\} = \{\alpha \leq a_n\} \in \mathcal{N}_t^+$; if $t \geq n + 1$, obviously, $\{\alpha^{(n)} \leq t\} = \Omega \in \mathcal{N}_t^+$; if $t < \frac{1}{2^n}$, obviously, $\{\alpha^{(n)} \leq t\} = \emptyset \in \mathcal{N}_t^+$. So $\alpha^{(n)}(\omega)$ is a stopping time. Similarly as above we obtain that $\bar{\alpha}^{(n)}(\omega)$ is also a stopping time.

Lemma 4.8
Let $X(t, \omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, let $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on $(E, \mathcal{E})$, $\alpha(\omega)$ be a stopping time. Then

$$E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big], \quad P_{\{\alpha \leq \bar{T}_n\}}\text{-a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = \mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big], \quad P_{\mathcal{N}_{\bar{\alpha}^{(n)}_-}}\text{-a.e..} \tag{4.12}$$

In particular, if $X(t, \omega)$ is a homogeneous Markov process,

$$E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = E\big\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big\}, \quad P_{\{\alpha \leq \bar{T}_n\}}\text{-a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big] = \mathcal{X}_{\{\alpha \leq \bar{T}_n\}} E\big\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)|\mathcal{N}_{\bar{\alpha}^{(n)}_-}\big\}, \quad P_{\mathcal{N}_{\bar{\alpha}^{(n)}_-}}\text{-a.e..}$$

Here $\bar{T}_n = \max(a_k^{(n)} : t \geq a_k^{(n)}, 1 \leq k \leq n2^n)$; $\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega) = \sum_{x\in E} K(0, t - \bar{\alpha}^{(n)}(\omega); x, f(x_{t-\bar{\alpha}^{(n)}(\omega)})\mathcal{X}_{\{x_{\bar{\alpha}^{(n)}} = x\}}(\omega) = K(0, t - \bar{\alpha}^{(n)}(\omega); x_{\bar{\alpha}^{(n)}}, f(x_{t-\bar{\alpha}^{(n)}(\omega)})$.

Remark 4.9

If $\bar{\alpha}^{(n)}(\omega) > t$ for some ω, then $\Theta_{\bar{\alpha}^{(n)}(\omega)} Z(\bar{\alpha}^{(n)}(\omega), \omega)$ might not be well defined. In this case, we may give $\Theta_{\bar{\alpha}^{(n)}(\omega)} Z(\bar{\alpha}^{(n)}(\omega), \omega)$ an arbitrary value. Obviously, it does not affect our conclusion. So we plight it in this way throughout this paper.

Proof. For every $n \geq 1$, set

$$B^{(n)}_{(t_1,A_1)(t_2,A_2)\dots(t_m,A_m)s} = \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\bar{\alpha}^{(n)}_{-} > s\},$$

$$\Pi_n = \{B^{(n)}_{(t_1,A_1)(t_2,A_2)\dots(t_m,A_m)s} : \ m \geq 1,\, t_1 \leq \dots \leq t_m \leq s, \\ A_1, \dots, A_m \in \mathcal{E}\},$$

$$N = \max\big(k : \ t \geq a^{(n)}_k,\, 1 \leq k \leq n2^n\big).$$

Using the abbreviation $B \triangleq \{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{\bar{\alpha}^{(n)}_{-} > s\}$ we have

$$\int\limits_{B\{\bar{\alpha}^{(n)}_{-} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} f(x_t)\, P(\mathrm{d}\omega). \tag{4.13}$$

Since $\alpha(\omega)$ is a stopping time,

$$\big\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\big\} = \big\{\alpha \leq a^{(n)}_k\big\} - \big\{\alpha \leq a^{(n)}_{k-1}\big\} \in \mathcal{N}^{+}_{a^{(n)}_k}. \tag{4.14}$$

Let $K_n = \min(k : \ 1 \leq k \leq n2^n,\, a^{(n)}_k \geq s)$. When $k \geq K_n$, by $t_1 \leq t_2 \leq \dots t_m \leq s \leq a^{(n)}_k$, it follows that $\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \in \mathcal{N}^{+}_{a^{(n)}_k}$. This and (4.14) give $\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\} \cap \{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\} \in \mathcal{N}^{+}_{a^{(n)}_k}$ for every $k \geq K_n$. Again, for $k < K_n$, we obviously have $B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\} = \emptyset \in \mathcal{N}^{+}_{a^{(n)}_k}$. Therefore, using Markov property, (4.13) is changed into

$$\begin{aligned}
&\int\limits_{B\{\bar{\alpha}^{(n)}_{-} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) \\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} E\big[f(x_t) | \mathcal{N}^{+}_{a^{(n)}_k}\big]\, P(\mathrm{d}\omega) \\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} E\big[f(x_t) | x_{a^{(n)}_k}\big]\, P(\mathrm{d}\omega) \qquad (4.15) \\
&= \sum_{k=1}^{N} \int\limits_{B\{a^{(n)}_{k-1} < \alpha \leq a^{(n)}_k\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega) \\
&= \int\limits_{B\{\bar{\alpha}^{(n)}_{-} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega).
\end{aligned}$$

Let

$$\Lambda = \left\{ B : \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega),\ B \in \mathcal{N}_{\bar{\alpha}_-^{(n)}} \right\}.$$

If $B = \Omega$,

$$\begin{aligned}\int\limits_{\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) &= \sum_{k=1}^{N} \int\limits_{\{a_{k-1}^{(n)} < \alpha \le a_k^{(n)}\}} E\big[f(x_t) | x_{a_k^{(n)}}\big]\, P(\mathrm{d}\omega) \\ &= \int\limits_{\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega),\end{aligned}$$

where the first equality follows from the Markov property and the definition of conditional expectation and (4.14). Again it could be easily verified that Λ satisfies the other conditions of λ-system. Therefore, Λ is a λ-system. Hence, by λ-π-system method, it follows that $\Lambda \supseteq \mathcal{F}(\Pi_n) = \mathcal{N}_{\bar{\alpha}_-^{(n)}}$, namely,

$$\int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} f(x_t)\, P(\mathrm{d}\omega) = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} Z(\bar{\alpha}^{(n)}(\omega), \omega)\, P(\mathrm{d}\omega)$$

for any $B \in \mathcal{N}_{\bar{\alpha}_-^{(n)}}$. From which and definition of conditional expectation we get

$$\begin{aligned}&\int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} E\big[f(x_t) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]\, P(\mathrm{d}\omega) \\ &\qquad = \int\limits_{B\{\bar{\alpha}_-^{(n)} < \bar{T}_n\}} E\big[Z(\bar{\alpha}^{(n)}(\omega), \omega) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]\, P(\mathrm{d}\omega).\end{aligned} \tag{4.16}$$

If $X(t, \omega)$ satisfies homogeneity, the last integrand of (4.15) is $\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega)$. So the last integrand of (4.16) is changed into $E\{\Theta_{\bar{\alpha}^{(n)}} Z(\bar{\alpha}^{(n)}(\omega), \omega) | \mathcal{N}_{\bar{\alpha}_-^{(n)}}\}$. By Radon–Nikodym Theorem we obtain the lemma.

Lemma 4.10

Let $X(t, \omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then, for any $t_1 \le \ldots \le t_m \le s$ and $A_1, \ldots, A_m \in \mathcal{E}$,

$$\begin{aligned}&\int\limits_{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha = s\}} E[f(x_t) | x_s]\, P(\mathrm{d}\omega) \\ &\qquad = \int\limits_{\{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m, \alpha = s\}} E[f(x_t) | x_\alpha]\, P(\mathrm{d}\omega).\end{aligned}$$

Proof. Since $E[f(x_t)|x_s](\omega) = Z(s,\omega)$ is a martingale relative to σ-algebra filtration $\{\mathcal{N}_s^+;\ s \le t\}$, by [7, Theorem 3.5], $E[f(x_t)|x_s] = E[\bar{Z}(s,\omega)|\mathcal{N}_s^+]$. Again, $\{x_{t_1} \in A_1, \dots, x_{t_m} \in A_m,\ \alpha = s\} \in \mathcal{N}_s^+$, so

$$\begin{aligned}\int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} & E[f(x_t)|x_s]\,P(\mathrm{d}\omega)\\ &= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} E[\bar{Z}(s,\omega)|\mathcal{N}_s^+]\,P(\mathrm{d}\omega)\\ &= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} \bar{Z}(s,\omega)\,P(\mathrm{d}\omega)\\ &= \int\limits_{\{x_{t_1}\in A_1,\dots,x_{t_m}\in A_m,\alpha=s\}} E[f(x_t)|x_\alpha]\,P(\mathrm{d}\omega),\end{aligned}$$

where the last equality follows from Lemma 4.6.

LEMMA 4.11
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_t)|\mathcal{N}_\alpha] = E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha], \quad P_{\{\alpha\le t\}}\text{–a.e..} \tag{4.17}$$

Namely,

$$\mathcal{X}_{\{\alpha\le t\}}E[f(x_t)|\mathcal{N}_\alpha] = \mathcal{X}_{\{\alpha\le t\}}E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha], \quad P_{\mathcal{N}_\alpha}\text{–a.e..} \tag{4.18}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_t)|\mathcal{N}_\alpha] = E\{\Theta_{\alpha(\omega)}\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}, \quad P_{\{\alpha\le t\}}\text{–a.e..}$$

Namely,

$$\mathcal{X}_{\{\alpha\le t\}}E[f(x_t)|\mathcal{N}_\alpha] = \mathcal{X}_{\{\alpha\le t\}}E\{\Theta_{\alpha(\omega)}\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}, \quad P_{\mathcal{N}_\alpha}\text{–a.e..}$$

Proof. $\mathcal{N}_{\bar{\alpha}_-^{(n)}} \subseteq \mathcal{N}_{\bar{\alpha}_-^{(n+1)}}$ for every $n \ge 1$ by Lemma 4.5. Set

$$Z_n = \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[f(x_t)|\mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]; \qquad X_n = \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(n)}}\big].$$

Then $\{Z_n;\ n \ge N\}$ is a martingale with respect to σ-algebra family $\{\mathcal{N}_{\bar{\alpha}_-^{(n)}};\ n \ge N\}$. From above and (4.12) it follows that $\{X_n;\ n \ge N\}$ is also a martingale with respect to σ-algebra family $\{\mathcal{N}_{\bar{\alpha}_-^{(n)}};\ n \ge N\}$. So, by the property of conditional expectation we get, for any $n \ge m \ge N$,

$$\begin{aligned}X_m &= E\big[X_n|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big] = E\big\{\mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(n)}}\big]|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big\}\\ &= \mathcal{X}_{\{\alpha\le\bar{T}_N\}}E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big].\end{aligned} \tag{4.19}$$

Here the third equality is a consequence of the fact that $\mathcal{X}_{\{\alpha\le\bar{T}_N\}}$ is $\mathcal{N}_{\bar{\alpha}_-^{(m)}}$-measurable if $m \ge N$. Next, take $\{a_k^{(n)} : n \ge 1, 1 \le k \le n2^n\} \subseteq D$, where D is given by [7, Theorem 3.5], then

$$\lim_{n\to\infty} \mathcal{X}_{\{\alpha\le\bar{T}_N\}} Z(\bar{\alpha}^{(n)}(\omega),\omega) = \mathcal{X}_{\{\alpha\le\bar{T}_N\}} \bar{Z}(\alpha(\omega),\omega), \quad P\text{–a.e.}. \tag{4.20}$$

By the convergence theorem of a martingale (see [3, Corollary 2.13]) and Lemma 4.5,

$$\lim_{N\to\infty}\lim_{n\to\infty} Z_n = \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha], \quad P\text{–a.e.}. \tag{4.21}$$

Again, from (4.19), (4.20) and the convergence theorem of a martingale it follows that

$$\begin{aligned}\lim_{N\to\infty}\lim_{m\to\infty} & X_m \\ &= \lim_{N\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \mathcal{X}_{\{\alpha\le\bar{T}_N\}} E\big[Z(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big] \\ &= \mathcal{X}_{\{\alpha\le t\}} E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha].\end{aligned} \tag{4.22}$$

By (4.12), (4.21), (4.22) we obtain (4.17). By (4.17), for any $B \in \mathcal{N}_\alpha$,

$$\int_B \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega) = \int_B \mathcal{X}_{\{\alpha\le t\}} E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega),$$

this yields (4.18). If $X(t,\omega)$ is a homogeneous Markov process, then the right-hand side of (4.19) is changed into

$$\mathcal{X}_{\{\alpha\le\bar{T}_N\}} E\big\{\Theta_{\bar{\alpha}^{(n)}} \bar{Z}(\bar{\alpha}^{(n)}(\omega),\omega)|\mathcal{N}_{\bar{\alpha}_-^{(m)}}\big\}.$$

Note that $\bar{Z}(s,\omega)$ is right continuous, so (4.22) is changed into

$$\lim_{N\to\infty}\lim_{m\to\infty} X_m = \mathcal{X}_{\{\alpha\le t\}} E\{\Theta_{\alpha(\omega)} \bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha\}.$$

Theorem 4.12 (the strong Markov property)
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_t)|\mathcal{N}_\alpha^+] = E[f(x_t)|x_\alpha], \quad P_{\{\alpha\le t\}}\text{–a.e.}.$$

Further,

$$\mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|x_\alpha], \quad P_{\mathcal{N}_\alpha^+}\text{–a.e.}. \tag{4.23}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_t)|\mathcal{N}_\alpha^+] = [E_{x_\alpha}(f(x_{t-\alpha}))], \quad P_{\{\alpha\le t\}}\text{–a.e.}. \tag{4.24}$$

Further,

$$\mathcal{X}_{\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha\le t\}}[E_{x_\alpha}(f(x_{t-\alpha}))], \quad P_{\mathcal{N}_\alpha^+}\text{–a.e.}, \tag{4.25}$$

where $E_{x_{\alpha(\omega)}}(f(x_{t-\alpha(\omega)})) = \sum_{x\in E} K(0,t-\alpha(\omega);x,f(x_{t-\alpha(\omega)}))\mathcal{X}_{\{x_{\alpha(\omega)}=x\}}(\omega) = K(0,t-\alpha(\omega);x_{\alpha(\omega)},f(x_{t-\alpha(\omega)}))$ for every $\omega\in\Omega$ with $\alpha(\omega)\le t$.

Proof. For any $t_1 \le t_2 \le \ldots \le t_m \le s \le t$ set

$$B^+ = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha \ge s\};$$
$$C = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha = s\};$$
$$B = \{x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\} \cap \{\alpha > s\}.$$

Then $B^+ = B + C$. From Theorem 2.3 and Theorem 2.2 it follows that $B \cap \{\alpha \le t\} \in \mathcal{N}_\alpha$. Again, $C \cap \{\alpha \le t\} = C \in \mathcal{N}_s^+$, from above we have

$$\begin{aligned}
&\int_{B^+\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha^+]\, P(\mathrm{d}\omega)\\
&\quad= \int_{B\{\alpha\le t\}} f(x_t)\, P(\mathrm{d}\omega) + \int_{C\{\alpha\le t\}} f(x_t)\, P(\mathrm{d}\omega)\\
&\quad= \int_{B\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega) + \int_{C\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_s^+]\, P(\mathrm{d}\omega)\\
&\quad= \int_{B\{\alpha\le t\}} E[\bar{Z}(\alpha(\omega),\omega)|\mathcal{N}_\alpha]\, P(\mathrm{d}\omega) + \int_{C\{\alpha\le t\}} E[f(x_t)|x_s]\, P(\mathrm{d}\omega)\\
&\quad= \int_{B\{\alpha\le t\}} \bar{Z}(\alpha(\omega),\omega)\, P(\mathrm{d}\omega) + \int_{\{x_{t_1}\in A_1,\ldots,x_{t_m}\in A_m\}\cap\{\alpha(\omega)=s\}} E[f(x_t)|x_s]\, P(\mathrm{d}\omega)\\
&\quad= \int_{B\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega) + \int_{C\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega)\\
&\quad= \int_{B^+\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega),
\end{aligned}$$

where the third equality follows from Lemma 4.11 and Markov property; the fifth equality follows from Lemma 4.6 and Lemma 4.10. By the λ-π-system method,

$$\int_{B\{\alpha\le t\}} E[f(x_t)|\mathcal{N}_\alpha^+]\, P(\mathrm{d}\omega) = \int_{B\{\alpha\le t\}} E[f(x_t)|x_\alpha]\, P(\mathrm{d}\omega)$$

for any $B \in \mathcal{N}_\alpha^+$. Next, it is required to verify that $E[f(x_t)|x_\alpha]$ is $\mathcal{N}_\alpha^+$-measurable by the definition of conditional expectation. Since $E[f(x_t)|x_\alpha]$ is $\mathcal{F}(x_\alpha)$-measurable by the definition of conditional expectation, $E[f(x_t)|x_\alpha]$ is $\mathcal{N}_\alpha^+$-measurable from $\mathcal{F}(x_\alpha) \subseteq \mathcal{N}_\alpha^+$ according to Theorem 3.6. Again, if $X(t,\omega)$ is a homogeneous Markov process, similarly to the above proof we obtain (4.24) and (4.25).

Note that if $\alpha(\omega) \equiv s$ (constant), then $\mathcal{F}(x_\alpha) = \mathcal{F}(x_s)$ and $\mathcal{N}_\alpha^+ = \mathcal{N}_s^+$. The following corollary is a consequence of Theorem 4.12.

Corollary (Markov property)
Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, f be a $\mathcal{E}$-measurable bounded

real-valued function defined on a measurable space $(E,\mathcal{E})$. *If* $X(t,\omega)$ *satisfies* (4.23), *then* $X(t,\omega)$ *is a Markov process, that is,* $X(t,\omega)$ *satisfies*

$$E[f(x_t)|\mathcal{N}_s^+] = E[f(x_t)|x_s], \quad P_{\mathcal{N}_s^+}\text{-}a.e.$$

for any $0 \le s \le t$.

In particular, if $X(t,\omega)$ *satisfies* (4.25), *then* $X(t,\omega)$ *is a homogeneous Markov process, that is,* $X(t,\omega)$ *has property:*

$$E[f(x_t)|\mathcal{N}_s^+] = E_{x_s}[f(x_{t-s})], \quad P_{\mathcal{N}_s^+}\text{-}a.e.$$

for any $0 \le s \le t$.

By the same method used in the proof of Theorem 4.12, Theorem 4.12 is extended as follows:

Theorem 4.12' (the strong Markov property)
Let $X(t,\omega)$ *be an arbitrary Markov process defined on a probability space* $(\Omega,\mathcal{F},P)$ *and valued in measurable space* $(E,\mathcal{E})$, $f(x_1,\ldots,x_n)$ *be a* n*-dimensional* $\mathcal{E}^n$*-measurable bounded real-valued function defined on a measurable space* $(E^n,\mathcal{E}^n)$ *and let* $\alpha(\omega)$ *be a stopping time. Then*

$$E[f(x_{t_1},\ldots,x_{t_n})|\mathcal{N}_\alpha^+] = E[f(x_{t_1},\ldots,x_{t_n})|x_\alpha], \quad P_{\{\alpha\le\min(t_1,\ldots,t_n)\}}\text{-}a.e..$$

Further,

$$\mathcal{X}_{\{\alpha\le\min(t_1,\ldots,t_n)\}}E[f(x_{t_1},\ldots,x_{t_n})|\mathcal{N}_\alpha^+] \\ = \mathcal{X}_{\{\alpha\le\min(t_1,\ldots,t_n)\}}E[f(x_{t_1},\ldots,x_{t_n})|x_\alpha], \quad P_{\mathcal{N}_\alpha^+}\text{-}a.e..$$

In particular, if $X(t,\omega)$ *is a homogeneous Markov process, then*

$$E[f(x_{t_1},\ldots,x_{t_n})|\mathcal{N}_\alpha^+] = [E_{x_\alpha}(f(x_{t_1-\alpha},\ldots,x_{t_n-\alpha}))], \quad P_{\{\alpha\le\min(t_1,\ldots,t_n)\}}\text{-}a.e..$$

Further,

$$\mathcal{X}_{\{\alpha\le\min(t_1,\ldots,t_n)\}}E[f(x_{t_1},\ldots,x_{t_n})|\mathcal{N}_\alpha^+] \\ = \mathcal{X}_{\{\alpha\le\min(t_1,\ldots,t_n)\}}[E_{x_\alpha}(f(x_{t_1-\alpha},\ldots,x_{t_n-\alpha}))], \quad P_{\mathcal{N}_\alpha^+}\text{-}a.e..$$

Theorem 4.13 (the strong Markov property)
Let $X(t,\omega)$ *be an arbitrary Markov process defined on a probability space* $(\Omega,\mathcal{F},P)$ *and valued in a measurable space* $(E,\mathcal{E})$, $\xi(\omega)$ *be* ${}_\alpha\mathcal{N}^+$*-measurable, and* $E|\xi|<\infty$. *Then*

$$E[\xi|\mathcal{N}_\alpha^+] = E[\xi|x_\alpha], \quad P_{\Omega_\alpha}\text{-}a.e.. \tag{4.26}$$

Proof. If $\xi(\omega) = \mathcal{X}_{\{\alpha\le s\}}\mathcal{X}_{\{x_{t_1}\in A_1\}\ldots\{x_{t_n}\in A_n\}}$, where $s \le t_1 \le \ldots \le t_n$, taking $f(x_{t_1},\ldots,x_{t_n}) = \mathcal{X}_{\{x_{t_1}\in A_1\}\ldots\{x_{t_n}\in A_n\}}$ in Theorem 4.12' yields

$$\mathcal{X}_{\{\alpha\le s\}}E[\mathcal{X}_{\{x_{t_1}\in A_1\}\ldots\{x_{t_n}\in A_n\}}|\mathcal{N}_\alpha^+] \\ = \mathcal{X}_{\{\alpha\le s\}}E[\mathcal{X}_{\{x_{t_1}\in A_1\}\ldots\{x_{t_n}\in A_n\}}|x_\alpha], \quad P_{\Omega_\alpha}\text{-a.e..}$$

By Theorem 2.2 and Theorem 3.3,

$$\begin{aligned} &E[\mathcal{X}_{\{\alpha\leq s, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}}|\mathcal{N}_\alpha^+] \\ &\quad = E[\mathcal{X}_{\{\alpha\leq s, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}}|x_\alpha],\ P_{\Omega_\alpha}\text{–a.e..} \end{aligned} \tag{4.27}$$

Set

$$\begin{aligned} \mathcal{L} &= \{\text{all integrable functions}\}; \\ \mathcal{H} &= \{\text{all } \xi(\omega) \text{ which satisfy (4.26)}\}. \end{aligned}$$

Then $\mathcal{H}$ is $\mathcal{L}$-system. Since $\mathcal{X}_{\{\alpha\leq s,\, x_{t_1}\in A_1,\ldots,x_{t_n}\in A_n\}} \in \mathcal{H}$ for any $n \geq 1$ and $0 \leq s \leq t_1 \leq \ldots \leq t_n$ and $A_1, \ldots, A_n \in \mathcal{E}$ from (4.27), again, ${}_\alpha\mathcal{N}^+ = \mathcal{F}(\{\alpha \leq s,\, x_{t_1} \in A_1, \ldots, x_{t_n} \in A_n\} :\ n \geq 1,\, 0 \leq s \leq t_1 \leq \ldots \leq t_n,\, A_1, \ldots, A_n \in \mathcal{E})$ from Theorem 3.4, by $\mathcal{L}$-system method it follows that $\mathcal{H}$ includes all ${}_\alpha\mathcal{N}^+$-measurable functions in $\mathcal{L}$.

Theorem 4.14 (the strong Markov property)
Let $X(t,\omega)$ be an arbitrary Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $(E, \mathcal{E})$, $f(x)$ be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E, \mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|x_\alpha], \quad P_{\Omega_\alpha}\text{–a.e..} \tag{4.28}$$

In particular, if $X(t,\omega)$ is a homogeneous Markov process, then

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E_{x_\alpha}[f(x_t)], \quad P_{\Omega_\alpha}\text{–a.e..} \tag{4.29}$$

Proof. By Theorem 3.2 and Theorem 3.4, similarly to the proof of (3.5), it follows that

$$\begin{aligned} \{x_{t+\alpha} \in A\} &= \{\alpha \leq t+\alpha,\, x_{t+\alpha} \in A\} \\ &= \bigcup_{s<\infty} (\{\alpha \leq t+s,\, x_{t+s} \in A\} \cap \{\alpha = s\}) + \{\beta \in A,\, \alpha = \infty\} \\ &\in {}_\alpha\mathcal{N}^+ \end{aligned}$$

for every $A \in \mathcal{E}$ and $t \geq 0$, that is, $x_{t+\alpha}$ is ${}_\alpha\mathcal{N}^+$-measurable. Therefore, $f(x_{t+\alpha})$ is ${}_\alpha\mathcal{N}^+$-measurable from [2, Theorem 2.2.13]. So $f(x_{t+\alpha})$ is also $\mathcal{F}$-measurable. Hence $f(x_{t+\alpha})$ is a random variable, that is, for every $B \in \mathcal{B}((-\infty, \infty))$,

$$\{\omega :\ f(x_{t+\alpha}) \in B\} \in \mathcal{F}. \tag{4.30}$$

Again, $E|f(x_{t+\alpha})| < \infty$, which follows from $f(x)$ is bounded. Hence, from Theorem 4.13 we get (4.28). Next, if $X(t,\omega)$ is a homogeneous Markov process, we shall prove (4.29). Set

$$f^{(n)}(x) = \sum_{k=-n2^n+1}^{n2^n} \frac{k}{2^n} \mathcal{X}_{\{\frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\}} + (n+1)\mathcal{X}_{\{f(x)>n\}} - n\mathcal{X}_{\{f(x)\leq -n\}};$$

$$A_k^{(n)} = \left\{x:\ \frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\right\} \quad (-n2^n+1 \le k \le n2^n);$$
$$A_{n2^n+1}^{(n)} = \{x:\ f(x) > n\};$$
$$A_{-n2^n}^{(n)} = \{x:\ f(x) \le -n\}.$$

Since $f(x)$ is $\mathcal{E}$-measurable, then $A_k^{(n)} \in \mathcal{E}$ for every $-n2^n \le k \le n2^n+1$. Again, because $x_t(\omega)$ values in a measurable space $(E,\mathcal{E})$, if $f(x)$ is replaced by $\mathcal{X}_{A_k^{(n)}}(x)$ in (4.30), it follows that $\mathcal{X}_{\{x_{t+\alpha}(\omega)\in A_k^{(n)}\}}$ is $\mathcal{F}$-measurable. Again, by (4.28), for every n and $-n2^n \le k \le n2^n+1$,

$$E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] = E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|x_\alpha\big]$$

for every $\omega \in \Omega_\alpha - N_{nk}$, where N_{nk} is a P-null measurable set and satisfies $N_{nk} \subseteq \Omega_\alpha$, from which it follows that

$$\begin{aligned}\mathcal{X}_{\{\alpha=s\}}E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] &= E\big[\mathcal{X}_{\{\alpha=s\}}\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= E\big[\mathcal{X}_{\{\alpha=s\}}\mathcal{X}_{\{x_{t+s}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= \mathcal{X}_{\{\alpha=s\}}E\big[\mathcal{X}_{\{x_{t+s}\in A_k^{(n)}\}}|x_\alpha\big]\\ &= \mathcal{X}_{\{\alpha=s\}}E_{x_\alpha}\big[\mathcal{X}_{\{x_t\in A_k^{(n)}\}}\big]\end{aligned}$$

for every $\omega \notin N_{nk}$, where the first equality follows from (4.28) and $\mathcal{X}_{\{\alpha=s\}}$ is $\mathcal{F}(x_\alpha)$-measurable according to Theorem 3.3; the last equality follows from (4.25). Note that N_{nk} does not depend on s. Then

$$\mathcal{X}_{\{\alpha<\infty\}}E\big[\mathcal{X}_{\{x_{t+\alpha}\in A_k^{(n)}\}}|\mathcal{N}_\alpha^+\big] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}\big[\mathcal{X}_{\{x_t\in A_k^{(n)}\}}\big]$$

for every $\omega \in \Omega_\alpha - N_{nk}$. Hence,

$$\mathcal{X}_{\{\alpha<\infty\}}E[f^{(n)}(x_{t+\alpha})|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}[f^{(n)}(x_t)]$$

for every $\omega \in \Omega_\alpha - N^{(n)}$, where $N^{(n)}$ is defined by $\bigcup_{k=-n2^n}^{n2^n+1} N_{nk}$. Further, by monotone convergence theorem we obtain

$$\mathcal{X}_{\{\alpha<\infty\}}E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha<\infty\}}E_{x_\alpha}[f(x_t)]$$

for every $\omega \in \Omega_\alpha - N$, where $N = \bigcup_{n=1}^\infty N^{(n)}$, thus yields (4.29).

By the above theorem and corollary we have the following statements.

Theorem 4.15

Let $X(t,\omega)$ be an arbitrary stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$ and valued in a measurable space $(E,\mathcal{E})$, f be a $\mathcal{E}$-measurable bounded real-valued function defined on a measurable space $(E,\mathcal{E})$ and let $\alpha(\omega)$ be a stopping time. Then the following statements are equivalent:

(1) *(Markov property) For any* $t \geq 0$,

$$E[f(x_t)|\mathcal{N}_s^+] = E[f(x_t)|x_s], \quad P_{\mathcal{N}_s^+}\text{-}a.e.$$

for any $0 \leq s \leq t$.

(2) *(the strong Markov property) For any* $t \geq 0$,

$$E[f(x_t)|\mathcal{N}_\alpha^+] = E[f(x_t)|x_\alpha], \quad P_{\{\alpha \leq t\}}\text{-}a.e..$$

Further, we have

$$\mathcal{X}_{\{\alpha \leq t\}} E[f(x_t)|\mathcal{N}_\alpha^+] = \mathcal{X}_{\{\alpha \leq t\}} E[f(x_t)|x_\alpha], \quad P_{\mathcal{N}_\alpha^+}\text{-}a.e.,$$

(3) *(the strong Markov property) Let* $\xi(\omega)$ *be* ${}_\alpha\mathcal{N}^+$*-measurable, and* $E|\xi| < \infty$. *Then*

$$E[\xi|\mathcal{N}_\alpha^+] = E[\xi|x_\alpha], \quad P_{\Omega_\alpha}\text{-}a.e..$$

(4) *(the strong Markov property) For any* $t \geq 0$,

$$E[f(x_{t+\alpha})|\mathcal{N}_\alpha^+] = E[f(x_{t+\alpha})|x_\alpha], \quad P_{\Omega_\alpha}\text{-}a.e..$$

Appendix A. Theorems and concepts cited in this paper

For convenience of the reader, we list all theorems used in this paper.

THEOREM A.1 ([2] PROPERTY 2.2.2)
Let f *be a mapping from* Ω *to* E, $\mathcal{H}$ *be a* σ*-algebra of* E. *Then* $f^{-1}(\mathcal{H})$ *is a* σ*-algebra of* Ω.

THEOREM A.2 ([2] THEOREM 2.2.13)
Let Ω *be a set,* $(E, \mathcal{E})$ *be a measurable space,* f *be a mapping from* Ω *to* E. *Then* φ *is a* $f^{-1}(\mathcal{E})$*-measurable function from* Ω *to* $\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty\}$ *if and only if there exists a* $\mathcal{E}$*-measurable real-valued function* g *on* $(E, \mathcal{E})$ *such that* $\varphi = g \circ f$. *And if* φ *is bounded or finite, then* g *is bounded or finite.*

THEOREM A.3 ([2] THEOREM 5.2.5)
Let ξ *be a random variable defined on a probability space* $(\Omega, \mathcal{F}, P)$, $\mathcal{C}$ *be a* σ*-subalgebra of* $\mathcal{F}$, B *be an arbitrary atom of* $\mathcal{C}$. *Then, for any* $\omega \in B$,

$$E(\xi|\mathcal{C})(\omega) \equiv constant.$$

Further, if $P(B) > 0$, *then*

$$E(\xi|\mathcal{C})(\omega) = \frac{1}{P(B)} \int_B \xi \, \mathrm{d}P$$

for every $\omega \in B$.

Theorem A.4 ([2] Theorem 5.3.1)
Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, $E\xi$ exist, f be a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$. Then, there exists a $\mathcal{E}$-measurable function g, which is P_f-almost everywhere uniquely determined by $E(\xi|\mathcal{F}(f))$, defined on $(E, \mathcal{E})$ such that

$$E(\xi|\mathcal{F}(f)) = g \circ f, \quad P_{\mathcal{F}(f)}\text{-a.e.},$$

where g satisfies

$$\int_A g\, P_f(\mathrm{d}x) = \int_{f^{-1}(A)} \xi\, P(\mathrm{d}\omega)$$

for every $A \in \mathcal{E}$, where P_f is a probability measure derived by f, that is, P_f satisfies $P_f(A) = P(f^{-1}(A))$ for every $A \in \mathcal{E}$.

Theorem A.5 (Integrable Transform Theorem; [2] Theorem 3.4.1)
Let f be a measurable transformation from the a measurable space $(\Omega, \mathcal{F})$ to the measurable space $(E, \mathcal{E})$; g be a measurable function defined on $(E, \mathcal{E})$; μ be a measure on $(\Omega, \mathcal{F})$; μ_f be a derived measure on $(E, \mathcal{E})$ by f, that is, $\mu_f(B) \triangleq \mu(f^{-1}(B))$ for every $B \in \mathcal{E}$. Then

$$\int_{f^{-1}(B)} g \circ f\, \mathrm{d}\mu = \int_B g\, \mathrm{d}\mu_f,$$

which means: if one of the two integrals exists, then the other also exists, and the two integrals are equal.

Theorem A.6 (Extended Föllmer Lemma; [7] Theorem 3.5)
Let $X(t, \omega)$ be a martingale with respect to σ-algebra filtration $\{\mathcal{F}_t;\ t \geq 0\}$, D be a countable dense subset of $\mathbb{R}_+$. Then there exists a $\mathcal{F}_{t+}$-adaptive process $\bar{X}(t, \omega)$, which satisfies the following properties:

(1) *The every trajectory of $\bar{X}(t, \omega)$ is right continuous, and there exists a null measurable ω-set N such that*

$$\bar{X}(t, \omega) = \lim_{s \in D, s \downarrow t} X(s, \omega)$$

for every $t \geq 0$ and $\omega \in \Omega - N$.

(2) *There exists a null measurable ω-set N_1 such that, for every $t > 0$ and $\omega \in \Omega - N_1$,*

$$\bar{X}(t-, \omega) = \lim_{s \in R_+, s \uparrow t} \bar{X}(s, \omega)$$

exists and is finite, and

$$\bar{X}(t-, \omega) = \lim_{s \in D, s \uparrow t} X(s, \omega).$$

(3) *For every* $t \geq 0$, $X(t,\omega) = E[\bar{X}(t,\omega)|\mathcal{F}_t]$, P*–a.e..*

(4) $\bar{X}(t,\omega)$ *is a martingale with respect to σ-algebra filtration* $\mathcal{F}_{t+}$.

Here $\mathbb{R}_+ = [0,\infty)$; $\{\mathcal{F}_t;\ t \geq 0\}$ *is a σ-algebra filtration, that is, if* $s \leq t$, *then* $\mathcal{F}_s \subseteq \mathcal{F}_t$; $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

Theorem A.7 ([3] Corollary 2.13)
Let $\{\mathcal{F}_n;\ n \geq 0\}$ *be a monotone increasing σ-subalgebra family of* $\mathcal{F}$, Y *be an integrable random variable,* $\mathcal{F}_\infty \triangleq \mathcal{F}(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$. *Set*

$$X_n = E[Y|\mathcal{F}_n]$$

for every $n \geq 0$. *Then we have*

(1) $\{X_n,\ n \geq 0\}$ *is uniformly integrable.*

(2) $X_n \to E(Y|\mathcal{F}_\infty)$, P*–a.e., and* $E|X_n - E(Y|\mathcal{F}_\infty)| \to 0$ *as* $n \to \infty$.

Theorem A.8 (Radon–Nikodym Theorem; [2] Theorem 3.7.6)
Let μ be a σ-finite measure on σ-algebra $\mathcal{A}$ of Ω. If the set function φ defined on $\mathcal{A}$ is σ-finite and σ-additive and μ-continuous, then there exists a $\mathcal{A}$-measurable finite function f defined on $(\Omega,\mathcal{A})$ such that φ is the indefinite integral of f on a measurable space $(\Omega,\mathcal{A},\mu)$, and f is $\mu_{\mathcal{A}}$-almost surly uniquely determined by φ.

Theorem A.9 (Tulcea Theorem; [2] theorem 5.4.5)
Let $(\Omega_n,\mathcal{A}_n)$, $n = 1,2,\ldots$ *be sequence of measurable spaces. Set* $\Omega^{(n)} = \prod_{k=1}^{n} \Omega_k$, $\mathcal{A}^{(n)} = \prod_{k=1}^{n} \mathcal{A}_k$, $\Omega^{(\infty)} = \prod_{k=1}^{\infty} \Omega_k$, $\mathcal{A}^{(\infty)} = \prod_{k=1}^{\infty} \mathcal{A}_k$. *Let* $P_n(\omega_1,\ldots,\omega_{n-1},A_n)$, $(\omega_1,\ldots,\omega_{n-1},A_n) \in \Omega^{(n-1)} \times \mathcal{A}_n$, $n = 2,3,\ldots$ *be the transition probabilities;* $P_1(A), A \in \mathcal{A}_1$ *be the probability on* $\mathcal{A}_1$. *Then there exists only one probability measure* $P^{(\infty)}$ *on* $\mathcal{A}^{(\infty)}$ *such that*

$$P^{(\infty)}(C(B^{(n)})) = P^{(n)}(B^{(n)})$$

and

$$P^{(n)}(B^{(n)}) = \int\limits_{\Omega_1} \ldots \int\limits_{\Omega_n} \mathcal{X}_{B^{(n)}}(\omega_1,\ldots,\omega_n) P_n(\omega_1,\ldots,\omega_{n-1},\mathrm{d}\omega_n)\ldots P_1(\mathrm{d}\omega_1).$$

Here $C(B^{(n)})$ *indicates the cylinder set based on* $B^{(n)}$; $B^{(n)} \in \mathcal{A}^{(n)}$.

Theorem A.10 (Fubini Theorem; [2] Theorem 4.2.1)
Let $(\Omega_i,\mathcal{A}_i,\mu_i)$, $i = 1,2$ *be two σ-finite measurable spaces, f be nonnegative $\mathcal{A}_1 \times \mathcal{A}_2$-measurable function. Then*

$$\int\limits_{\Omega_1\times\Omega_2} f\,\mathrm{d}\mu_1 \times \mu_2 = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} f(\omega_1,\omega_2)\,\mathrm{d}\mu_2(\omega_2) \right) \mathrm{d}\mu_1(\omega_1)$$
$$= \int\limits_{\Omega_2} \left(\int\limits_{\Omega_1} f(\omega_1,\omega_2)\,\mathrm{d}\mu_1(\omega_1) \right) \mathrm{d}\mu_2(\omega_2).$$

Dfinition A.11 ([2] Definition 5.1.3)
Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\{B_n\} \subseteq \mathcal{A}$ be a countable subdivision of Ω, that is, $\Omega = \sum_{n=1}^{\infty} B_n$ and $B_i \cap B_j = \emptyset$, $i \neq j$. Put $\mathcal{G} = \mathcal{F}(B_n; n = 1, 2, \ldots)$. Suppose $E\xi$ exists. The following $\mathcal{G}$-measurable function in the sense of equivalence (that is, we may give an arbitrary value on null measurable set of $\mathcal{G}$, such as. If $P(B_n) = 0$, then $E(\xi|B_n)$ may be given arbitrarily.)

$$E(\xi|\mathcal{G}) = \sum_{n=1}^{\infty} E(\xi|B_n)\mathcal{X}_{B_n}(\omega)$$

is called the conditional expectation of ξ given $\mathcal{G}$.

Appendix B. The concepts of λ-system and $\mathcal{L}$-system

Here we will introduce the concepts of λ-system and $\mathcal{L}$-system, the λ-π-system method and $\mathcal{L}$-system method mentioned in this paper, which are taken from [1, Appendix].

Dfinition B.1
A system Π of subsets of a set Ω is called a π-system, if $A_1 \in \Pi, A_2 \in \Pi \implies A_1 A_2 \in \Pi$.

Dfinition B.2
A system Λ of subsets of a set Ω is called a λ-system, if it has the following properties:

(1) $\Omega \in \Lambda$;

(2) $A_1 \in \Lambda, A_2 \in \Lambda, A_1 \cap A_2 = \emptyset \implies A_1 \cup A_2 \in \Lambda$;

(3) $A_1 \in \Lambda, A_2 \in \Lambda, A_1 \supset A_2 \implies A_1 - A_2 \in \Lambda$;

(4) $A_n \in \Lambda, A_n \uparrow A, n = 1, 2, \ldots \implies A \in \Lambda$.

Theorem B.3
(1) *If the system $\mathcal{M}$ of subsets of a set Ω is a π-system, and is also a λ-system, then $\mathcal{M}$ is a σ-algebra.*

(2) *If λ-system Λ contains π-system Π, then $\Lambda \supseteq \mathcal{F}(\Pi)$.*

When we make use of Theorem B.3, we call this method λ-π-system method.
Let $\mathcal{L}$ be a family of functions defined on Ω, and satisfies:
if $\xi(\omega) \in \mathcal{L}$, set

$$\eta(\omega) = \begin{cases} \xi(\omega) & \text{if } \xi(\omega) \geq 0, \\ 0 & \text{if } \xi(\omega) < 0, \end{cases}$$

then $\eta(\omega)$ and $\eta(\omega) - \xi(\omega)$ lie in $\mathcal{L}$.

Dfinition B.4
A set L of functions is called $\mathcal{L}$-system, if it satisfies the following conditions:

(1) $\mathbb{1} \in L$, where the $\mathbb{1}$ is the function whose functional value is equal to 1;

(2) For two arbitrary functions in L, their linear combination lies in L;

(3) If $\xi_n(\omega) \in L, 0 \leq \xi_n(\omega) \uparrow \xi(\omega)$, and $\xi(\omega)$ is bounded or lies in $\mathcal{L}$, then $\xi(\omega) \in L$.

Theorem B.5
If a $\mathcal{L}$-system L contains the indicator function $\mathcal{X}_A(\omega)$ of every set A of π-system Π, then L contains all $\mathcal{F}(\Pi)$-measurable function in $\mathcal{L}$.

When we make use of Theorem B.5, we call this method $\mathcal{L}$-system method.

Appendix C. The concepts of partial ordering

We recall the concepts of partial ordering and three important theorems from real analysis (such as [8]).

Dfinition C.1
Let $\mathcal{S}$ be an arbitrary set. $\mathcal{S}$ is said to be a partially ordered set, if there is a binary relation "$\preceq$" called a partial ordering, defined on $\mathcal{S}$ with the following properties:

(1) $x \preceq x$ for all $x \in \mathcal{S}$ (reflexive),

(2) $x \preceq y, y \preceq z \implies x \preceq z$ for all $x, y, z \in \mathcal{S}$ (transitive),

(3) $x \preceq y, y \preceq x \implies x = y$ for all $x, y \in \mathcal{S}$ (antisymmetric).

Dfinition C.2
A partially ordered set $\mathcal{S}$ is called a totally ordered set if it follows $x \preceq y$ or $x \preceq y$ for any $x, y \in \mathcal{S}$.

Dfinition C.3
Let $\mathcal{S}$ be a partially ordered set, x_0 lies in $\mathcal{S}$. x_0 is said to be the maximal element of $\mathcal{S}$ if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x_0 \preceq x$; x_0 is said to be the minimal element of $\mathcal{S}$ if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x \preceq x_0$.

Dfinition C.4
Let $\mathcal{S}$ be a partially ordered set, $\mathcal{M}$ be a subset of $\mathcal{S}$, α lies in $\mathcal{S}$. α is said to be an upper bound of $\mathcal{M}$ in $\mathcal{S}$ if it follows $x \preceq \alpha$ for all $x \in \mathcal{M}$; α is said to be a lower bound of $\mathcal{M}$ in $\mathcal{S}$ if it follows $\alpha \preceq x$ for all $x \in \mathcal{M}$.

Dfinition C.5
Let $\mathcal{S}$ be a partially ordered set, $\mathcal{A}$ be a subset of $\mathcal{S}$. α is called a minimum element of $\mathcal{A}$ if α is a lower bound of $\mathcal{A}$ and α lies in $\mathcal{A}$; α is called a maximum element of $\mathcal{A}$ if α is an upper bound of $\mathcal{A}$ and α lies in $\mathcal{A}$.

Dfinition C.6
A partial ordering "$\preceq$" on $\mathcal{S}$ is said to be a well ordering if for every nonempty subset of $\mathcal{S}$ has the minimum element. $\mathcal{S}$ is called well-ordered set if there is a well ordering defined on $\mathcal{S}$.

Theorem C.7 (Zorn Lemma)
Let $\mathcal{S}$ be a partially ordered set. If every totally ordered subset $\mathcal{A}$ of $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal element.

Theorem C.8 (Well Order Theorem)
Every set can be well ordered.

Theorem C.9 (Principle of Transfinite Induction)
Let $(W, \preceq)$ be a well-ordered set. For any $a \in W$, let

$$I(a) = \{x \in W : \ x \prec a\}.$$

If A is a subset of W such that $a \in A$ whenever $I(a) \subset A$, then $A = W$.

References

[1] Wang Zikun, *The general theory of stochastic process*, Beijing Normal University Press, Beijing, 1996 (in Chinese).

[2] Yan Shijian etc., *The foundation of probability*, Science Press, Beijing, 1982 (in Chinese).

[3] Wan Chenggao, *The limit theory of martingale*, Science Press, Beijing, 2002 (in Chinese).

[4] H. Bauer, *Probability theory and elements of measure theory*, Academic Press, London-New York, 1981, Second english edition.

[5] Kai Lai Chung, *Markov chains with stationary transition probabilities*, Springer-Verlang, New York, 1967.

[6] J.L. Doob, *Stochastic processes*, John Wiley&Sons, New-York, 1953.

[7] Yan Jiaan, *The theory of martingale and stochastic integral*, Shanghai Science and Technology Press, Shanghai, 1981 (in Chinese).

[8] A. Mukherjea, K. Pothoven, *Real and functional analysis*, Plenum Press, New York-London, 1978.

[9] Robert B. Ash, Catherine A. Doléans-Dade, *Porbability & measurable theory*, (print in China), Post & telecom press, Beijing, 2007.

[10] Hou Zhenting, Zou Jiezhong etc., *The Q-matrix Problem of Markov Process*, Hunan scientific Press, Hunan, 1994 (in Chinese).

[11] W. Feller, *On the Integro-differential Equation of Purely Discontinuous Markoff Process*, Trans. Amer. Math. Soc. **48** (1940), 488–515.

[12] Iosif I. Gikhman, Anatoli V. Skorokhol (translater Samuel Kotz), *The theory of stochastic process(II)*, Springer-Verlag, Berlin, 2004.

[13] E.B Dynkin, *Markov processes*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

[14] Qian Mingpin, Gong Guanglu, *The theory stocastic processes*, Peking University Press, Beijing, 1997 (in Chinese).

[15] D. Ray, *Stationary Markov processes with continuous paths*, Trans. Amer. Math. Soc. **82** (1956), 452–493.

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