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The regular density on the plane

Abstract. In the note [1] the notion of the regular density point of the measurable subset of the real line was introduced. Then it was shown that the new definition is equivalent to the definition of O'Malley points, which has been examined in [2]. In this note we demonstrate that the analogous definitions for measurable subsets of the plane are not equivalent.

1. Notation

In the sequel we use the following symbols:
χ_A – the characteristic function of the set A,
$\mu(A)$ – the two dimensional Lebesgue measure of the set A,
$\mu_1(B)$ – the linear Lebesgue measure of the set $B \subset \mathbb{R}$,
$\bigvee_a^b f$ – the total variation of the function f on $[a, b]$,
$\mathrm{D}(A, B) = \mu(A \cap B)/\mu(B)$ – the average density of A on the set B,
$A_{x_0} = \{y : (x_0, y) \in A\}$ – the vertical cut of the set A,
$A - (x, y) = \{(p - x, q - y) : (p, q) \in A\}$ – the translation of the set A,
$\lambda A = \{(\lambda p, \lambda q) : (p, q) \in A\}$ – the homothety of the set A.

2. Introduction

In the paper [2] W. Poreda and W. Wilczyński considered the operator Φ_{OM} defined on the class $\mathcal{S}$ of Lebesgue measurable subsets of the real line. The definition of Φ_{OM} was suggested by R. O'Malley in oral communication:

Definition 1
Let A be a measurable subset of $\mathbb{R}$, $x \in \mathbb{R}$. We say that x is an O'Malley point of A iff

$$\int_0^1 \frac{\chi_{A'}(x+t) + \chi_{A'}(x-t)}{t}\, dt < \infty.$$

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The set of all O'Malley points of A we denote by $\Phi_{OM}(A)$. In [2] the authors established, between others that

- every O'Malley point of A is the density point of A,
- the operator Φ_{OM} has the properties similar to the properties of the density operator Φ, however the analogue of the Lebesgue Theorem does not hold,
- the family $\mathcal{T}_0 = \{A \in \mathcal{S} : \ A \subset \Phi_{OM}(A)\}$ forms the topology stronger than the natural topology but coarser than the density topology on the real line,
- the analogue of the Lusin–Menchoff Theorem for Φ_{OM} holds.

In the note [1] the notion of the regular density was introduced:

Definition 2
Let $A \in \mathcal{S}$ and $x \in \mathbb{R}$. Put $f_x(h) = \frac{\mu_1(A\cap[x-h,x+h])}{2h}$ for $h > 0$ and $f_x(0) = 1$. We say that x is the regular density point of the set A if and only if the following conditions are satisfied:

1. $x_0 \in \Phi(A)$,
2. $\bigvee_0^1 f_x < +\infty$.

The main result of the paper [1] states, that the notions of the regular density point and the O'Malley point are equivalent.

In this note we are going to demonstrate, that the situation on the plane is not analogous. In order to do this we are going to redefine the notions of the regular density point and the O'Malley point for the planar sets.

3. The two dimensional case

Definition 3
Let $A \subset \mathbb{R}^2$ be a measurable set. Let us define the function $f\colon [0,1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} D(A, [-x,x]^2) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

We say, that $(0,0)$ is the point of the ordinary regular density of A ($(0,0) \in \Phi_R(A)$) if

1. the function f is continuous at the point 0,
2. $\bigvee_0^1 f < +\infty$.

We say that $(x,y) \in \Phi_R(A)$ iff $(0,0) \in \Phi_R(A - (x,y))$.

The first condition means that $(0,0)$ is the ordinary density point of A. Let us examine the second condition:

Proposition 1
For each positive ε the function f restricted to the interval $[\varepsilon, 1]$ satisfies the Lipschitz condition, so it is absolutely continuous.

Proof. For $x, x+h \in [\varepsilon, 1]$, $h > 0$ we have

$$\begin{aligned} f(x+h) - f(x) &= \frac{\mu(A \cap [-x-h, x+h]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x, x]^2)}{4x^2} \\ &\le \frac{\mu(A \cap [-x, x]^2) + 4h(2x+2h)}{4x^2} - \frac{\mu(A \cap [-x, x]^2)}{4x^2} \\ &= \frac{4h(2x+2h)}{4x^2} \le 2\frac{h}{x^2} \\ &\le \frac{2h}{\varepsilon^2}. \end{aligned}$$

At the same time

$$\begin{aligned} f(x+h) - f(x) &= \frac{\mu(A \cap [-x-h, x+h]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x, x]^2)}{4x^2} \\ &\ge \frac{\mu(A \cap [-x, x]^2)}{4(x+h)^2} - \frac{\mu(A \cap [-x, x]^2)}{4x^2} \\ &= \frac{x^2 - (x+h)^2}{(x+h)^2} = h\frac{-2x-h}{(x+h)^2} \\ &\ge -\frac{2h}{\varepsilon^2}. \end{aligned}$$

Corollary 1

As f is absolutely continuous on $[\varepsilon, 1]$ and $f([0,1]) \subset [0,1]$, we have

$$\bigvee_{\varepsilon}^{1} f = \int_{\varepsilon}^{1} |f'(x)|\, dx$$

and, consequently

$$\bigvee_{0}^{1} f \le \int_{0}^{1} |f'(x)|\, dx + 1.$$

In order to simplify calculations divide the square $[-x, x]^2$ into four triangles: $T_1(x)$ having the vertices $(0,0)$, (x,x) and $(x,-x)$; $T_2(x)$ having the vertices $(0,0)$, $(-x,x)$ and (x,x); $T_3(x)$ having the vertices $(0,0)$, $(-x,-x)$ and $(-x,x)$ and $T_4(x)$ having the vertices $(0,0)$, $(x,-x)$ and $(-x,-x)$. It is easy to observe, that for $x \in (0,1]$, $f(x)$ is the arithmetic average of the average densities of A on the corresponding triangles.

Let us consider the triangle $T_1(x)$. Let

$$g_1(x) = \begin{cases} D(A, T_1(x)) & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Our objective is to estimate from above the number $\int_0^1 |g'_1(x)|\, dx$.

Proposition 2
Let $x, x+h \in (0,1]$, $h > 0$. Let $P(x,h) = T_1(x+h) \setminus T_1(x)$. Then

$$\lim_{h\to 0+} \frac{\mu(A \cap P(x,h))}{h} = \mu_1(A_{(x)} \cap [-x,x])$$

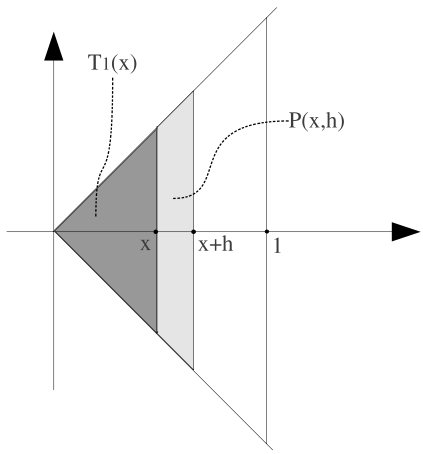
for μ_1–a.e. $x \in (0,1)$.

Proof. Let $s(x) = \mu_1(A_{(x)} \cap [-x,x]) = \mu_1((A \cap T_1(1))_{(x)})$. From the Fubini Theorem the function s is measurable. As the set $A \cap T_1(1)$ is bounded, the function s is also summable. Let

$$S(x) = \int_0^x s(t)\,dt.$$

From the Lebesgue Differentiation Theorem $S'(x) = s(x)$ for μ_1–a.e. $x \in (0,1)$. But

$$S'(x) = \lim_{h\to 0+} \frac{S(x+h) - S(x)}{h} = \lim_{h\to 0+} \frac{\mu(A \cap P(x,h))}{h}.$$



Now we are ready to give a more convenient formula for the function g_1':

Proposition 3
The following formula holds for μ_1–almost every $x \in [0,1]$:

$$g_1'(x) = \frac{2}{x}\left(\frac{\mu_1(A_{(x)} \cap [-x,x])}{2x} - g_1(x)\right).$$

Proof. Let the point $x \in (0,1]$ fulfill the thesis of the previous proposition. Then

$$\begin{aligned}
g_1'(x) &= \lim_{h\to 0+}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0+}\frac{1}{h}\left(\frac{\mu(A\cap T_1(x+h))}{(x+h)^2}-\frac{\mu(A\cap T_1(x))}{x^2}\right)\\
&= \lim_{h\to 0+}\frac{1}{h}\frac{1}{x^2}\left(\mu(A\cap T_1(x+h))\left(1-\frac{h}{(x+h)}\right)^2-\mu(A\cap T_1(x))\right)\\
&= \lim_{h\to 0+}\frac{1}{h}\frac{1}{x^2}\left(\mu(A\cap P(x,h))-\left(\frac{2h}{(x+h)}+\frac{h^2}{(x+h)^2}\right)\mu(A\cap T_1(x+h))\right).
\end{aligned}$$

By virtue of the previous proposition, and the fact that $\mu(A\cap T_1(x+h))$ tends to $\mu(A\cap T_1(x))$, when h tends to 0 we have that

$$g_1'(x) = \frac{1}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{x}-2g_1(x)\right) = \frac{2}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{2x}-g_1(x)\right).$$

Using the last proposition we are able to estimate the number $\int_0^1 |g_1'(x)|\,dx$:

$$\begin{aligned}
&\int_0^1 |g_1'(x)|\,dx\\
&= \int_0^1\left|\frac{2}{x}\left(\frac{\mu_1(A_{(x)}\cap[-x,x])}{2x}-g_1(x)\right)\right|dx = \int_0^1\left|\frac{2}{x}\left(\int_{-x}^{x}\frac{\chi_A(x,t)}{2x}\,dt-g_1(x)\right)\right|dx\\
&= \int_0^1\left|\frac{2}{x}\left(\int_{-x}^{x}\frac{\chi_A(x,t)-g_1(x)}{2x}\,dt\right)\right|dx = \int_0^1\left|\left(\int_{-x}^{x}\frac{\chi_A(x,t)-g_1(x)}{x^2}\,dt\right)\right|dx\\
&\le \int_0^1\int_{-x}^{x}\left|\frac{\chi_A(x,t)-g_1(x)}{x^2}\right|dt\,dx = \int_{T_1(1)}\left|\frac{\chi_A(x,t)-g_1(x)}{x^2}\right|d\mu\\
&= \int_{T_1(1)\cap A}\frac{1-g_1(x)}{x^2}\,d\mu + \int_{T_1(1)\cap A'}\frac{g_1(x)}{x^2}\mathrm{d}\mu.
\end{aligned}$$

Let us denote the last two integrals by C and D, respectively.

Proposition 4

$$C<+\infty \iff D<+\infty.$$

In fact

$$1\ge g(1) = \int_0^1 g_1'(x)\,dx = \int_0^1\int_{-x}^{x}\frac{\chi_A(x,t)-g_1(x)}{x^2}\,dt\,dx = C-D.$$

Definition 4
Let $A\subset\mathbb{R}^2$ be a measurable set. Let $\|(x,y)\| = \max(|x|,|y|)$. We say that $(0,0)$ is an O'Malley point of the set A ($(0,0)\in\Phi_{OM}(A)$) iff

$$\int_{[-1,1]^2\cap A'}\|(x,y)\|^{-2}\,d\mu < +\infty.$$

We say that $(x,y) \in \Phi_{OM}(A)$ iff $(0,0) \in \Phi_{OM}(A-(x,y))$.

Theorem 1
Let $A \subset \mathbb{R}^2$ be a measurable set. Then

$$\Phi_{OM}(A) \subset \Phi_R(A).$$

Proof. Assume that $(0,0) \in \Phi_{OM}(A)$. At first we are going to show, that $(0,0)$ is the ordinary density point of A. Suppose conversely that there exist the sequence (h_n) of positive numbers, tending to 0, and $\varepsilon > 0$ such that $D(A,[-h_n,h_n]^2) < 1-\varepsilon$ for $n = 1,2,\ldots$. We can assume that $(\frac{h_{n+1}}{h_n})^2 < \frac{\varepsilon}{2}$ and $h_0 = 1$. Let $Z_n = [-h_n,h_n]^2 \setminus [-h_{n+1},h_{n+1}]^2$. Then

$$\int\limits_{[-1,1]^2 \cap A'} \|(x,y)\|^{-2}\, d\mu = \sum_{n=0}^{\infty} \int\limits_{Z_n \cap A'} \|(x,y)\|^{-2}\, d\mu.$$

But

$$\int\limits_{Z_n \cap A'} \|(x,y)\|^{-2}\, d\mu > h_n^{-2} \int\limits_{Z_n \cap A'} d\mu \geq h_n^{-2} \cdot h_n^2 \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Hence

$$\int\limits_{[-1,1]^2 \cap A'} \|(x,y)\|^{-2}\, d\mu = +\infty,$$

the contradiction.

From the assumption that $\int_{[-1,1]^2 \cap A'} \|(x,y)\|^{-2}\, d\mu < +\infty$ it follows that $\int_{T_1(1) \cap A'} \|(x,y)\|^{-2}\, d\mu < +\infty$. But for $(x,y) \in T_1(1)$ we have $\|(x,y)\| = x$. Hence

$$D = \int\limits_{T_1(1) \cap A'} \frac{g_1(x)}{x^2}\, d\mu \leq \int\limits_{T_1(1) \cap A'} \frac{1}{x^2}\, d\mu < +\infty.$$

Then, by virtue of Proposition 4, $C < +\infty$, and consequently

$$\int\limits_0^1 |g_1'(x)|\, dx < +\infty.$$

The same holds for the functions g_i corresponding with triangles T_i, $i = 2,3,4$. From

$$f(x) = \frac{1}{4}(g_1(x) + g_2(x) + g_3(x) + g_4(x))$$

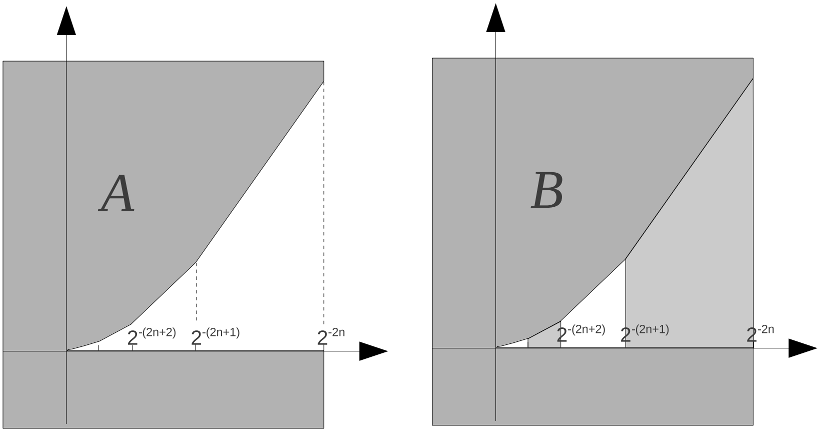
it follows that

$$\bigvee_0^1 f \leq \int\limits_0^1 |f'(x)|\, dx + 1 \leq \int\limits_0^1 \frac{1}{4}(|g_1'(x)| + |g_2'(x)| + |g_3'(x)| + |g_4'(x)|)\, dx + 1 < +\infty.$$

Remark 1
In the one dimensional case the theorem analogous to the Theorem 1 gives the necessary and sufficient condition for x to be the point of regular density. The following example shows that in two dimensions the opposite implication does not hold.

Example 1
In the following example we construct two planar sets, A and B such that

1. $A \subset B$,
2. $(0,0)$ is the regular density point of A,
3. $(0,0)$ is not the regular density of B.



Let

$$x_n = \frac{1}{2^n}, \quad y_n = \frac{1}{(n+1)2^n} \qquad \text{for } n = 0, 1, 2 \ldots.$$

Let $h\colon [0,1] \to [0,1]$ be a function such that

a. $h(0) = 0$,

b. $h(x_n) = y_n$ for $n = 0, 1, 2, \ldots,$

c. h is linear on each interval $[x_{n+1}, x_n]$.

It is easy to observe, that the function h is convex and continuous on the interval $[0,1]$. Let

$$A = [-1,1]^2 \setminus \{(x,y) : \; x \in (0,1) \wedge y \in (0, h(x))\}$$

and

$$P_n = \left\{(x,y) : \; x \in \left[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}\right]\right\}.$$

Finally, let $B = A \cup \bigcup_{n=0}^{\infty} P_n$.

Since the function h is convex, for every $t \in (0,1)$ the set $\{(x,y):\ x \in (0,t) \wedge y \in (0,h(x))\}$ is included in the triangle Z_t with vertices $(0,0)$, $(t,0)$ and $(t,h(t))$. Hence

$$f_A(t) = D(A,[-t,t]^2) \geq \frac{1}{4t^2}(4t^2 - \mu(Z_t)) \longrightarrow 1$$

when t tends to 0.

We shall show that the function f_A is decreasing on the interval $(0,1)$. Let $t \in (0,1)$ and $\lambda \in (0,1)$. We have

$$f_A(\lambda t) = D(A,[-\lambda t,\lambda t]^2) = D\Big(\frac{1}{\lambda}A,[-t,t]^2\Big)$$

and

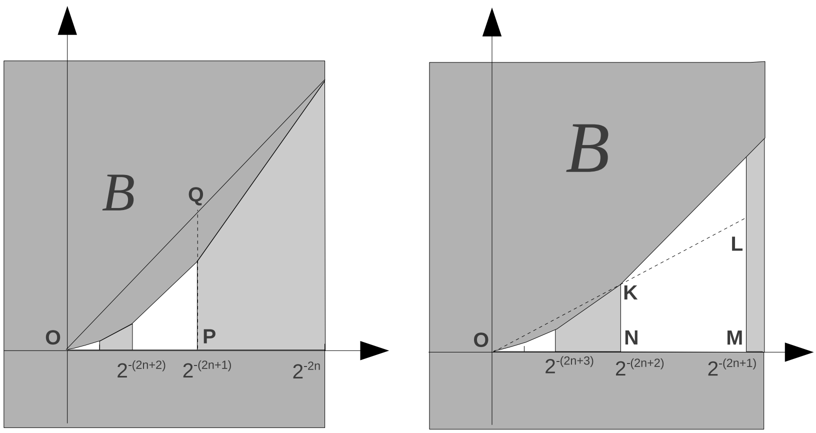
$$\frac{1}{\lambda}A = \Big[-\frac{1}{\lambda},\frac{1}{\lambda}\Big] \setminus \Big\{(x,y):\ x \in (0,1) \wedge y \in \Big(0,\frac{1}{\lambda}h(\lambda x)\Big)\Big\}.$$

Since h is convex and $h(0) = 0$ we have that $\frac{1}{\lambda}h(\lambda x) < h(x)$. Hence $\frac{1}{\lambda}A \cap [-t,t]^2 \supset A \cap [-t,t]^2$ and at last $f_A(\lambda t) \geq f_A(t)$.

By virtue of the last observation and Proposition 1 we have that the function f_A is of bounded variation on $[0,1]$. Hence $(0,0)$ is the regular density point of A.

Now we shall show that $(0,0)$ is NOT the regular density point of B. First we estimate from below the value of the function f_B in point 2^{-2n}. In order to do this we estimate the measure of the complement of B laying on the left of the point $2^{-(2n+1)}$ by the area of the triangle $\triangle OPQ$. (see the picture)

$$f_B(2^{-2n}) > \frac{(2\cdot 2^{-2n})^2 - \frac{1}{8}\cdot(2^{-2n})^2\cdot\frac{1}{2n+1}}{(2\cdot 2^{-2n})^2} = 1 - \frac{1}{32\cdot(2n+1)}.$$



Now we shall estimate from above the value of the function f_B in the point $2^{-(2n+1)}$. In order to do this we estimate the measure of the complement of B

lying on the left of the point $2^{-(2n+1)}$ by the area of the trapezoid $\triangle\!\!\!\nabla\!\!\triangle KLMN$. (again see the picture)

$$f_B(2^{-(2n+1)}) < \frac{(2\cdot 2^{-(2n+1)})^2 - \mu(\triangle\!\!\!\nabla\!\!\triangle KLMN)}{(2\cdot 2^{-(2n+1)})^2} = \frac{(2\cdot 2^{-(2n+1)})^2 - 3\mu(\triangle OKN)}{(2\cdot 2^{-(2n+1)})^2}$$
$$= \frac{(2\cdot 2^{-(2n+1)})^2 - \frac{3}{2}\cdot(2^{-(2n+2)})^2\cdot\frac{1}{2n+3}}{(2\cdot 2^{-(2n+1)})^2} = 1 - \frac{3}{32\cdot(2n+3)}.$$

Hence

$$f_B(2^{-(2n+2)}) - f_B(2^{-(2n+1)}) > \frac{1}{16\cdot(2n+3)}.$$

Since the last expression is a term of the divergent series, the total variation of the function f_B is unbounded.

Directly from the definition of an O'Malley point of the set A it follows that $\Phi_{OM}(A) \subset \Phi_{OM}(B)$ when $A \subset B$. Hence, by virtue of Theorem 1, $(0,0)$ is not an O'Malley point of A.

References

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