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On Minkowski decomposition of Okounkov bodies on a Del Pezzo surface

Abstract. We show that on a blow up of $\mathbb{P}^2$ in 3 general points there exists a finite set of nef divisors $P_1, \ldots, P_s$ such that the Okounkov body $\Delta(D)$ of an arbitrary effective $\mathbb{R}$–divisor D on X is the Minkowski sum

$$\Delta(D) = \sum_{i=1}^{s} a_i \Delta(P_i) \tag{1}$$

with non-negative coefficients $a_i \in \mathbb{R}_{\geqslant 0}$.

1. Introduction

Okounkov bodies form a new and rapidly developing research area in algebraic geometry. They are convex bodies associated to algebraic varieties in a very general setting, and may be viewed as a vast generalization of toric geometry. The idea is to associate to a big divisor D on a variety X a convex body $\Delta(D)$ (the Okounkov body of D) in such a way that questions about the original variety and D can be answered from the geometry of this polytope.

A systematic development of the theory has been initiated in [8] and [5] and we refer to these articles for details and motivations. Here we recall the basic construction.

Let X be an irreducible projective variety of dimension n and

$$Y_\bullet : \; X = Y_0 \supset Y_1 \supset \ldots \supset Y_{n-1} \supset Y_n = \{p\}$$

be a flag of irreducible subvarieties of X such that $\mathrm{codim}_X(Y_i) = i$ and p is a smooth point of each Y_i for $i = 0, \ldots, n$.

Let D be a Cartier divisor on X. The flag $Y_\bullet$ defines an order n valuation-type mapping

$$\nu_{Y_\bullet} : H^0(X, kD) \to \mathbb{Z}^n \cup \{\infty\}$$

in the following way. Given a section $0 \neq s \in H^0(X, kD)$ we set

$$\nu_1 = (\nu_{Y_\bullet})_1(s) := \mathrm{ord}_{Y_1}(s).$$

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This determines a section

$$\widetilde{s} \in H^0(X, kD - \nu_1 Y_1),$$

which does not vanish identically along Y_1, and thus restricts to a non-zero section

$$s_1 \in H^0(Y_1, (kD - \nu_1 Y_1)|_{Y_1}).$$

We repeat the above construction for s_1 and so on. In this way we produce a valuation vector

$$\nu_{Y_\bullet}(s) = ((\nu_{Y_\bullet})_1(s), \dots, (\nu_{Y_\bullet})_n(s)) \in \mathbb{Z}^n$$

and an element

$$(\nu_{Y_\bullet}(s), k) \in \Gamma_{Y_\bullet}(D) \subset \mathbb{Z}^{n+1}$$

in the *graded semigroup* of the linear series $|D|$. Let $S(D) \subset \mathbb{R}^n$ be the set of all normalized valuation vectors obtained as above, i.e.,

$$S(D) = \left\{ \frac{1}{k}\nu_{Y_\bullet}(s) : \; s \in H^0(X, kD), \, k = 1, 2, 3, \dots \right\}.$$

Definition 1.1 (Okounkov body)
The *Okounkov body* $\Delta_{Y_\bullet}(D)$ associated to the divisor D is the closed convex hull of the set $S(D)$.

Note that the shape of the Okounkov body depends on the flag $Y_\bullet$. However some invariants, for example its volume, are independent of $Y_\bullet$. This is in fact the main result of [8].

Computing the Okounkov body explicitly is in general not an easy task. We address this question here for Del Pezzo surfaces. First, we need to recall some properties of Okounkov bodies on arbitrary surfaces.

2. Okounkov bodies on surfaces

A remarkable fact about divisors on arbitrary smooth surfaces is the existence of the Zariski decomposition. This fact goes back to Zariski [11]. We refer to [1] for a modern proof.

Theorem 2.1 (Zariski decomposition)
Let D be an effective divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{Q}$–divisors P_D and N_D such that

$$D = P_D + N_D$$

and

(i) *P_D is nef,*

(ii) *N_D is zero or has negative definite intersection matrix,*

(iii) $P_D \cdot C = 0$ *for all irreducible components* C *of* N.

Assume that p *is the smallest positive integer such that* pN_D *is a divisor defined by the section* n_D *of the line bundle* $\mathcal{O}_X(pN_D)$. *Then, multiplication by the section* $n_D^{\frac{k}{p}}$ *induces an isomorphism*

$$H^0(X, kP_D) \simeq H^0(X, kD) \tag{2}$$

for all k *divisible by* p.

We take a flag $Y_\bullet : X = Y_0 \supset Y_1 \supset Y_2 = \{p\}$ with the curve Y_1 not contained in the augmented base locus $B_+(D)$ (which in particular implies that Y_1 is not a component of N_D). Then because of (2) we have for an arbitrary section $s \in H^0(X, kD)$ with k divisible enough

$$s = t \cdot n_D^{\frac{k}{p}}$$

for some section $t \in H^0(X, kP_D)$ and

$$\nu_1(s) = \nu_1(t) + \nu_1\big(n_D^{\frac{k}{p}}\big) = \nu_1(t) + \frac{k}{p}\nu_1(n_D) = \nu_1(t). \tag{3}$$

Similarly, we have

$$\nu_2(s) = \nu_2(t) + \nu_2\big(n_D^{\frac{k}{p}}\big) = \nu_2(t) + k \cdot \frac{1}{p}\nu_2(n_D). \tag{4}$$

It follows that the Okounkov body of D is up to translation by $\frac{1}{p}\nu_2(n_D)$ equal to that of P_D.

Corollary 2.2

Let D *be an effective divisor on a smooth algebraic surface* X *with Zariski decomposition* $D = P_D + N_D$ *and let* $Y_\bullet$ *be a flag as above. Then*

$$\Delta(D) = \Delta(P_D) + (0, \mathrm{ord}_p(N_D)).$$

In the view of the above corollary, it is sufficient to know what Okounkov bodies of nef effective divisors are. It turns out that on Del Pezzo surfaces there are only finitely many building blocks. This is made precise in the next section.

3. Del Pezzo surfaces

Let r be a fixed integer $0 \leqslant r \leqslant 8$. We fix r points $p_1, \ldots, p_r$ in the projective plane $\mathbb{P}^2$ in general position. More precisely we assume that

a) no three of these points are collinear,

b) no six of them are on the same conic,

c) a cubic curve passing through 6 of them and singular in the seventh point, is not passing through the eighth.

Let $f_r\colon X_r \to \mathbb{P}^2$ be the blowing up of $P_1, \ldots, P_r$ with exceptional divisors $E_1, \ldots, E_r$. Under the above assumptions, X_r is a smooth Del Pezzo surface, i.e., the anticanonical divisor $-K_{X_r}$ is ample, see [4]. We denote the class of the pullback by f_r of a line in $\mathbb{P}^2$ by H.

From now on, we fix also the following flag. Let Y_1 be a line in $\mathbb{P}^2$ not passing through any of the points $P_1, \ldots, P_r$ and let $p \in Y_1$ be a point not lying on the image under f_r in $\mathbb{P}^2$ of any of the (-1)–curves on X_r. This assumption, in view of (4) guarantees that

$$\Delta(D) = \Delta(P_D)$$

for an arbitrary big divisor D on X_r.

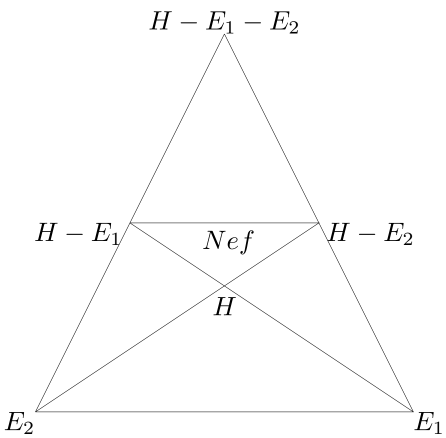
Del Pezzo surfaces are two-dimensional Fano varieties. It is well known from the Mori theory, see [3] and [10, Theorem 1.1.5], that the nef cone of a Fano variety is finitely generated.

For Del Pezzo surfaces it is easier to write down generators of the pseudo-effective cone than those of the nef cone. The effective cone is generated by classes of irreducible (-1)–curves on X_r for $r \geqslant 2$. For $r \leqslant 1$ one has to include also H in the set of generators.

One could naively expect, that in order to get the decomposition claimed in (1), one could take as the divisors P_i the generators of the nef cone. The following two simple examples show that this wouldn't work.

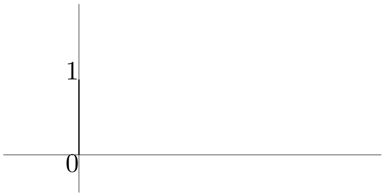
Example 3.1

Let X_2 be the blowup of $\mathbb{P}^2$ in two points. A slice of the effective cone of X_2 looks like in the following picture



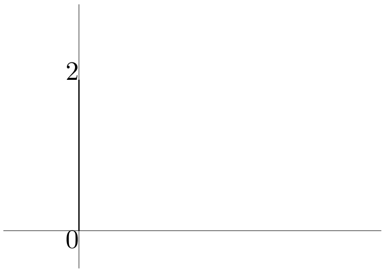
Picture 1. A slice of the effective cone of X_2.

We consider the generators $H - E_1$ and $H - E_2$ of the nef cone of X_2. The Okounkov bodies, constructed with respect to the flag, given in Section 2 coincide for both divisors. They are presented in the Picture 2.



Picture 2. Okounkov body of $H - E_i$ for $i = 1, 2$.

The Minkowski sum of two such segments is, again, a segment, presented in the next picture.

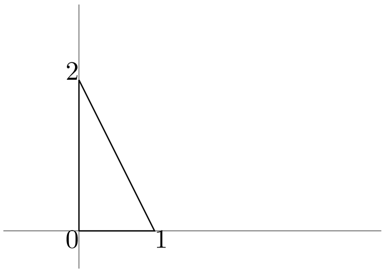


Picture 3. Minkowski sum of $\Delta(H - E_1)$ and $\Delta(H - E_2)$.

On the other hand

$$H - E_1 + H - E_2 = 2H - E_1 - E_2 = H + (H - E_1 - E_2)$$

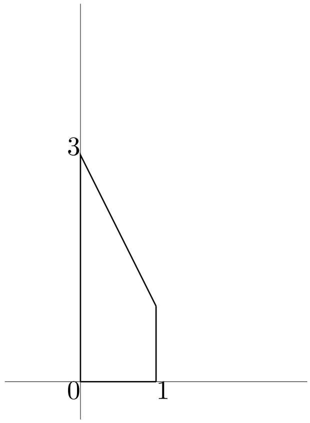
is a big (and nef) divisor, so its Okounkov body has in any case some positive volume. In fact, it is the triangle presented in the Picture 4.



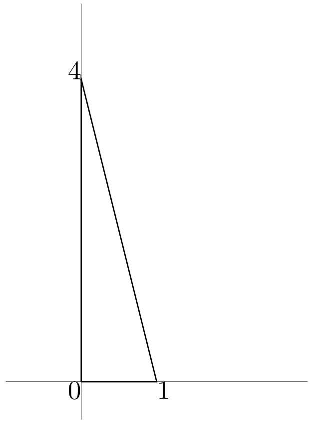
Picture 4. Okounkov body of $2H - E_1 - E_2$.

One might suspect that the reason for the bad behavior of the generators $H - E_1$ and $H - E_2$ is caused by them not being big. The next example shows that even for big and nef divisors the Okounkov bodies might not be additive.

Example 3.2
Now we look at X_3 with the flag fixed as explained in Section 2. We consider divisors $D_1 = 3H - 2E_1 - E_2$ and $4H - 2E_1 - 2E_2 - 2E_3$. They are both big and nef, with Okounkov bodies represented on Pictures 5 and 6, respectively.



Picture 5. Okounkov body of D_1.

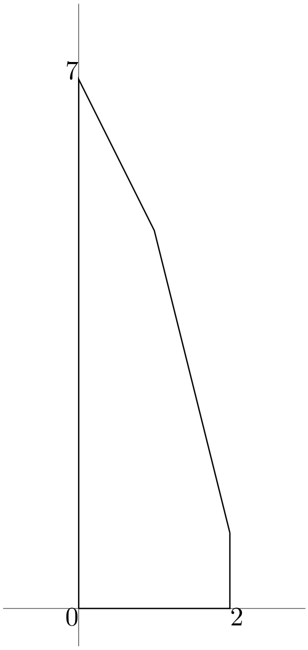


Picture 6. Okounkov body for D_2.

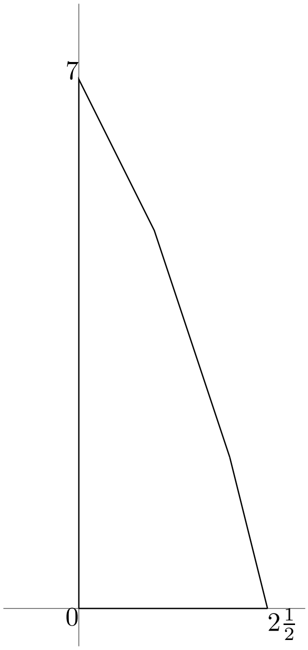
The Minkowski sum of $\Delta(D_1)$ and $\Delta(D_2)$ is presented in the Picture 7. On the other hand the Okounkov body of the sum

$$3H - 2E_1 - E_2 + 4H - 2E_1 - 2E_2 - 2E_3 = 7H - 4E_1 - 3E_2 - 2E_3$$

is presented in the Picture 8. The two figures do not agree.



Picture 7. Minkowski sum of $\Delta(D_1) + \Delta(D_2)$.



Picture 8. Okounkov body of $D_1 + D_2$.

These examples show that our main result stated in the next section is by no means obvious.

4. Minkowski decomposition

Theorem 4.1 (Minkowski decomposition on Del Pezzo surfaces)
Let X be a smooth Del Pezzo surface, i.e., $X = X_r$ for some r, or $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then there exists a finite set of nef divisors $P_1, \ldots, P_s$ such that for any big $\mathbb{R}$–divisor D we have

$$D = \sum_{i=0}^{s} a_i P_i + N_D \quad \textit{and} \quad \Delta(D) = \sum_{i=1}^{s} a_i \Delta(P_i)$$

with non-negative real numbers $a_i \in \mathbb{R}_{\geqslant 0}$.

Note that in the first equality there is the sum of divisors, whereas in the second equality the sum stands for the Minkowski sum of convex sets. We call the set $\{P_i\}$ the *Minkowski basis* of X_r, even though this is strictly speaking not a basis.

The complete proof of this theorem will appear in the forthcoming paper [9]. In this announcement we restrict our attention to the case $r = 3$ as this is already interesting enough.

Before we can proceed with the actual proof, we need to establish some notation. Following [2] we write

$$\text{Null}(D) = \{C \subset X : \ C \text{ irreducible curve with } C \cdot D = 0\}$$

for the set of all irreducible curves orthogonal to a given $\mathbb{R}$–divisor D with respect to the intersection form on X.

We write

$$\text{Null}^*(D) := \text{Null}(D) \setminus \{E_1, \ldots, E_r\}$$

for the set $\text{Null}(D)$ with $E_1, \ldots, E_r$ excluded.

The Neron-Severi group on X_r is generated by H and $E_1, \ldots, E_r$. We abbreviate

$$P(a; b_1, \ldots, b_r) := aH - b_1 E_1 - \ldots - b_r E_r.$$

5. Proof of Theorem 4.1 for $\mathbf{r = 3}$

We claim that as a set $\{P_i\}$ we can take the following divisors:

a) $P(1; 0, 0, 0)$,

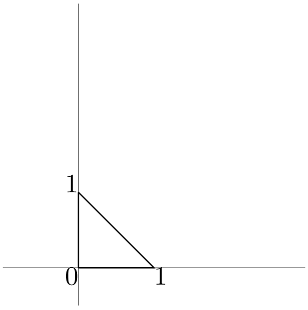
b) $P(1; 1, 0, 0)$, $P(1; 0, 1, 0)$, $P(1; 0, 0, 1)$,

c) $P(2; 1, 1, 0)$, $P(2; 1, 0, 1)$, $P(2; 0, 1, 1)$,

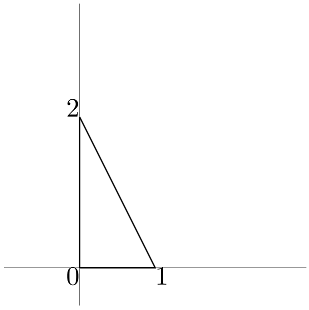
d) $P(2; 1, 1, 1)$,

e) $P(3; 2, 1, 1)$, $P(3; 1, 2, 1)$, $P(3; 1, 1, 2)$.

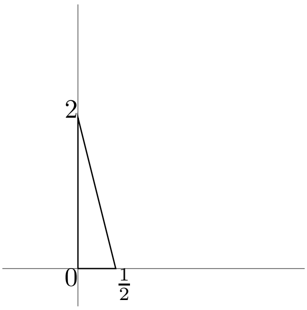
The divisors above are grouped in the obvious manner. The Okounkov bodies of divisors in each group are the same and they are depicted below. For the Okounkov bodies of the divisors of type b) see Picture 2 above.



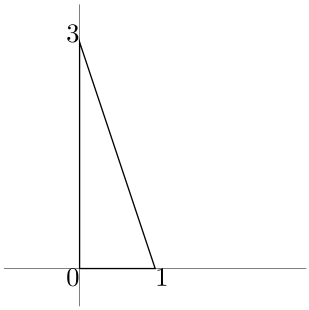
a) The Okounkov body of H



c) The Okounkov body of $2H - E_1 - E_2$



d) The Okounkov body of $2H - E_1 - E_2 - E_3$



e) The Okounkov body of $3H - 2E_1 - E_2 - E_3$

A nef divisor P can be written as a combination with non-negative coefficients of the divisors above (because the set contains the generators of the nef cone) but not in a unique way. It is in fact crucial for the Theorem to pick up the right decomposition. To this end we first list the space $\mathrm{Null}^*(\cdot)$ for each type of divisors in the Minkowski basis.

D	$\mathrm{Null}^*(D)$
H	$\emptyset$
$H - E_i$	$H - E_i,\ H - E_i - E_j,\ H - E_i - E_k$
$2H - E_i - E_j$	$H - E_i - E_j$
$2H - E_1 - E_2 - E_3$	$H - E_1 - E_2,\ H - E_1 - E_3,\ H - E_2 - E_3$
$3H - 2E_i - E_j - E_k$	$H - E_i - E_j,\ H - E_i - E_k$

The convention in this table is that i, j, k stay for mutually distinct indices.

In order to establish the theorem for arbitrary $\mathbb{Q}$–divisors, it is enough to work with the integral divisors, as the Okounkov bodies scale well. The claim for

$\mathbb{R}$–divisors follows then from the existence of the global Okounkov body, see [8, Theorem 4.5]. So we assume that P is an integral nef divisor on X_3. Next we compute the coefficients $\{a_i\}$ according to the following algorithm.

Let M be the divisor in the Minkowski basis given above with the property

$$\mathrm{Null}^*(M) = \mathrm{Null}^*(P).$$

Such an element exists and is unique. Indeed, it follows from the Index Theorem that $\mathrm{Null}^*(P)$ has a negative semi-definite intersection matrix. There are only finitely many such matrices possible on X_3 and each one of them appears in our list exactly once.

Then we set $P' := P - M$ and we claim that

$$\Delta(P) = \Delta(P') + \Delta(M). \tag{5}$$

Taking this for granted for a moment, we are finished with the proof of the Theorem, as we now apply our algorithm to P' and so on. This procedure terminates since we lower the absolute value of the coefficients of P in the basis $H, E_1, \ldots, E_r$ in every step.

The equality in (5) follows from observing that P and M lie on the same face (in the sense of convex geometry) of the nef cone of X_3. Moreover, subtracting M from P results in a divisor P' which either lies on the same face or on its boundary. Hence we can assume that

$$\mathrm{Null}^*(P) = \mathrm{Null}^*(M) = \{N_1, \ldots, N_s\}$$

and

$$\mathrm{Null}^*(P') = \{N_1, \ldots, N_s, N_{s+1}, \ldots, N_{s+t}\}.$$

Then

$$M = \mu_{Y_1}(M) \cdot Y_1 + \sum_{i=1}^{s} \alpha_i N_i \quad \text{and} \quad P' = \mu_{Y_1}(P') \cdot Y_1 + \sum_{j=1}^{s+t} \beta_j N_j,$$

where

$$\mu_{Y_1}(F) := \sup\{t \in \mathbb{R} : \ F - tY_1 \text{ is effective}\}.$$

Note that the exceptional divisors $E_1, \ldots, E_r$ do not appear in decompositions of nef divisors.

We claim that

$$\mu_{Y_1}(P' + M) = \mu_{Y_1}(P') + \mu_{Y_1}(M). \tag{6}$$

It is clear that we have the $\geqslant$ inequality in (6). Assume for the contrary that this inequality is sharp, i.e., $P = P' + M = \gamma \cdot Y_1 + R$ with some $\gamma > \mu_{Y_1}(P') + \mu_{Y_1}(M)$ and R, a pseudo-effective divisor. Comparing the two presentations of the sum $P' + M$ we have

$$(\gamma - \mu_{Y_1}(P') - \mu_{Y_1}(M)) \cdot Y_1 + R = \sum_{i=1}^{s} (\alpha_i + \beta_i) N_i + \sum_{i=s+1}^{s+t} \beta_i N_i.$$

In our case Y_1 is a big divisor, whereas the divisor on the right is not big and this contradiction shows (6).

Then we use the description of Okounkov bodies on surfaces from [8, Theorem 6.4]. For $P = P' + M$ we have

$$\Delta(P) = \{(x, y) \in \mathbb{R}^2 : \ 0 \leqslant x \leqslant \mu_{Y_1}(P) \text{ and } \alpha(x) \leqslant y \leqslant \beta(x)\}$$

with

$$\alpha(x) = \operatorname{ord}_p(N_{P-xY_1}) \quad \text{and} \quad \beta(x) = \operatorname{ord}_p(N_{P-xY_1}) + (Y_1 \cdot P_{P-xY_1}).$$

By our choice of the point p in the flag, $\alpha(x)$ is zero for all $0 \leqslant x \leqslant \mu_{Y_1}(D)$. The same is true for the α–functions for P' and M, so everything amounts to the computation of P_{P-xY_1}. We have

$$P_{P-xY_1} = \begin{cases} P' + P_{M-xY_1} & \text{for } 0 \leqslant x \leqslant \mu_{Y_1}(M), \\ P_{P'-(x-\mu_{Y_1}(M))Y_1} & \text{for } \mu_{Y_1}(M) \leqslant x \leqslant \mu_{Y_1}(P) = \mu_{Y_1}(P') + \mu_{Y_1}(M). \end{cases}$$

This shows the equality

$$\Delta(P' + M) = \Delta(P') + \Delta(M).$$

References

[1] Th. Bauer, *A simple proof for the existence of Zariski decompositions on surfaces*, J. Algebraic Geom. **18** (2009), 789–793.

[2] Th. Bauer, A. Küronya, T. Szemberg, *Zariski chambers, volumes, and stable base loci*, J. Reine Angew. Math. **576** (2004), 209–233.

[3] H. Clemens, J. Kollár, S. Mori, *Higher-dimensional complex geometry*, Astérisque **166** (1989), 144 pp.

[4] M. Demazure, *Surfaces de del Pezzo,I,II,III,IV,V. Singularites des surfaces*, Lecture Notes in Math. **777** (1980), 21–69.

[5] K. Kaveh, A. Khovanskii, *Convex bodies and algebraic equations on affine varieties*, preprint, arXiv:0804.4095.

[6] A. Küronya, V. Lozovanu, C. Maclean, *Convex bodies appearing as Okounkov bodies of divisors*, arXiv:1008.4431v1.

[7] R. Lazarsfeld, *Positivity in Algebraic Geometry I, II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **48, 49**, Springer–Verlag, Berlin, 2004.

[8] R. Lazarsfeld, M. Mustata, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835.

[9] P. Łuszcz-Świdecka, T. Szemberg, *Minkowski decomposition of Okounkov bodies on Del Pezzo surfaces*, in preparation.

[10] Y. Tschinkel, *Algebraic varieties with many rational points*, Arithmetic geometry, 243–334, Clay Math. Proc., **8**, Amer. Math. Soc., Providence, RI, 2009.

[11] O. Zariski, *The theorem of Riemann–Roch for high multiples of an effective divisor on an algebraic surface*, Ann. Math. **76** (1962), 560–616.

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