Summary

We analyse two works by Bernard Bolzano about the real numbers: *Rein analytischer Beweis* (Bolzano 1817) and *Reine Zahlenlehre* (Bolzano 1962/1976). In the first one, real numbers are treated axiomatically, while in the second one, they are constructed. In both works, the supremum (least upper bound) principle and Cauchy completeness are proven, although, as we demonstrate, the proofs in (Bolzano 1817) are incorrect.

In (Bolzano 1817), the axioms of an ordered field are implicitly used when formulas are manipulated in the proofs of the theorems. The axiom of continuity is explicitly formulated as the least upper bound principle and Cauchy completeness, conjoined with the Archimedean axiom. Based on this, we deduce that an axiomatic approach to real numbers is present in the paper. The proof of the supremum principle in (Bolzano 1817) utilizes the Cauchy completeness, which Bolzano attempts to derive from the axioms of an ordered field, as we demonstrate.

The work (Bolzano 1962/1976) is a theory of so-called measurable numbers, which includes infinitesimal numbers. In this paper, Bolzano also employs infinitely large numbers, defined as the inverses of infinitesimal numbers. We interpret this paper based on extending rational numbers using an ultrafilter. The set of hyperrational numbers $\mathbb{Q}^*$ is defined as $\mathbb{Q}^{\mathbb{N}}/_{\mathcal{U}}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. In the set $\mathbb{Q}^*$, we introduce operations and order in such a way that a non-Archimedean field is formed. Within the field of hyperrational numbers, we distinguish the ring of bounded numbers $\mathbb{L}_Q$ and the ideal of infinitesimals Ω_Q. Real numbers are defined as the quotient ring $\mathbb{L}_Q/_{\Omega_Q}$. We demonstrate that a measurable number can be interpreted as an element of the ring $\mathbb{L}_Q/_{\Omega_Q}$. We further show that, in the adopted interpretation, the proofs of the least upper bound principle and Cauchy completeness in (Bolzano 1962/1976) are correct.