# Geometric and homological properties of smooth curve arrangements in $\mathbb{P}^2_{\mathbb{C}}$
# Summary of the PhD thesis

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The main goal of my doctoral thesis is to examine the algebraic-combinatorial properties of smooth curve arrangements in the complex projective plane. This line of research has been present in world mathematics for nearly 40 years, namely since the formulation of Terao's conjecture (although some authors suggest that this conjecture should bear the name Terao-Saito conjecture), which predicts that for arrangements of hyperplanes contained in $\mathbb{P}^n_{\mathbb{C}}$, the algebraic property of the freeness is fully determined by its intersection lattice. This conjecture is considered one of the most important and most difficult open problems in combinatorial algebraic geometry. In the plane case, i.e., for $n = 2$, this conjecture has been verified (only!) for arrangements of $d \leqslant 14$ lines, which also shows the difficulty and complexity of the problem faced by researchers around the world. As it turned out later, Terao's conjecture is just one of many difficult research problems lying on the borderline of algebra and combinatorics, and we can recall a long list of similar problems that additionally touch topology, such as the problem of non-triviality of Alexander polynomials determined by arrangements of lines, the containment problems for symbolic powers of homogeneous ideals associated with point configurations, or the study of Zariski pairs problems. In this paper, we focus on arrangements of smooth plane curves degrees 1, 2, and 4. This setting, i.e., admitting curves of degree greater than one, is fundamental, mainly due to the complications that arise. To illustrate the situation, we will mention just a few of the difficulties, namely:

- ordinary singularities for curve arrangements, contrary to the case of line arrangements, need not be quasi-homogeneous,
- many techniques that work for line arrangements are no longer applicable in a general situation, e.g., when we admit smooth conics,
- there are very few known tools that can apply effectively for curve arrangements, e.g., in the context of limiting the weak combinatorics of such arrangements,
- the computational complexity of homological algebra problems increases significantly, e.g., when we want to determine the minimal free resolutions of certain quotient algebras.

In this doctoral dissertation, we will focus on two important aspects: namely on the creation of tools that allow to study curve properties from a combinatorial perspective, and the construction of curve arrangements with interesting algebraic-combinatorial properties. During our research, we noticed certain gaps in the combinatorial description of arrangement of bitangents associated with smooth plane quartics, which motivated us to fill these gaps. Now, let us briefly present the most significant research results obtained from our studies and the structure of this doctoral dissertation.
The first result of our research concerns smooth conics arrangements that admit only certain ADE singularities. We are interested in the mutual relationships between the number of singularities $A_1$, $A_3$, $D_4$, $A_5$, $A_7$ and the number of conics . Using the Bogomolov-Miyaoka-Yau inequality and Bézout's theorem, we determine the relationship between these quantities in the form of the following inequality.

Theorem **A** (see Twierdzenie 2.3). *Let $\mathcal{C}$ be an arrangement of $k \geqslant 3$ smooth conics admitting $n_2$ nodes, $t_3$ tacnodes, $n_3$ ordinary triple, $t_5$ points of type $A_5$ and $t_7$ points of type $A_7$. Then the following Hirzebruch-type inequality holds*

$$560k + 100n_2 + 75n_3 \geqslant 608t_7 + 404t_5 + 184t_3.$$

The obtained inequality allows us to determine upper bounds on the number of $A_5$ and $A_7$ singularities in conic arrangements that have only certain types of singularities. If we denote by $t_{2m+1}$ the number of singular points of type $A_{2m+1}$ for $m \geqslant 0$, these estimates, depending on the number of conics $k$ in a given arrangement, take the following form:

$$t_5 \leqslant \frac{25}{88}k^2 + \frac{45}{88}k,$$

$$t_7 \leqslant \frac{25}{126}k^2 + \frac{5}{14}k.$$

In the further part we focus on arrangements of smooth quartics and lines contained in the complex projective plane. We begin our considerations with a general result admitting us to bound the weak combinatorics for arrangements of quartics and lines, having only certain quasi-homogeneous singularities.

Theorem **B** (see Twierdzenie 3.1). *Let $\mathcal{QL} = \{\ell_1, ..., \ell_d, Q_1, ..., Q_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geqslant 1$ lines and $k \geqslant 1$ smooth quartics such that $4k + d \geqslant 6$. Assume that $\mathcal{QL}$ admits $n_2$ nodes, $t_3$ tacnodes, $t_5$ singularities of type $A_5$, $d_6$ singularities of type $D_6$, $t_7$ singularities of type $A_7$, $n_3$ ordinary triple and $n_4$ ordinary quadruple points. Then one has*

$$56k + n_2 + \frac{3}{4}n_3 \geqslant d + \frac{13}{8}d_6 + \frac{5}{2}t_3 + 5t_5 + \frac{29}{4}t_7.$$

In the further part of this chapter we recall some known facts regarding bitangent lines to smooth quartics, and then we prove several interesting facts describing arrangements of bitangents to Klein, Dyck and Komiya-Kuribayashi quartics. In the next part of this chapter, we present a complete description of the weak combinatorics of arrangements of 28 bitangents to very symmetric smooth quartics, also providing the respective equations of these lines, as well as the coordinates of the quadruple points, and the incidences this points and lines.

Next we will discuss the properties of freeness, nearly-freeness and plus-one generation for arrangements consisting of lines and smooth quartics. In the final part of the chapter, we will also discuss this for arrangements composed of elements of a certain pencil of quartics. At the beginning, we prove proposition, which assert that there does not exist a free or nearly-free arrangement composed of a smooth quartic and a single line. Moreover, if we add a second line to the arrangement and restrict ourselves to certain quasi-homogeneous singularities, such arrangement also cannot be free. In the following propositions, we present specific free, nearly-free and plus-one generated arrangements of the Dyck quartic and selected bitangent lines. Further analysis of arrangements of bitangent concerns arrangements associated with the Klein quartic. The peculiarity of arrangements consisting of Klein quartic and bitangents arises from the fact that every bitangent to Klein quartic has exactly two intersection points with it, which are $A_3$ singularties. The absence of $A_7$ singularities results in fewer possibilities for constructing arrangements with a relatively large total Tjurina number. For this reason, we provide only one type of plus-one generated arrangements of Klein quartic and four bitangent lines intersecting at quadruple point. However, these arrangements are interested compared to some of the results from Dimca and Sticlar papers, as they achieve the maximum exponent $d_3$ with respect to the degree of the arrangement for 3-syzygy curves. The final part of the fourth chapter is about the study of the pencil of curves

$$\mathcal{P}: \quad uK_1 + vK_2, \quad (u:v) \in \mathbb{P}^1_{\mathbb{C}},$$

where, respectively, $K_1$ is Komiya-Kuribayashi quartic and $K_2$ is reducible quartic composed of certain four hyperosculating lines. From the curves belonging to the pencil, we construct a free arrangement $C_3$ consisting of our base quartics $K_1, K_2$ and two specially chosen irreducible singular quartics from the pencil. Analyzing this arrangement, we can show that our pencil consists a non-reduced curve. We prove a result which summarizes these analyses, not only finding this non-reduced conic curve but also stating that it is the only one such non-reduced curve in this pencil. Lastly, inspired by a theorem proved by Dimca and Măcinic and Pokora, we examined arrangements resulting from free arrangements by removing one hyperosculating line, which is a component of the quartic $K_2$, demonstrating that the resulting arrangements are free.