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Katarzyna Domańska

An analytic description of the class of rational associative functions

Abstract. We deal with the following problem: which rational functions of two variables are associative? We provide a complete answer to that question.

1. Introduction

By an associative function on a nonempty set A usually we understand a binary operation $F\colon A \times A \to A$ satisfying for all $x, y, z \in A$ an equation

$$F(x, F(y, z)) = F(F(x, y), z). \tag{E1}$$

Rational functions, which are defined as elements of a field of fractions of polynomials in two variables have a form of a quotient of two polynomials in two variables. Since the natural domain of such a function is usually not a rectangular $A \times A$, the associativity is defined by a conditional form of (E1). We propose the following definition.

Definition
A rational function $F\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a given nonempty set is called *associative* if and only if it satisfies

$$F(x, F(y, z)) = F(F(x, y), z) \tag{E}$$

for all $(x, y, z) \in \mathbb{R}^3$ such that $(x, y), (y, z), (x, F(y, z)), (F(x, y), z) \in D$. An associative rational function is often called an *associative operation.*

In the literature associative rational functions appear among others while considering the functional equations of the form

$$f(x + y) = F(f(x), f(y))$$

or

$$f(F(x, y)) = f(x) + f(y),$$

where F is a given associative operation (see cf. [1], [3], [4], [5]). It is worth

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mentioning the following associative functions (see Aczél [1], Losonczi [5]), all of them we consider on their natural domains:

$$G_1(x,y)=\frac{x+y}{1-xy},\quad G_2(x,y)=\frac{x+y}{1+xy},\quad G_3(x,y)=\frac{1+xy}{x+y},$$
$$G_4(x,y)=\frac{xy-1}{x+y},\quad G_5(x,y)=\frac{x+y+2xy}{1-xy},\quad G_6(x,y)=\frac{x+y-1}{2(x+y)-2xy-1}.$$

An associative rational function were studied by Chéritat in [2]. Among others, he has proved the following

Theorem C1
If a rational function F is associative, then there exist $a,b,c,d,e,f,g,h\in\mathbb{R}$, $e^2+f^2+g^2+h^2>0$ such that

$$F(x,y)=\frac{axy+bx+cy+d}{exy+fx+gy+h}. \tag{$\star\star$}$$

The form ($\star\star$) is necessary but not sufficient for a rational function to be associative. One can see this considering the following functions (on natural domains):

$$K(x,y)=\frac{axy}{exy+h},\qquad a,e,h\in\mathbb{R}\setminus\{0\},$$
$$M(x,y)=\frac{axy+b(x+y)}{f(x+y)+h},\qquad a,b,f,h\in\mathbb{R}\setminus\{0\},$$
$$N(x,y)=\frac{axy+d}{exy+f(x+y)+h},\qquad a,d,e,f,h\in\mathbb{R}\setminus\{0\},$$
$$H(x,y)=\frac{xy+2(x+y)+2}{2xy+(x+y)+2}.$$

In [2] Chéritat gave also a representation theorem describing the class of associative rational functions.

Theorem C2
For every associative operation $F\colon D\to\mathbb{R}$ there exist a homografic function φ and a set $P\subset D$ such that

$$F(x,y)=\varphi(G(\varphi^{-1}(x),\varphi^{-1}(y))),\qquad (x,y)\in D\setminus P,$$

where $G\in\{\cdot,+,G_1\}$, $\cdot(x,y)=xy$, $+(x,y)=x+y$, $G_1(x,y)=\frac{x+y}{1-xy}$.

Each such an operation G is called a *multiplicative, additive and rational generator*, respectively.

Theorem C2 allows us to formulate the assertion of Theorem C1 as follows: *there exist constants $a,b,d,e,f,h\in\mathbb{R}$, $e^2+f^2+h^2>0$ such that*

$$F(x,y)=\frac{axy+b(x+y)+d}{exy+f(x+y)+h}. \tag{$\star$}$$

This result shows more precisely which rational functions are associative. But it is still only a necessary condition. The same examples show that it is not sufficient (the functions K, M, N, H are of the form $(\star)$).

In the present paper we give a sufficient condition for a rational function to be associative. Moreover, we describe the class of those functions in such a way that in each individual case the question whether a given function belongs to the considered class, is immediately and uniquely answered.

2. Main result

The sufficient condition spoken of is given by the following

Theorem 1
Let rational function $F\colon D \to \mathbb{R}$ be of the form

$$F(x,y) = \frac{axy + b(x+y) + d}{exy + f(x+y) + h}, \tag{$\star$}$$

with $D \subset \mathbb{R}^2$ being its natural domain, where $a, b, d, e, f, h \in \mathbb{R}$, $e^2 + f^2 + h^2 > 0$. Then F satisfies equation (E) *if and only if the following system of equalities is satisfied:*

$$2abef + a^3 f + a^2 eh + be^2 h = a^2 be + ade^2 + de^2 f + a^2 f^2 + b^2 e^2, \tag{I}$$
$$ab^2 e + bde^2 + def^2 = a^2 bf + abeh + befh, \tag{II}$$
$$\begin{split} b^3 e + bdef + df^3 + abde + d^2 e^2 + defh \\ = ab^2 f + abfh + bf^2 h + a^2 df + adeh + beh^2, \end{split} \tag{III}$$
$$2bdef + df^3 + b^3 e = adf^2 + b^2 eh + ab^2 f + bf^2 h, \tag{IV}$$
$$b^2 de + df^2 h + d^2 ef = abdf + adfh + bfh^2, \tag{V}$$
$$bd^2 e + d^2 eh + b^2 h^2 + d^2 f^2 + dfh^2 = 2bdfh + ad^2 f + adh^2 + bh^3. \tag{VI}$$

Proof. Assume that a rational function F given by $(\star)$ is associative. We obtain equations (I)–(VI) from (E) by a comparison of the coefficients of two polynomials in three variables. Obviously, if a function of the form $(\star)$ fulfills the above conditions, then it is associative. Thus the proof has been completed.

For a particular rational function it is much easier to verify the system (I)–(VI) then to check (E) directly. This allows to build a simple computer program in order to test the associativity of the functions of the form $(\star)$.

The system (I)–(VI) has infinitely many nonzero solutions. Among others, it is satisfied by (c, c, c, c, c, c), where $c \in \mathbb{R} \setminus \{0\}$. But such solutions lead to a constant rational function and they lead to trivial associative functions. In what follows, these solution will be excluded from our consideration.

Among all such nontrivial nonzero solutions there are also infinitely many solutions with all coefficients nonvanishing. For instance, that is the case where

$$\begin{split}(a,b,d,e,f,h) \in \{&(2,2,4,1,2,2), (2,-5,-10,2,4,-1), (12,18,27,-8,-12,-18),\\ &(3,-1,-1,1,1,-3), (3,1,-1,-1,1,3), (4,-1,-2,1,2,-5), (1,1,-1,-1,1,1)\}.\end{split}$$

After substitution to $(\star)$, these solutions give associative functions with six nonzero coefficients.

Other examples of such solutions of (I)–(VI) are members of the following set

$$\left\{\left(a, b, \frac{b^2}{a}, -\frac{a^2}{b}, -a, -b\right), (ad, bd, d^2, ab, ad, bd) : \; a, b, d \neq 0\right\}$$

It is worth emphasizing that the associative functions of the form $(\star)$ with all nonzero coefficients are hard to find in the literature.

Obviously, if (a, b, d, e, f, h) is a solution to the system (I)–(VI), then so is (ca, cb, cd, ce, cf, ch), where $c \in \mathbb{R} \setminus \{0\}$.

The above considered nonassociative functions K, M, N, H do not satisfy equalities (I), (II), (II), (I), respectively. For the function H each of the equalities (I)–(VI) fails to hold.

We will use the system (I)–(VI) to classify all operations which are associative.

In the class of functions given by $(\star)$ there are also such ones which have polynomials of degree zero in denominators (for $e = f = 0$, $h \neq 0$). These particular rational functions are polynomials of the form $W(x, y) = Axy + B(x + y) + C$, where $A = \frac{a}{h}$, $B = \frac{b}{h}$, $C = \frac{d}{h}$, $h \neq 0$. The following corollary gives a necessary and sufficient condition for associativity for them

Corollary

A polynomial

$$W(x, y) = Axy + B(x + y) + C, \tag{$\diamondsuit$}$$

$A, B, C \in \mathbb{R}$, *is an associative function if and only if*

$$AC = B(B - 1).$$

Proof. The polynomial $(\diamondsuit)$ is a function of the form $(\star)$, where $a = hA$, $b = hB$, $d = hC$, $e = f = 0$, $h \neq 0$. On account of Theorem 1, the polynomial $(\diamondsuit)$ satisfies (E) if and only if (a, b, d, e, f, h) satisfies the system (I)–(VI). As a matter of fact, equalities (I)–(V) become, in fact, identities, and (VI) takes the form

$$B^2h^4 = ACh^4 + Bh^4,$$

whence

$$AC = B(B - 1).$$

This ends the proof.

Let $\mathcal{F}$ denote the class of all nonpolynomial functions of the form $(\star)$ and let $\mathcal{F}_0$ be the subclass of $\mathcal{F}$ consisting of all members of $\mathcal{F}$ with all the coefficients occurring in $(\star)$ nonvanishing. Put $\breve{\mathcal{F}} := \mathcal{F} \setminus \mathcal{F}_0$.

In what follows, we shall determine the associative elements of the class $\breve{\mathcal{F}}$.

Theorem 2

The following functions (with natural domains in question) are the only associative members of the class $\breve{\mathcal{F}}$*:*

$$F(x,y) = \frac{x+y+\beta}{\alpha xy + \alpha\beta(x+y) + \alpha\beta^2 + 1}, \qquad \alpha \neq 0;$$
$$F(x,y) = \frac{(1+\alpha\beta)xy + \alpha(x+y) + \frac{\alpha}{\beta}}{\beta xy + x + y}, \qquad \beta \neq 0;$$
$$F(x,y) = \frac{\alpha xy}{xy + \beta(x+y) + \beta(\beta-\alpha)}, \qquad \alpha, \beta \neq 0;$$
$$F(x,y) = \frac{xy+\alpha}{x+y+\beta}, \qquad \alpha, \beta \in \mathbb{R};$$
$$F(x,y) = \frac{\alpha xy + x + y}{\beta xy + 1}, \qquad \beta \neq 0.$$

Proof. Let us split the class $\breve{\mathcal{F}}$ into the following 39 subclasses:

a) 6 classes

$$\mathcal{F}_i = \Big\{ F(x,y) = \frac{a_1 xy + a_2(x+y) + a_3}{a_4 xy + a_5(x+y) + a_6} : \ a_i = 0,\ a_n \neq 0,\ n \in \{1,\dots,6\} \setminus \{i\} \Big\},$$

for $i = 1, \dots, 6$;

b) 14 classes

$$\mathcal{F}_{i,j} = \Big\{ F(x,y) = \frac{a_1 xy + a_2(x+y) + a_3}{a_4 xy + a_5(x+y) + a_6} : \ a_i = a_j = 0,\ a_n \neq 0,\ n \in \{1,\dots,6\} \setminus \{i,j\} \Big\},$$

for $i,j = 1,\dots,6$, $i \neq j$, $\{i,j\} \neq \{4,5\}$;

c) 15 classes

$$\mathcal{F}_{i,j,k} = \Big\{ F(x,y) = \frac{a_1 xy + a_2(x+y) + a_3}{a_4 xy + a_5(x+y) + a_6} : \ a_i = a_j = a_k = 0,\ a_n \neq 0,\ n \in \{1,\dots,6\} \setminus \{i,j,k\} \Big\},$$

for $i,j,k = 1,\dots,6$ pairwise different, $\{i,j,k\} \notin \{\{4,5,6\},\{1,2,3\},\{1,4,5\}, \{2,4,5\},\{3,4,5\}\}$;

d) 4 one-parameter classes

$$\Big\{F(x,y) = c\frac{xy}{x+y}\Big\},\ \Big\{F(x,y) = c\frac{x+y}{xy}\Big\},\ \Big\{F(x,y) = \frac{c}{xy}\Big\},\ \Big\{F(x,y) = \frac{c}{x+y}\Big\}$$

with $c \neq 0$.

We shall show that the classes $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$ from a) contain no associative functions. Indeed, if $F \in \mathcal{F}_2$, i.e.

$$F(x,y) = \frac{axy + d}{exy + f(x+y) + h}, \qquad a,d,e,f,h \neq 0,$$

then putting $b = 0$ in (II) we get $def^2 = 0$, which is impossible in view of the fact that $d, e, f \neq 0$. This means that F fails to be associative.

Similarly, for $F \in \mathcal{F}_3$ we have $a, b, e, f, h \neq 0$, and by setting $d = 0$ in (V) we see that $bfh^2 = 0$. For $F \in \mathcal{F}_4$ is $a, b, d, f, h \neq 0$ and by (II), with $e = 0$, we obtain $a^2bf = 0$. In the case $F \in \mathcal{F}_5$ we have $a, b, d, e, h \neq 0$, and putting $f = 0$ in (V) we get $b^2de = 0$.

Let $F \in \mathcal{F}_1$, i.e.

$$F(x, y) = \frac{b(x + y) + d}{exy + f(x + y) + h}, \qquad b, d, e, f, h \neq 0.$$

Substituting $a = 0$ in the system (I)–(VI) we get by (I),

$$be^2h = de^2f + b^2e^2,$$

whence

$$bh = df + b^2$$

and by (II),

$$bde^2 + def^2 = befh.$$

Since

$$befh = bhef = (df + b^2)ef,$$

we have by (II),

$$bde^2 = b^2ef,$$

i.e.

$$de = bf.$$

Equalities (III)–(VI) jointly with $bh = df + b^2$ and $de = bf$ become identities. We show that (V) is satisfied. Indeed,

$$\begin{aligned} b^2de + df^2h + d^2ef = bfh^2 &\iff b^2de + df^2h + d^2ef = bhfh \\ &\iff b^2de + df^2h + d^2ef = (df + b^2)fh \\ &\iff b^2de + d^2ef = b^2fh \\ &\iff b^2bf + dbff = bhbf \\ &\iff (b^2 + df)bf = bhbf \\ &\iff b^2 + df = bh. \end{aligned}$$

Thus $F \in \mathcal{F}_1$ is associative if and only if $bh = df + b^2$, $de = bf$.

Since $b, d, e, f, h \neq 0$ and equalities $bh = df + b^2$ and $de = bf$ lead to

$$d = \frac{bf}{e}, \qquad h = \frac{df}{b} + b = \frac{f^2 + be}{e},$$

we have

$$F(x, y) = \frac{b(x + y) + \frac{bf}{e}}{exy + f(x + y) + \frac{be+f^2}{e}} = \frac{x + y + \frac{f}{e}}{\frac{e}{b}xy + \frac{f}{b}(x + y) + 1 + \frac{f^2}{be}},$$
$$b, e, f \neq 0,\ be \neq -f^2,$$

whence, by setting $\alpha := \frac{e}{b}$, $\beta := \frac{f}{e}$, we conclude that

$$F(x,y) = \frac{x+y+\beta}{\alpha xy + \alpha\beta(x+y) + 1 + \alpha\beta^2}, \qquad \alpha, \beta \neq 0,\ \alpha\beta^2 \neq -1.$$

Let further $F \in \mathcal{F}_6$, i.e.

$$F(x,y) = \frac{axy + b(x+y) + d}{exy + f(x+y)} \qquad \text{with } a, b, d, e, f \neq 0.$$

Putting $h = 0$ in (V) and in (VI) we obtain

$$b^2e + def = abf, \qquad be + f^2 = af,$$

respectively. Therefore

$$b^2e + def = abf = b^2e + bf^2,$$

whence

$$de = bf.$$

Using $de = bf$ and $be + f^2 = af$, one may easily show that the system (I)–(VI) is satisfied. This means that $F \in \mathcal{F}_6$ is associative if and only if $af = be + f^2$, $de = bf$. Since $a, b, d, e, f \neq 0$ and equalities $de = bf$, $be + f^2 = af$ lead to

$$d = \frac{bf}{e}, \qquad a = \frac{be}{f} + f,$$

we have

$$F(x,y) = \frac{(f + \frac{be}{f})xy + b(x+y) + \frac{bf}{e}}{exy + f(x+y)} = \frac{(1 + \frac{b}{f}\frac{e}{f})xy + \frac{b}{f}(x+y) + \frac{b}{e}}{\frac{e}{f}xy + x + y},$$
$$b, e, f \neq 0,\ be \neq -f^2$$

and putting $\alpha := \frac{b}{f}$, $\beta := \frac{e}{f}$ we obtain

$$F(x,y) = \frac{(1+\alpha\beta)xy + \alpha(x+y) + \frac{\alpha}{\beta}}{\beta xy + x + y}, \qquad \alpha, \beta \neq 0,\ \alpha\beta \neq -1.$$

Let $F \in \mathcal{F}_{1,2}$, i.e.

$$F(x,y) = \frac{d}{exy + f(x+y) + h}, \qquad d, e, f, h \neq 0.$$

After setting $a = b = 0$ in (I) we have $de^2f = 0$, which is impossible because of $d, e, f \neq 0$. Therefore F fails to be associative.

Similarly for $F \in \mathcal{F}_{1,3}$ we have $b, e, f, h \neq 0$, and putting $a = d = 0$, in (II) we get $befh = 0$, a contradiction, and for $F \in \mathcal{F}_{1,5}$, after setting $a = f = 0$ in (II) we obtain $bde^2 = 0$, which is impossible by means of $b, d, e \neq 0$.

Further, let $F \in \mathcal{F}_{1,4}$, i.e.

$$F(x,y) = \frac{b(x+y)+d}{f(x+y)+h}, \qquad b,d,f,h \neq 0.$$

Putting $a = e = 0$ in (III), we have

$$df = bh$$

and therefore

$$F(x,y) = \frac{b^2(x+y)+bd}{bf(x+y)+bh} = \frac{b^2(x+y)+bd}{bf(x+y)+df} = \frac{b}{f}\frac{b(x+y)+d}{b(x+y)+d},$$

which implies that F is a constant function. Thus in this class there are no associative functions.

Let $F \in \mathcal{F}_{1,6}$, i.e.

$$F(x,y) = \frac{b(x+y)+d}{exy+f(x+y)}, \qquad b,d,e,f \neq 0.$$

Putting $a = h = 0$ in (I) and in (II), we obtain

$$df = -b^2, \qquad be = -f^2,$$

respectively. Applying these equalities we show that equalities (III),…,(VI) are fullfiled. Therefore F is associative if and only if $be + f^2 = 0$, $df + b^2 = 0$. Since $b,d,e,f \neq 0$ and $df = -b^2$, $be = -f^2$, we conclude that

$$F(x,y) = \frac{b(x+y) - \frac{b^2}{f}}{-\frac{f^2}{b}xy + f(x+y)} = \frac{x+y-\frac{b}{f}}{-\frac{f^2}{b^2}xy + \frac{f}{b}(x+y)}, \qquad b,f \neq 0,$$

whence by setting $\alpha := -\frac{f^2}{b^2}$, $\beta := -\frac{b}{f}$, we obtain

$$F(x,y) = \frac{x+y-\beta}{\alpha xy + \alpha\beta(x+y) + \alpha\beta^2 + 1}, \qquad \alpha,\beta \neq 0.$$

Now, assume that $F \in \mathcal{F}_{2,3}$, i.e.

$$F(x,y) = \frac{axy}{exy+f(x+y)+h}, \qquad a,e,f,h \neq 0.$$

Putting $b = d = 0$ in (I) we get

$$af + eh = f^2.$$

On account of $b = d = 0$ equalities (II),…,(VI) hold true. This means that F is associative if and only if $af + eh = f^2$. Therefore, since $a,e,f,h \neq 0$, we infer that

$$F(x,y) = \frac{axy}{exy+f(x+y)+\frac{f^2-af}{e}} = \frac{\frac{a}{e}xy}{xy+\frac{f}{e}(x+y)+\frac{f^2}{e^2}-\frac{af}{e^2}},$$
$$a,e,f \neq 0,\ a \neq f,$$

which after putting $\alpha := \frac{a}{e}$, $\beta := \frac{f}{e}$ leads to

$$F(x,y) = \frac{\alpha xy}{xy + \beta(x+y) + \beta(\beta - \alpha)}, \qquad \alpha, \beta \neq 0, \alpha \neq \beta.$$

For $F \in \mathcal{F}_{2,4}$, i.e. for

$$F(x,y) = \frac{axy + d}{f(x+y) + h}, \qquad a, d, f, h \neq 0,$$

after setting $b = e = 0$ in (I), we conclude that

$$a = f.$$

Applying $b = e = 0$ and $a = f$ in (II), ..., (VI) we obtain identities. Therefore F is associative if and only if $a = f$. Consequently,

$$F(x,y) = \frac{axy + d}{a(x+y) + h} = \frac{xy + \frac{d}{a}}{x + y + \frac{h}{a}}, \qquad a, d, h \neq 0,$$

where, on setting $\alpha := \frac{d}{a}$, $\beta := \frac{h}{a}$, we get

$$F(x,y) = \frac{xy + \alpha}{x + y + \beta}, \qquad \alpha, \beta \neq 0$$

For $F \in \mathcal{F}_{2,5}$, i.e. for

$$F(x,y) = \frac{axy + d}{exy + h}, \qquad a, d, e, h \neq 0$$

substitution $b = f = 0$ in (I) leads to

$$ah = de,$$

whence

$$F(x,y) = \frac{a^2xy + ad}{aexy + ah} = \frac{a^2xy + ad}{aexy + de} = \frac{a}{e}\frac{axy + d}{axy + d}.$$

This implies that F is a constant function. Thus in this class there are no associative functions.

Similarly, we conclude that in $\mathcal{F}_{3,6}$ the only associative functions are the constant ones.

Let $F \in \mathcal{F}_{2,6}$, i.e.

$$F(x,y) = \frac{axy + d}{exy + f(x+y)}, \qquad a, d, e, f \neq 0.$$

After setting $b = h = 0$ in (II), we get $def^2 = 0$, which contradicts the fact that $d, e, f \neq 0$.

Similarly, for $F \in \mathcal{F}_{3,4}$, we have $a, b, f, h \neq 0$, and putting $d = e = 0$ in (V) we obtain $bfh^2 = 0$.

Let $F \in \mathcal{F}_{3,5}$, i.e.

$$F(x,y) = \frac{axy + b(x+y)}{exy + h}, \qquad a, b, e, h \neq 0.$$

Substituting $d = f = 0$ to (VI) we arrive at

$$b = h.$$

On account of $b = h$ and $d = f = 0$, equalities (I), ..., (V) hold true. This means that F is associative if and only if $b = h$. Thus

$$F(x,y) = \frac{axy + b(x+y)}{exy + b} = \frac{\frac{a}{b}xy + x + y}{\frac{e}{b}xy + 1}, \qquad a, b, e \neq 0,$$

and, similarly as above, setting $\alpha := \frac{a}{b}$, $\beta := \frac{e}{b}$ we infer that

$$F(x,y) = \frac{\alpha xy + x + y}{\beta xy + 1}, \qquad \alpha, \beta \neq 0.$$

For $F \in \mathcal{F}_{4,6}$ we have $a, b, d, f \neq 0$ and, simultaneously, $e = h = 0$. This implies that equality (V) fails to hold because of the fact that $e = h = 0$ would then imply that $abdf = 0$, a contradiction. Therefore F is not associative.

Similarly, for $F \in \mathcal{F}_{5,6}$, for $F \in \mathcal{F}_{1,2,4}$, for $F \in \mathcal{F}_{1,2,5}$, for $F \in \mathcal{F}_{1,2,6}$, for $F \in \mathcal{F}_{1,3,4}$, for $F \in \mathcal{F}_{1,3,6}$, for $F \in \mathcal{F}_{1,4,6}$, for $F \in \mathcal{F}_{1,5,6}$ and for $F \in \mathcal{F}_{2,3,5}$ we conclude that every from among functions considered is not associative.

Let now $F \in \mathcal{F}_{1,3,5}$, i.e.

$$F(x,y) = \frac{b(x+y)}{exy + h}, \qquad b, e, h \neq 0.$$

Setting $a = d = f = 0$ in (I) we see that

$$h = b.$$

With $a = d = f = 0$ and $h = b$ equalities (II), ..., (VI) hold true. So F is associative if and only if $h = b$. This means that

$$F(x,y) = \frac{b(x+y)}{exy + b} = \frac{x+y}{\frac{e}{b}xy + 1}, \qquad b, e \neq 0.$$

Putting $\beta := \frac{e}{b}$, $\alpha := 0$ we get

$$F(x,y) = \frac{\alpha xy + x + y}{\beta xy + 1}, \qquad \beta \neq 0.$$

Further, let $F \in \mathcal{F}_{2,3,4}$, i.e.

$$F(x,y) = \frac{axy}{f(x+y) + h}, \qquad a, f, h \neq 0.$$

From (I) with $b = d = e = 0$ we have

$$a = f.$$

By $b = d = e = 0$ and $a = f$ equalities (II), ..., (VI) hold true. This means that F is associative if and only if $a = f$. Therefore

$$F(x,y) = \frac{axy}{a(x+y)+h} = \frac{xy}{x+y+\frac{h}{a}}, \qquad a,h \neq 0.$$

Setting $\alpha := 0$, $\beta := \frac{h}{a}$, we get

$$F(x,y) = \frac{xy+\alpha}{x+y+\beta}, \qquad \beta \neq 0.$$

Assume that $F \in \mathcal{F}_{2,3,6}$, i.e.

$$F(x,y) = \frac{axy}{exy+f(x+y)}, \qquad a,e,f \neq 0.$$

By $b = d = h = 0$ equalities (II), ..., (VI) hold true and from (I) we have

$$a = f,$$

whence F is associative if and only if $a = f$ and we have

$$F(x,y) = \frac{axy}{exy+a(x+y)} = \frac{xy}{\frac{e}{a}xy+x+y}, \qquad a,e \neq 0,$$

whence by setting $\alpha := 0$, $\beta := \frac{e}{a}$, we arrive at

$$F(x,y) = \frac{(1+\alpha\beta)xy+\alpha(x+y)+\frac{\alpha}{\beta}}{\beta xy+x+y}, \qquad \beta \neq 0.$$

Further, let $F \in \mathcal{F}_{2,4,6}$, i.e.

$$F(x,y) = \frac{axy+d}{f(x+y)} \qquad a,d,f \neq 0.$$

With $b = e = h = 0$ equality (I) takes the form

$$a = f$$

whereas (II), ..., (VI) become identities. This means that F is associative if and only if $a = f$. Consequently,

$$F(x,y) = \frac{axy+d}{a(x+y)} = \frac{xy+\frac{d}{a}}{x+y}, \qquad a,d \neq 0,$$

i.e.

$$F(x,y) = \frac{xy+\alpha}{x+y+\beta}, \qquad \alpha \neq 0,$$

where $\alpha := \frac{d}{a}$, $\beta := 0$.

The function $F \in \mathcal{F}_{2,5,6}$ fails to be associative on account of $a,d,e \neq 0$ whereas by (I) (with $b = f = h = 0$) we have $ad^2e = 0$.

Similarly, functions $F \in \mathcal{F}_{3,4,6} \cup \mathcal{F}_{3,5,6}$ are nonassociative as well.

Let $F \in \mathcal{F}_{2,3,4,6}$, i.e.

$$F(x,y) = \frac{axy}{f(x+y)}, \qquad a, f \neq 0,$$

where $a = cf$, and $c \neq 0$. Putting $b = d = e = h$ in (I) we obtain

$$cf = f.$$

Therefore $c = 1$ and (II), ..., (VI) become identities. This means that F is associative if and only if $c = 1$.

One can easily show that

$$F(x,y) = \frac{xy + \alpha}{x + y + \beta}$$

with $\alpha = \beta = 0$.

For $F \in \mathcal{F}_{1,3,5,6}$ we have $b, e \neq 0$ and by (IV) (with $a = d = f = h = 0$) we obtain $b^3 e = 0$ implying that in $\mathcal{F}_{1,3,5,6}$ are nonassociative functions.

Similarly, every function $F \in \mathcal{F}_{1,2,5,6} \cup \mathcal{F}_{1,2,4,6}$ is nonassociative as well.

A simple calculation shows that each function F in the assertion of Theorem 2 is associative. By using system (I)-(VI) it is easier to check that for every F of the above form equation (E) holds true. Thus the proof has been completed.

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Institute of Matematics and Computer Science
Jan Długosz Academy of Częstochowa
Armii Krajowej 13/15
42-201 Częstochowa,
Poland
E-mail: k.domanska@ajd.czest.pl

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