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Iterations of homographic functions and recurrence equations involving a homographic function*

Abstract. The formulas for the m -th iterate ($m \in \mathbb{N}$) of an arbitrary homographic function H are determined and the necessary and sufficient conditions for a solution of the equation $y_{m+1} = H(y_m)$, $m \in \mathbb{N}$ to be an infinite n -periodic sequence are given. Based on the results from this paper one can easily determine some particular solutions of the Babbage functional equation.

1. Preliminaries

The recurrence equations involving a homographic function were studied in (Graham, Knuth, Patashnik, 2002). The authors stated that the only known examples of such equations possessing periodic solutions are

$$y_{m+1} = 2i \sin \pi r + \frac{1}{y_m}, \quad m \in \mathbb{N},$$

where r is a rational number such that $0 \leq r < \frac{1}{2}$.

Various approaches to the sequences given by the recurrence equation

$$y_{m+1} = H(y_m), \quad m \in \mathbb{N}, \quad (1)$$

where H is a homographic function may be found in (Koźniewska, 1972; Levy, Lessman, 1966; Uss, 1966; Wachniczy, 1966).

In this paper we prove formulas determining all solutions of (1). We also give the necessary and sufficient conditions for a solution of (1) to be periodic.

We also determine some particular solutions of the Babbage functional equation

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$$\varphi^m(x) = x, \quad x \in X, \quad (2)$$

where m is an arbitrary fixed integer. Recall that ψ^n for $n \in \mathbb{N}$ denotes the n -th iterate of a function $\psi: X \rightarrow X$, i.e. $\psi^0 = \text{Id}_X$ and $\psi^n = \psi \circ \psi^{n-1}$ for integer $n \geq 1$.

Some results concerning (2) may be found in (Kuczma, 1968). In particular the following

THEOREM 1 (KUCZMA, 1968, P. 291)

If φ is a meromorphic solution of equation (2), then

$$\varphi(x) = \frac{a'x + b'}{c'x + d'}$$

for some $a', b', c', d' \in \mathbb{C}$.

THEOREM 2 (KUCZMA, 1968, P. 291)

If $L(x) = \alpha x + \beta$, where $\alpha \neq 0$, then φ satisfies (2) if and only if $L^{-1} \circ \varphi \circ L$ does so.

THEOREM 3 (KUCZMA, 1968, P. 291)

Let $K_{-1} = 0$, $K_0 = 1$, $K_m = \gamma K_{m-1} + \delta K_{m-2}$ for $m \in \mathbb{N}_+$ and let $S(x) = \gamma + \frac{\delta}{x}$, where $\gamma, \delta \in \mathbb{C}$ and $\delta \neq 0$, then

$$S^m(x) = \frac{K_mx + \delta K_{m-1}}{K_{m-1}x + \delta K_{m-2}} \quad \text{for } m \in \mathbb{N}_+. \quad (3)$$

Let

$$H(x) := \frac{ax + b}{cx + d}, \quad \text{where } a, b, c, d \in \mathbb{C}, \quad c \neq 0, \quad ad - bc \neq 0 \quad (4)$$

In the sequel we assume that the domain of H is the set D defined as follows

$$D := \bigcap_{m \in \mathbb{N}_+} D_{H^m},$$

where D_{H^m} denotes the domain of H^m and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. We also set $H^0 := \text{Id}_D$.

Let $H: D \rightarrow \mathbb{C}$ be a function given by (4) and let

$$y_0 = x_0 \quad \text{and} \quad y_{m+1} = H(y_m) \quad \text{for an } x_0 \in D \text{ and } m \in \mathbb{N}.$$

Notice that

$$y_m = H^m(x_0) \quad m \in \mathbb{N} \quad (5)$$

and $x_0, H(x_0), H^2(x_0), \dots, H^{m-1}(x_0) \in D$.

Based on the theory of recurrence linear equations of order 2 with constant coefficients (Koźniewska, 1972, p. 59) we get

LEMMA 1

Let $K_{-1} = 0$, $K_0 = 1$, $K_m = \gamma K_{m-1} + \delta K_{m-2}$ for $m \in \mathbb{N}_+$, $\delta \neq 0$ and let $\Delta = \gamma^2 + 4\delta$. Then for $m \in \mathbb{N} \cup \{-1\}$,

$$1^o \ K_m = (m+1)\left(\frac{\gamma}{2}\right)^m, \text{ if } \Delta = 0,$$

$$2^o \ K_m = \frac{1}{\sqrt{\Delta}} \left(\left(\frac{\gamma+\sqrt{\Delta}}{2} \right)^{m+1} - \left(\frac{\gamma-\sqrt{\Delta}}{2} \right)^{m+1} \right), \text{ if } \Delta \neq 0,$$

where $\sqrt{\Delta}$ denotes one of the complex square roots of Δ .

A consequence of Lemma 1 is

LEMMA 2

If $K_{-1} = 0$, $K_0 = 1$, $K_m = \gamma K_{m-1} + \delta K_{m-2}$ for $m \in \mathbb{N}_+$, $\delta \neq 0$, $\gamma, \delta \in \mathbb{R}$ and $\Delta = \gamma^2 + 4\delta$, then for $m \in \mathbb{N} \cup \{-1\}$,

$$1^o \ K_m = (m+1)\left(\frac{\gamma}{2}\right)^m, \text{ if } \Delta = 0,$$

$$2^o \ K_m = \frac{1}{\sqrt{\Delta}} \left(\left(\frac{\gamma+\sqrt{\Delta}}{2} \right)^{m+1} - \left(\frac{\gamma-\sqrt{\Delta}}{2} \right)^{m+1} \right), \text{ if } \Delta > 0,$$

$$3^o \ K_m = (\sqrt{-\delta})^m \cos \frac{m\pi}{2}, \text{ if } \gamma = 0 \text{ and } \Delta < 0,$$

$$4^o \ K_m = (\sqrt{-\Delta})^m (\cos m\psi + \cot \psi \sin m\psi), \text{ if } \gamma \neq 0 \text{ and } \Delta < 0,$$

where ψ is the principal value of an argument of the complex number $\frac{\gamma}{2} + i\frac{\sqrt{-\Delta}}{2}$.

2. Periodic solutions of the recurrence equation

DEFINITION 1

An infinite sequence $(y_m)_{m \in \mathbb{N}}$ is called periodic with period n (or n -periodic), where $n \in \mathbb{N}$, $n \geq 1$, if $y_{m+n} = y_m$ for every $m \in \mathbb{N}$.

Consider equation (1) with the initial condition $y_0 = x_0$, where $H: D \rightarrow \mathbb{C}$ is a function defined by (4) and $x_0 \in D$. By (5) we get

LEMMA 3

Let $H: D \rightarrow \mathbb{C}$ be a function defined by (4) and let $n \geq 2$ be a fixed integer. Every solution of (1) is periodic with period n if and only if

$$H^n = \text{Id}_D. \quad (6)$$

Proof. Assume that for some integer $n \geq 2$ equation (6) holds, then by (5) for every $m \in \mathbb{N}$ we have

$$y_{m+n} = H^{m+n}(x_0) = H^m(H^n(x_0)) = H^m(x_0) = y_m,$$

where $y_0 = x_0 \in D$. For the converse suppose that every solution of (1) is n -periodic. Let $x_0 \in D$, so $H^m(x_0) \in D$ for every $m \in \mathbb{N}$. Put $y_m := H^m(x_0)$, $m \in \mathbb{N}$. The sequence $(y_m)_{m \in \mathbb{N}}$ satisfies (1), so it is n -periodic. Thus

$$H^n(x_0) = y_n = y_0 = H^0(x_0) = x_0.$$

Hence (6) holds.

Observe that Lemma 3 holds true if H is an arbitrary function with a proper domain satisfying (2).

THEOREM 4

Let $S: D' \rightarrow \mathbb{C}$ be a function defined as $S(x) = \gamma - \frac{\delta}{x}$, where $\gamma, \delta \in \mathbb{C}$, $\delta \neq 0$ and $D' := \bigcap_{m \in \mathbb{N}_+} D_{S^m}$, where D_{S^m} denotes the domain of S^m . Every sequence $(y_m)_{m \in \mathbb{N}}$ satisfying the following recurrence relation

$$y_{m+1} = S(y_m), \quad m \in \mathbb{N} \quad (7)$$

is 2-periodic if and only if $\gamma = 0$.

Proof. In view of Lemma 3 it follows that every sequence $(y_m)_{m \in \mathbb{N}}$ satisfying (7) is 2-periodic if and only if

$$S^2(x) = \gamma + \frac{\delta x}{\gamma x + \delta} = \text{Id}_{D'}(x), \quad x \in D'.$$

Which is equivalent to the fact that $\gamma = 0$.

Now we prove the following results.

THEOREM 5

Let S be as in Theorem 4, $\Delta = \gamma^2 + 4\delta$ and let $n \in \mathbb{N}$ be such that $n \geq 3$.

- (i) If every sequence $(y_m)_{m \in \mathbb{N}}$ satisfying (7) is n -periodic, then $\Delta \neq 0$ and $\delta = \frac{-\gamma^2}{4 \cos^2 \frac{k\pi}{n}}$ for some $k \in \{1, 2, 3, \dots, n-1\}$.
- (ii) If $k \in \{1, 2, 3, \dots, n-1\}$ and $\gamma^2 + 4\delta \neq 0$ and $4\delta \cos^2 \frac{k\pi}{n} = -\gamma^2$, then every sequence $(y_m)_{m \in \mathbb{N}}$ satisfying (7) is n -periodic.

Proof. To show (i) observe that by Lemma 3 we get

$$S^n(x) = x, \quad x \in D'. \quad (8)$$

By Theorem 3 and Lemma 1, (8) is equivalent to the following conditions

$$\begin{aligned} \frac{K_n x + \delta K_{n-1}}{K_{n-1} x + \delta K_{n-2}} - x \frac{K_{n-1} x + \delta K_{n-2}}{K_{n-1} x + \delta K_{n-2}} &= 0, \quad x \in D', \\ \frac{\gamma K_{n-1} x + \delta K_{n-2} x + \delta K_{n-1} - K_{n-1} x^2 - \delta K_{n-2} x}{K_{n-1} x + \delta K_{n-2}} &= 0, \quad x \in D', \\ \frac{K_{n-1}(-x^2 + \gamma x + \delta)}{K_{n-1} x + \delta K_{n-2}} &= 0, \quad x \in D', \\ K_{n-1} &= 0, \\ \Delta \neq 0 \quad \text{and} \quad \left(\frac{\gamma + \sqrt{\Delta}}{2} \right)^n &= \left(\frac{\gamma - \sqrt{\Delta}}{2} \right)^n, \\ \Delta \neq 0 \quad \text{and} \quad (\gamma + \sqrt{\Delta})^n &= (\gamma - \sqrt{\Delta})^n, \end{aligned}$$

$$\Delta \neq 0 \text{ and } \exists k \in \{1, \dots, n-1\} : \gamma + \sqrt{\Delta} = (\gamma - \sqrt{\Delta}) \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right). \quad (9)$$

Now notice that

$$\gamma + \sqrt{\Delta} = (\gamma - \sqrt{\Delta}) \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right)$$

is equivalent to the following conditions

$$\begin{aligned} \sqrt{\Delta} \left(1 + \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right) &= \gamma \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} - 1 \right), \\ 2\sqrt{\Delta} \cos \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n} \right) &= 2\gamma \sin \frac{k\pi}{n} \left(i \cos \frac{k\pi}{n} - \sin \frac{k\pi}{n} \right), \\ 2\sqrt{\Delta} \cos \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n} \right) &= 2\gamma i \sin \frac{k\pi}{n} \left(\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n} \right). \end{aligned} \quad (10)$$

Thus condition (9) is equivalent to

$$\begin{aligned} \Delta \neq 0 \text{ and } \exists k \in \{1, \dots, n-1\} \quad \sqrt{\Delta} \cos \frac{k\pi}{n} &= \gamma i \sin \frac{k\pi}{n}, \\ \Delta \neq 0 \text{ and } \exists k \in \{1, \dots, n-1\} \quad \sqrt{\Delta} &= \gamma i \tan \frac{k\pi}{n}, \\ \Delta \neq 0 \text{ and } \exists k \in \{1, \dots, n-1\} \quad \Delta &= -\gamma^2 \tan^2 \frac{k\pi}{n}, \\ \Delta \neq 0 \text{ and } \exists k \in \{1, \dots, n-1\} \quad \delta &= \frac{-\gamma^2}{4 \cos^2 \frac{k\pi}{n}}, \end{aligned}$$

which completes the proof of (i).

For the implication (ii) consider two cases:

- a. n is an even number and $k = \frac{n}{2}$,
- b. n is an even number and $k \neq \frac{n}{2}$ or n is odd and $k \in \{1, 2, 3, \dots, n-1\}$.

In the case a, we get $\gamma = 0$ and according to Theorem 4 every sequence satisfying (7) is 2-periodic and hence n -periodic.

For the case b, notice that for every $k \in \{1, 2, 3, \dots, n-1\}$, $\cos \frac{k\pi}{n} \neq 0$ we have $\Delta \neq 0$ and $\delta = \frac{-\gamma^2}{4 \cos^2 \frac{k\pi}{n}}$ which yields $\Delta = -\gamma^2 \tan^2 \frac{k\pi}{n}$. Denote by $\sqrt{\Delta}$ the number $\gamma i \tan \frac{k\pi}{n}$, thus $\sqrt{\Delta} \cos \frac{k\pi}{n} = \gamma i \sin \frac{k\pi}{n}$ which is equivalent to (10). Now reversing the reasoning from the case (i) – from condition (10) to (8) (without condition (9)) – finishes the proof.

The results obtained above will be now applied to examine the sequences defined by (1).

THEOREM 6

If $H: D \rightarrow \mathbb{C}$ is a function given by (4) and $(K_{-1}, K_0, K_1, \dots)$ is a sequence defined in Theorem 3 for which $\gamma = a + d$ and $\delta = bc - ad$, then

$$H^m(x) = \frac{1}{c} \frac{cK_mx + dK_m + \delta K_{m-1}}{cK_{m-1}x + dK_{m-1} + \delta K_{m-2}} - \frac{d}{c} \quad \text{for } m \in \mathbb{N}_+, L(x) \in D. \quad (11)$$

Proof. Let $\gamma = a + d$, $\delta = bc - ad$, $L(x) = \frac{x}{c} - \frac{d}{c}$ and $S(x) = (L^{-1} \circ H \circ L)(x)$, $L(x) \in D$. We have

$$\begin{aligned} L^{-1}(x) &= cx + d, \\ (H \circ L)(x) &= \frac{ax + bc - ad}{cx}, \\ S(x) &= (L^{-1} \circ H \circ L)(x) = a + d + \frac{bc - ad}{x} = \gamma + \frac{\delta}{x}, \end{aligned}$$

for $L(x) \in D$, where $\delta \neq 0$. By Theorem 3 we obtain

$$S^m(x) = \frac{K_mx + \delta K_{m-1}}{K_{m-1}x + \delta K_{m-2}} \quad \text{for } m \in \mathbb{N}_+, L(x) \in D.$$

Now observe that

$$S^m = L^{-1} \circ H^m \circ L \quad \text{for } m \in \mathbb{N}_+,$$

thus

$$H^m = L \circ S^m \circ L^{-1} \quad \text{for } m \in \mathbb{N}_+,$$

which gives (11).

Lemma 3 and Theorem 2 yield

THEOREM 7

Let $H: D \rightarrow \mathbb{C}$ be a function defined by (4), $L(x) = \frac{x}{c} - \frac{d}{c}$, $S(x) = (L^{-1} \circ H \circ L)(x)$, $L(x) \in D$ and let $n \geq 2$ be a fixed integer. Then every solution of (1) is n -periodic if and only if

$$S^n = \text{Id}_D.$$

3. Examples

From the proof of Theorem 5 it follows that condition

$$\gamma^2 + 4\delta \neq 0 \quad \text{and} \quad \exists k \in \{1, \dots, n-1\} \quad \sqrt{\gamma^2 + 4\delta} = -\gamma i \tan \frac{k\pi}{n} \quad (12)$$

is equivalent to the fact that every sequence $(y_m)_{m \in \mathbb{N}}$ satisfying equation

$$y_{m+1} = \gamma + \frac{\delta}{y_m}$$

or equation

$$y_{m+1} = \frac{ay_m + b}{cy_m + d},$$

where $a + d = \gamma$ and $bc - ad \neq 0$, is n -periodic with $n \geq 3$.

Moreover, it is easy to find numbers γ, δ satisfying (12) and a, b, c, d - solutions of the system $a + d = \gamma, bc - ad \neq 0$. Namely, for $n = 3, \gamma = 1, \delta = -1$ (12) is fulfilled and numbers $a = 2, b = -3, c = 1, d = -1$ satisfy the system $a + d = 1, bc - ad = -1$, thus in view of Theorem 5 and Lemma 3 the following functions

$$S(x) = 1 + \frac{-1}{x}, \quad H(x) = \frac{2x - 3}{x - 1}$$

fulfil the Babbage equation $\varphi^3(x) = x$ (which can be directly checked).

Now let $n = 4$, for $\gamma = 2, \delta = -2$ (12) holds true. Let $a = 3, b = -5, c = 1$ and $d = -1$, then $a + d = 2, bc - ad = -2$. Similarly as above we get that the mappings

$$S(x) = 2 + \frac{-2}{x}, \quad H(x) = \frac{3x - 5}{x - 1}$$

satisfy equation $\varphi^4(x) = x$.

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