

*Jan Górowski, Adam Łomnicki***Quasi-arithmetic means***

Abstract. We present a list of geometric problems with solutions that lead to known or less known means. We also prove, by elementary means, some property for so-called quasi-arithmetic means. We use the proved result to justify some inequalities between the means.

1. Introduction

Let $J \subset \mathbb{R}$ denote the open interval or respectively closed or half-closed. The sets $\mathbb{R}$, $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_+ \cup \{0\}$ will be also considered as intervals.

One of the most general definition of a mean is the following

DEFINITION 1

Every function $d: J \times J \rightarrow J$ satisfying

- (i) $\forall a, b \in J \quad \min\{a, b\} \leq d(a, b) \leq \max\{a, b\}$,
- (ii) d is a increasing function with respect to each variable

is called a mean.

In (Aczél, 1948) and (Kitagawa, 1934) it was proved that under some additional conditions on d there exists a strictly monotone function g defined on J such that

$$d(a, b) = g^{-1}(pg(a) + qg(b)), \quad a, b \in J,$$

for some $p, q \in (0, 1)$ such that $p+q = 1$. Such means will be called quasi-arithmetic means.

For the purposes of this paper we modify Definition 1.

*Średnie quasi-arytmetyczne

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DEFINITION 2

Let $T = \{(x, y) \in J \times J : x \geq y\}$. Every function $d: T \rightarrow J$ satisfying conditions (i) and (ii) of Definition 1 is said to be a mean.

In the sequel by a mean we understand a function in a sense of Definition 2.

There is a wide literature on means, some information may be found in (Aczél, 1948; Aczel, Dhombres, 1989; Galwani, 1927; Gładowska, Jarczyk, Matkowski, 2002; Górowski, Łomnicki, 2010; Kitagawa, 1934; Kołgomorov, 1930; Leach, Sholander, 1978; Leach, Sholander, 1983; Witkowski, 2009).

2. Geometric problems leading to means

Let us assume that a quadrangle $ABCD$ (see Fig. 1.) is a trapezium such that $AB \parallel CD$, $|AB| = a$, $|DC| = b$ and according to Fig. 1. $EF \parallel AB$, $|EF| = d$ and $DA' \parallel CB$. Denoting $\lambda = \frac{|AE|}{|ED|}$ we express d as a function of λ . By The Intercept

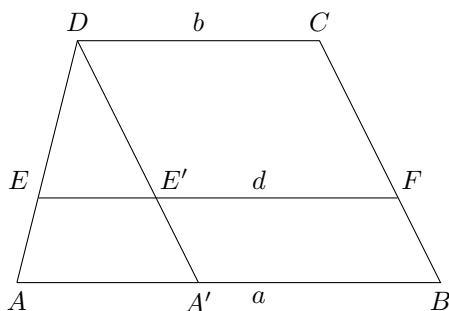


Fig. 1.

Theorem we get

$$\frac{|AA'|}{|EE'|} = \frac{|AD|}{|ED|} = \frac{|AE| + |ED|}{|ED|} = \lambda + 1.$$

Hence

$$|EE'| = \frac{|AA'|}{\lambda + 1} = \frac{a - b}{\lambda + 1} \quad \text{and} \quad d = \frac{a - b}{\lambda + 1} + b = \frac{a + \lambda b}{\lambda + 1}$$

and thus

$$d = \frac{a + \lambda b}{\lambda + 1}. \quad (1)$$

Now we formulate some geometric problems leading to means. Notice the well known problems P1-P4.

PROBLEM 1

Find the length d of the segment EF in the $ABCD$ (see Fig 1.) if

- P1.** E, F are the midpoints of the segments AD and BC , respectively;
- P2.** the diagonals AC and BD and the segment EF intersect at a point;

- P3.** the trapezes $ABFE$ and $EFCD$ are similar;
- P4.** the areas of the trapezes $ABFE$ and $EFCD$ are equal;
- P5.** the volumes of the solids of revolution obtained by rotating $ABFE$ and $EFCD$ around the line EF are equal;
- P6.** the volumes of the solids of revolution obtained by rotating $ABFE$ around the line AB and $EFCD$ around the line DC are equal;

It is easy to see that the solution of P1 is $d = \frac{a+b}{2}$.

Denote by S the intersection point of the diagonals AC and BD and the segment EF (problem P2). Then by Intersection Theorem we get

$$\lambda = \frac{|AE|}{|ED|} = \frac{|AS|}{|SC|} = \frac{|AB|}{|DC|} = \frac{a}{b}.$$

thus

$$d = \frac{2ab}{a+b}.$$

For the problem P3 notice that since the trapezes $ABFE$ and $EFCD$ are similar we obtain

$$\frac{|AE|}{|ED|} = \frac{|AB|}{|EF|} = \frac{|EF|}{|DC|},$$

hence $d^2 = ab$, and $d = \sqrt{ab}$.

To solve P4 denote by h_1, h_2 the altitudes of the trapezes $ABFE$ and $EFCD$, respectively. Let P denotes the area of the trapezium $ABFE$ (also trapezium $EFCD$). Then

$$\lambda = \frac{|AE|}{|ED|} = \frac{h_1}{h_2} = \frac{P}{a+d} \cdot \frac{d+b}{P} = \frac{d+b}{a+d}.$$

This and (1) give

$$d = \sqrt{\frac{a^2 + b^2}{2}}.$$

Now (problem P5) let h_1, h_2 be defined as above, then

$$\lambda^2 = \frac{h_1^2}{h_2^2} = \frac{\pi h_1^2}{\pi h_2^2}. \quad (2)$$

On the other hand, the volumes of the solids of revolution obtained by rotating $ABFE$ and $EFCD$ around the line EF are equal

$$\pi h_1^2 d + \frac{2}{3} \pi h_1^2 (a-d), \quad \pi h_2^2 b + \frac{1}{3} \pi h_2^2 (d-b), \quad (3)$$

respectively. From (2) and (3) we get

$$\lambda^2 = \frac{\pi h_1^2 (d + \frac{2}{3}(a-d))}{\pi h_2^2 (b + \frac{1}{3}(d-b))} \cdot \frac{b + \frac{1}{3}(d-b)}{d + \frac{2}{3}(a-d)} = \frac{b + \frac{1}{3}(d-b)}{d + \frac{2}{3}(a-d)} = \frac{d+2b}{d+2a},$$

which by (1) yields

$$d = \frac{2(a^2 + ab + b^2)}{3(a + b)} = \frac{\frac{a^3 - b^3}{3}}{\frac{a^2 - b^2}{2}}. \quad (4)$$

Finally, for the solution of P6 observe that

$$V_1 = \pi h_1^2 d + \frac{1}{3} \pi h_1^2 (a - d) \quad \text{and} \quad V_2 = \pi h_2^2 b + \frac{2}{3} \pi h_2^2 (d - b),$$

where h_1, h_2 denote the altitudes of the trapezes $ABFE$ and $EFCD$, resp., and V_1, V_2 are the volumes of the solids of revolution obtained by rotating $ABFE$ around the line AB and $EFCD$ around the line DC , resp. Similarly as above we get

$$\lambda^2 = \frac{b + 2d}{a + 2d},$$

thus

$$d = \sqrt{\frac{a^2 + ab + b^2}{3}} = \sqrt{\frac{a^3 - b^3}{3(a - b)}}. \quad (5)$$

Observe that (4) is one of the means introduced by Leach and Sholander in (Leach, Sholander, 1978), and (5) is a Stolarsky's mean from (Kolgomorov, 1930).

PROBLEM 2

Consider three pairwise homothetic squares with side length $a > d > b$, see Fig 2. Find d in terms of a and b .

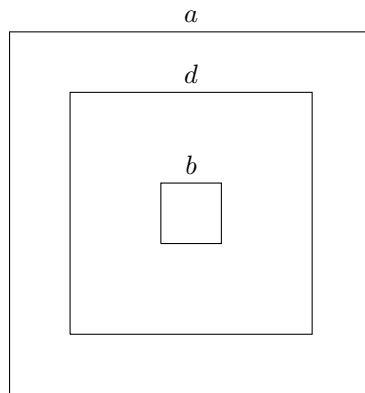


Fig. 2.

Let $\lambda = \frac{a^2 - d^2}{d^2 - b^2}$. Then

$$d = \sqrt{\frac{a^2 + \lambda b^2}{1 + \lambda}}.$$

This yields the following relationships:

$$\begin{aligned}
 d &= \sqrt{\frac{a^2 + b^2}{2}} && \text{for } \lambda = 1, \\
 d &= \sqrt{ab} && \text{for } \lambda = \frac{a}{b}, \\
 d &= \frac{a + b}{2} && \text{for } \lambda = \frac{3a + b}{a + 3b}, \\
 d &= \frac{2ab}{a + b} && \text{for } \lambda = \frac{a^2(a + 3b)}{b^2(3a + b)}, \\
 d &= \sqrt{\frac{a^3 + b^3}{a + b}} && \text{for } \lambda = \frac{b}{a}.
 \end{aligned}$$

3. Quasi-arithmetic means

Considerations from the previous section imply that, under some assumptions on a function $g: J \rightarrow \mathbb{R}$ and λ , it is worth to consider the function d_g^λ defined on $T = \{(x, y) \in J \times J : x \geq y\}$ and given by

$$d_g^\lambda(a, b) = g^{-1} \left(\frac{g(a) + \lambda g(b)}{1 + \lambda} \right), \quad (a, b) \in T. \quad (6)$$

we prove now the following result.

THEOREM 1

If $g: J \rightarrow \mathbb{R}$ is a strictly monotonic function, continuous on J and λ is a non-negative real number, then d_g^λ given by (6) is a mean (in a sense of Definition 2).

Proof. It is easy to see that d_g^λ is well defined. Indeed, if for some $(a, b) \in T$ and some $\lambda \in [0, +\infty)$ we had

$$\frac{g(a) + \lambda g(b)}{1 + \lambda} - g(x) \neq 0 \quad \text{for every } x \in J,$$

then since g is continuous,

$$\frac{g(a) + \lambda g(b)}{1 + \lambda} - g(x) > 0 \quad \text{for every } x \in J \quad (7)$$

or

$$\frac{g(a) + \lambda g(b)}{1 + \lambda} - g(x) < 0 \quad \text{for every } x \in J. \quad (8)$$

From (7) we obtain the following system of inequalities

$$\frac{g(a) + \lambda g(b)}{1 + \lambda} - \frac{(1 + \lambda)g(a)}{1 + \lambda} > 0 \quad \text{and} \quad \frac{g(a) + \lambda g(b)}{1 + \lambda} - \frac{(1 + \lambda)g(b)}{1 + \lambda} > 0,$$

which leads to a contradiction. The similar argument can be applied to (8).

The task is now to show that

$$\min\{a, b\} \leq d_g^\lambda(a, b) \leq \max\{a, b\}, \quad (a, b) \in T. \quad (9)$$

Observe that if g is a strictly increasing function, then so is g^{-1} and (9) is equivalent to

$$g(b) \leq \frac{g(a) + \lambda g(b)}{1 + \lambda} \leq g(a),$$

$$(1 + \lambda)g(b) \leq g(a) + \lambda g(b) \leq (1 + \lambda)g(a),$$

where the last inequality holds true. Similar argument applies to the case when g is strictly decreasing.

Finally, we prove that d_g^λ is an increasing function with respect to each variable. Fix $a \in J$ and suppose that g is strictly increasing. Let $b_1, b_2 \in J$ be such that $b_1 > b_2$ and $a \geq b_1$, then $g(b_1) > g(b_2)$, $\lambda g(b_1) \geq \lambda g(b_2)$, $g(a) + \lambda g(b_1) \geq g(a) + \lambda g(b_2)$ and in a consequence $d_g^\lambda(a, b_1) \geq d_g^\lambda(a, b_2)$. Now fix $b \in J$ and assume that $a_1 > a_2 \geq b$ for arbitrary $a_1, a_2 \in J$. We have $g(a_1) > g(a_2)$, $g(a_1) + \lambda g(b) > g(a_2) + \lambda g(b)$ and $d_g^\lambda(a_1, b) \geq d_g^\lambda(a_2, b)$.

For a strictly decreasing g the proof runs similarly.

DEFINITION 3

Let g satisfies the assumptions of Theorem 1. Every function defined by (6) will be called a mean generated by pair (g, λ) .

THEOREM 2

Let d_g^λ be a mean on a set T generated by pair (g, λ) . A function $\psi: [0, +\infty) \rightarrow \mathbb{R}^{T \setminus \{(a, a): a \in J\}}$ defined by

$$\psi(\lambda) = \bar{d}_g^\lambda,$$

where $\bar{d}_g^\lambda$ is a restriction of d_g^λ to the set $T \setminus \{(a, a) : a \in J\}$ is strictly decreasing.

Proof. Fix $a, b \in J$ such that $a > b$ and put

$$\phi(\lambda) := \frac{g(a) + \lambda g(b)}{1 + \lambda}, \quad \lambda \in [0, +\infty).$$

It follows that

$$\phi'(\lambda) := \frac{g(b) - g(a)}{(1 + \lambda)^2},$$

thus ϕ is strictly increasing (resp. strictly decreasing) if g is strictly decreasing (resp. strictly increasing). Hence for $\lambda_1 < \lambda_2$ we have $d_g^{\lambda_1}(a, b) > d_g^{\lambda_2}(a, b)$ and $\bar{d}_g^{\lambda_1} > \bar{d}_g^{\lambda_2}$, which completes the proof.

4. Means generated by the identity function

Suppose that $g = \text{Id}_{\mathbb{R}_+}$, where $\text{Id}_{\mathbb{R}_+}(x) = x$ for $x \in \mathbb{R}_+$, then (6) becomes

$$d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b) = \frac{a + \lambda b}{1 + \lambda}, \quad (a, b) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Some of the means of this kind appeared in Problem 1. (problems P1-P6).

Now using Theorem 2 we establish some inequalities between means generated by pair $(\text{Id}_{\mathbb{R}_+}, \lambda)$. Fix $a, b \in \mathbb{R}_+$ such that $a > b$, then

$$\frac{a}{b} > \sqrt{\frac{a}{b}} > 1 > \frac{\sqrt{\frac{a^2+b^2}{2}} + b}{\sqrt{\frac{a^2+b^2}{2}} + a},$$

which yields the following relation between the harmonic, geometric, arithmetic and quadratic mean of a and b ,

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}}.$$

Moreover, the means from problems P5 and P6 are greater than the arithmetic mean. Indeed, solving problem P5 we proved that for $a > b$,

$$\lambda^2 = \frac{2b + d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b)}{2a + d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b)}$$

which means that

$$\lambda < 1 \quad \text{and} \quad \frac{2a^3 - b^3}{3a^2 - b^2} > \frac{a+b}{2}.$$

By a similar argument, from equality obtained in the solution of problem P6,

$$\lambda^2 = \frac{b + 2d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b)}{a + 2d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b)}$$

it follows that for $a > b$,

$$\sqrt{\frac{a^3 - b^3}{3(a-b)}} > \frac{a+b}{2}.$$

To end this section let us remark that for arbitrary fixed $a, b \in \mathbb{R}_+$ such that $a > b$ we have

$$\left(\frac{a}{b}\right)^\mu > 1 \quad \text{for } \mu > 0 \quad \text{and} \quad \left(\frac{a}{b}\right)^\mu < 1 \quad \text{for } \mu < 0,$$

thus for $\lambda = \left(\frac{a}{b}\right)^\mu$,

$$d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b) = \frac{ab^\mu + ba^\mu}{a^\mu + b^\mu} < \frac{a+b}{2} \quad \text{for } \mu > 0$$

and

$$d_{\text{Id}_{\mathbb{R}_+}}^\lambda(a, b) = \frac{ab^\mu + ba^\mu}{a^\mu + b^\mu} > \frac{a+b}{2} \quad \text{for } \mu < 0.$$

5. Some other generated means

In this section we consider means generated by pair (g, λ) , where $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a power or a logarithmic function.

Let $g(x) = x^\nu$, $x \in \mathbb{R}_+$, $\nu \in \mathbb{R} \setminus \{0\}$. Then

$$d_g^\lambda(a, b) = \left(\frac{a^\nu + \lambda b^\nu}{1 + \lambda} \right)^{\frac{1}{\nu}}, \quad (a, b) \in \{(x, y) : x \in \mathbb{R}_+, x \geq y\}.$$

By Theorem 2 it follows that for $a > b$ and $\mu > 0$ we have $\left(\frac{a}{b}\right)^\mu > 1$ and

$$d_g^\lambda(a, b) < d_g^1(a, b),$$

hence for $\lambda = \left(\frac{a}{b}\right)^\mu$,

$$\left(\frac{a^\nu b^\mu + a^\mu b^\nu}{a^\mu + b^\mu} \right)^{\frac{1}{\nu}} < \left(\frac{a^\nu + b^\nu}{2} \right)^{\frac{1}{\nu}}.$$

Similarly, for $\mu < 0$ we get

$$\left(\frac{a^\nu b^\mu + a^\mu b^\nu}{a^\mu + b^\mu} \right)^{\frac{1}{\nu}} > \left(\frac{a^\nu + b^\nu}{2} \right)^{\frac{1}{\nu}}.$$

Now suppose that $g(x) = \ln x$, $x \in \mathbb{R}_+$. We have

$$d_g^\lambda(a, b) = \exp \frac{\ln a + \lambda \ln b}{1 + \lambda} = (ab^\lambda)^{\frac{1}{1+\lambda}}, \quad (a, b) \in \{(x, y) : x \in \mathbb{R}_+, x \geq y\}.$$

Setting again $\lambda = \left(\frac{a}{b}\right)^\mu$, $\mu \in \mathbb{R} \setminus \{0\}$ we get

$$d_g^\lambda(a, b) = a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}}$$

and from Theorem 2 the following inequalities

$$a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}} < \sqrt{ab} \quad \text{for } \mu > 0,$$

$$a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}} > \sqrt{ab} \quad \text{for } \mu < 0.$$

Notice that every strict inequality obtained by Theorem 2 for $(a, b) \in \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : x \geq y\}$ if replaced by a its corresponding non-strict inequality holds true for $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$.

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