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ON THE LAURICELLI PROBLEM FOR THE EQUATION $(\Delta - c^2)^2 u(xy) = 0$ IN THE HALF-PLANE

1. In the paper we shall give the solution of the equation

$$(\Delta - c^2)^2 u(x,y) = \Delta^2 u(x,y) - 2c^2 \Delta u(x,y) + c^4 u(x,y) = 0, \tag{1}$$

where c is a positive constant, in the half-plane $y>0$ satisfying the boundary data

$$\lim_{(x,y)\to(x_0,0)} u(x,y) = f_1(x_0), \tag{2a}$$

$$\lim_{(x,y)\to(x_0,0)} D_y u(x,y) = f_2(x_0), \tag{2b}$$

f_1, f_2 being given functions. We shall construct this solution using the convenient Green function.

2. Let $P(x,y)$, $Q(s,t)$, $P \neq Q$ be points of the half-plane $y > 0$. Let

$$r^2 = (x-s)^2 + (y-t)^2,$$

$$r_1^2 = (x-s)^2 + (y+t)^2,$$

$$R^2 = (x-s)^2 + y^2$$

and let $K_n(cr)$ denote the Mac Donald function with index n ([4], p.115). We shall prove

<u>Theorem 1</u>. The function

$$G(P,Q) = 2c^2 ytK_0(cr) - crK_1(cr) + cr_1K_1(cr_1) \tag{3}$$

is the Green function for the equation (1) in the half-plane $t>0$ with a pole P, with the boundary conditions

$$G(P,Q)\big|_{t=0} = 0, \qquad D_t G(P,Q)\big|_{t=0} = 0. \tag{4}$$

Proof. The function $cr_1K_1(cr_1) - crK_1(cr)$ is the solution of the equation (1) for $t > 0$ with respect to the point Q ([1]). Since $K_0(cr)$ satisfies the equation $\Delta u - c^2u = 0$ with respect to the point Q ([2], p.115) and

$$\Delta_Q(2c^2ytK_0(cr)) = 2c^2yt\,\Delta_Q K_0(cr) + 4c^2yD_tK_0(cr),$$

then

$$\Delta_Q(2c^2ytK_0(cr)) = 2c^4ytK_0(cr) + 4c^2yD_tK_0(cr) .$$

From the above formulae we get

$$(\Delta_Q - c^2)^2(2c^2\,ytK_0(cr)) = 0$$

for $t>0$. For $t = 0$, $r = r_1$ and $G(P,Q)\big|_{t=0} = 0$.

Applying the formulae ([4], p.117)

(5) $$\begin{cases} D_z(z^{-n}K_n(z)) = -z^{-n}K_{n+1}(z), \\ D_z(z^nK_n(z)) = -z^nK_{n-1}(z) \end{cases}$$

we obtain

$$D_tG(P,Q)\big|_{t=0} = \Big(2c^2yK_0(cr) + 2c^3ty(y-t)r^{-1}K_1(cr) - c^2K_0(cr)(y-t) + \\ - c^2K_0(cr_1)(y+t)\Big)\Big|_{t=0} = 0.$$

3. Let

(6) $$u(x,y) = u(P) = \frac{1}{4\pi c^2}\int_{-\infty}^{\infty}\Big(f_1(s)D_t\Delta_QG(P,Q) + f_2(s)\Delta_QG(P,Q)\Big)\Big|_{t=0}\,ds.$$

From (3) and (5) we have

$$\Delta_QG(P,Q) = 2c^4ytK_0(cr) + 4c^3y(y-t)r^{-1}K_1(cr) + 2c^2\big(K_0(cr) - K_0(cr_1)\big) + \\ - c^2\big(crK_1(cr) - cr_1K_1(cr_1)\big),$$

hence

$$\Delta_QG(P,Q)\big|_{t=0} = 4c^3y^2R^{-1}K_1(cR).$$

Similary

$$D_t \Delta_Q G(P,Q)\Big|_{t=0} = 4c^4 y^3 R^{-2} K_2(cR).$$

Hence we get

$$(7) \qquad u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(s) y^3 c^2 R^{-2} K_2(cR)\,ds + \frac{1}{\pi}\int_{-\infty}^{\infty} f_2(s)\, c y^2 R^{-1} K_1(cR)\,ds.$$

We shall using the following lemmas ([5]):

<u>Lemma 1</u>. If the function f is absolutely integrable in the interval $(-\infty, \infty)$, then the integral

$$I(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s) K_0(cR)\,ds$$

is of the class C^{∞} in the half-plane $y > 0$ and

$$D_{x^p y^q} I(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s) D_{x^p y^q} K_0(cR)\,ds ,$$

<u>Lemma 2.</u> If the function f is absolutely integrable in the interval $(-\infty, \infty)$ and continuous at the point x_0, then

$$\lim_{(x,y)\to(x_0,0)} \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\, c y R^{-1} K_1(cR)\,ds = f(x_0)\,.$$

Let $R_1^{\,2} = t^2 + y^2$. We shall prove

<u>Lemma 3.</u> For $y > 0$ and every x

$$(8) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} y^3 c^2 R^{-2} K_2(cR)\,ds = yce^{-cy} + e^{-cy},$$

$$(9) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} y^5 c^3 R^{-3} K_3(cR)\,ds = y^2c^2 + 3yce^{-cy} + 3e^{-cy},$$

$$(10) \qquad \int_{0}^{\infty} y^2 t R_1^{-2} K_1(cR_1)\,dt = yc^2 K_1(cy)\,,$$

$$(11) \qquad \int_{0}^{\infty} y^4 c^3 t R_1^{-3} K_3(cR_1)\,dt = c^3 y^2 K_2(cy)\,.$$

<u>Proof.</u> Introducing the transformation of the variable $s = y + t$, we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty} y^3c^2R^{-2}K_2(cR)\,ds = \frac{1}{\pi}\int_{-\infty}^{\infty} y^3c^2R_1{}^{-2}K_2(cR_1)\,dt =$$

$$= \frac{2}{t}\int_{0}^{\infty} y^3c^2R_1{}^{-2}K_2(cR_1)\,dt\,.$$

Since ([3] , p.719)

$$\int_0^{\infty} K_n\left(a\sqrt{x^2+z^2}\right) z^{2m+1}(x^2+z^2)^{-\frac{1}{2}}dz = \frac{2^m\,\Gamma(m+1)}{a^{m+1}x^{n-m-1}}\,K_{n-m-1}(ax)$$

for $n > 0$ and $m > -1$, then

$$\frac{2}{\pi}\int_0^{\infty} y^3c^2R_1{}^{-2}K_2(cR_1)\,dt = \frac{2}{\pi}y^2c^2\,\Gamma(\tfrac{1}{2})(2cy)^{-\frac{1}{2}}\,K_{\frac{3}{2}}(cy).$$

From [3] (p.975) we get

$$K_{\frac{3}{2}}(cy) = \sqrt{\frac{\pi}{2cy}}\,e^{-cy}\left(1 + \frac{1}{cy}\right),$$

hence

$$\frac{1}{\pi}\int_{-\infty}^{\infty} y^3c^2R^{-2}K_2(cR)\,ds = yce^{-cy} + e^{-cy}.$$

Similary we get the formulae (9)-(11).

<u>Lemma 4.</u> If the functions f_1, f_2 are absolutely integrable in the interval $(-\infty,\infty)$, then the function $u(x,y)$ defined by formula (6) resp. (7) is a solution of the equation (1) in the half-plane $y>0$.

<u>Proof.</u> From (5), we have

$$cy^2R^{-1}K_1(cR) = -\,yD_yK_o(cR),$$

$$c^2y^3R^{-2}K_2(cR) = -\,yD_{yy}K_o(cR) + D_yK_o(cR),$$

hence

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(s)\left(-yD_{yy}K_o(cR) + D_yK_o(cR)\right)ds - \frac{1}{\pi}\int_{-\infty}^{\infty} f_2(s)\,yD_yK_o(cR)\,ds.$$

From Lemma 1 follows that the function $u(x,y)$ given by the formula (6), resp.(7) is of the class C^{∞} in the half-plane $y>0$ and

$$(\Delta-c^2)^2u(x,y)=\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_1(s)(\Delta_P-c^2)^2D_t\Delta_Q G(P,Q)\Big|_{t=0}ds+$$

$$+\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_2(s)(\Delta_P-c^2)^2\Delta_Q G(P,Q)\Big|_{t=0}ds=$$

$$=\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_1(s)\,D_t\Delta_Q(\Delta_P-c^2)^2G(P,Q)\Big|_{t=0}ds+$$

$$+\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_1(s)\Delta_Q(\Delta_P-c^2)^2G(P,Q)\Big|_{t=0}ds=0,$$

because $(\Delta_P-c^2)^2G(P,Q)=0$ for $P\neq Q$.

Lemma 5. If the functions f_1, f_2 are absolutely integrable in the interval $(-\infty,\infty)$ and continuous at the point x_0, then the boundary condition (2a) is satisfied.

Proof. In view of lemma 2 we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f_2(s)cy^2R^{-1}K_1(cR)ds\to 0,\quad\text{when}\quad(x,y)\to(x_0,0).$$

We shall prove that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f_1(s)c^2y^3R^{-2}K_2(cR)ds\to f_1(x_0),\ \text{when}\ (x,y)\to(x_0,0).$$

Let

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f_1(s)c^2y^3R^{-2}K_1(cR)ds=A(x,y)+Q(x,y),$$

where

$$A(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f_1(x_0)c^2y^3R^{-2}K_2(cR)ds\ ,$$

$$Q(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}(f_1(s) - f_1(x_0))c^2y^3R^{-2}K_2(cR)ds.$$

From (8) follows that $A(x,y) \to f_1(x_0)$, when $(x,y)\to(x_0,0)$.

Let $\varepsilon > 0$ be an arbitrary positive number. From the continuity of the function f_1 at the point x_0 follows that there exists a number $\delta > 0$, such that

$$|f_1(s) - f_1(x_0)| < \frac{\varepsilon}{4}$$

for $|s - x_0| < \delta$. Let

$$Q_1(x,y) = \frac{1}{\pi} c^2y^3 \int_{|s-x_0|<\delta} [f_1(s) - f_1(x_0)] R^{-2}K_2(cR)ds ,$$

$$Q_2(x,y) = \frac{1}{\pi} c^2y^3 \int_{|s-x_0|\geq\delta} [f_1(s) - f_1(x_0)] R^{-2}K_2(cR)ds.$$

From (8) it follows that

$$|Q_1(x,y)| \leq \frac{\varepsilon}{4\pi} c^2y^3 \int_{-\infty}^{\infty} R^{-2}K_2(cR)ds \leq \frac{\varepsilon(yc+1)}{4} < \frac{\varepsilon}{2}$$

for $0 < y < \frac{1}{c}$. If $|x-x_0| < \frac{\delta}{2}$ and $|s-x_0| \geq \delta$, then $|s-x| > \frac{\delta}{2}$;

hence $$|Q_2(x,y)| \leq \frac{1}{\pi} c^2y^3 \int_{|s-x|\geq\frac{\delta}{2}} [f_1(s) - f_1(x_0)] R^{-2}K_2(cR)\, ds .$$

Using the formula ([6], p.174)

$$K_n(bz) = \frac{\Gamma(\frac{1}{2}+1)(2z)^n}{b^n\,\Gamma(\frac{1}{2})} \int_0^{\infty} (x^2 + z^2)^{-n-\frac{1}{2}} \cos(bx)\, dx$$

for $n > -\frac{1}{2}$, $b > 0$, $z > 0$, we get

(12) $$|K_n(z)| < k\, z^{-n}$$

for every n integer and positive, and $k>0$, constant.

Hence

$$|Q_2(x,y)| \leq \frac{1}{\pi} y^3k_1 \int_{|s-x|\geq\frac{\delta}{2}} |f_1(s)|R^{-4}ds + \frac{1}{\pi} y^3k_2|f_1(x_0)| \int_{|s-x|\geq\frac{\delta}{2}} R^{-4}ds.$$

Let

$$M = \max\left\{\int_{-\infty}^{\infty} |f_1(s)|\,ds,\ |f_1(x_0)|\right\},$$

thus

$$|Q_2(x,y)| \leqslant \frac{1}{\pi} Mk_3\Big(\sup_{|s-x|\geqslant \frac{\delta}{2}} y^3 R^{-4} + y^3 \int_{|s-x|\geqslant\frac{\delta}{2}} R^{-4}\,ds\Big) \leqslant$$

$$\leqslant \frac{1}{\pi} Mk_3\Big(y \sup_{|s-x|\geqslant\frac{\delta}{2}} R^{-2} + y \int_{|s-x|\geqslant\frac{\delta}{2}} R^{-2}\,ds\Big) < \frac{\varepsilon}{2}$$

for $0 < y < \delta_1$, where δ_1 is a convenient positive number. Hence $|Q(x,y)| < \varepsilon$ for $|x-x_0| < \frac{\delta}{2}$ and $0 < y < \min\left(\frac{1}{c},\ \delta_1\right)$.

<u>Lemma 6</u>. If the assumptions of lemma 3 are satisfied and the derivative f_1' is absolutely integrable in the interval $(-\infty,\infty)$ and continuous at the point x_0 then (2b) is satisfied.

<u>Proof</u>. From lemma 4 and the formula (5) we obtain

$$D_y u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(s)\Big(3y^2c^2R^{-2}K_2(cR) - y^4c^3R^{-3}K_3(cR)\Big)\,ds +$$

$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} f_2(s)\Big(2ycR^{-1}K_1(cR) - c^2y^3R^{-2}K_2(cR)\Big)\,ds.$$

In view of lemmas 2 and 3 we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f_2(s)\Big(2ycR^{-1}K_1(cR) - c^2y^3R^{-2}K_2(cR)\Big)\,ds \longrightarrow f_2(x_0),$$

when $(x,y)\to(x_0,0)$.

It follows from (8) and (9) that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f_1(s)\Big(3y^2c^2R^{-2}K_2(cR) - y^4c^3R^{-3}K_3(cR)\Big)\,ds =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\big(f_1(s) - f_1(x)\big)\Big(3y^2c^2R^{-2}K_2(cR) - y^4c^3R^{-3}K_3(cR)\Big)\,ds \;-\; 3yce^{-cy} f_1(x).$$

Since $3yce^{-cy} \longrightarrow 0$, when $(x,y) \longrightarrow (x_0,0)$, it is sufficient to prove that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\big(f_1(s)-f_1(x)\big)\big(3y^2c^2R^{-2}K_2(cR)-y^4c^3R^{-3}K_3(cR)\big)\,ds \longrightarrow 0,$$

when $(x,y) \longrightarrow (x_0,0)$. Let

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\big(f_1(s)-f_1(x)\big)\big(3y^2c^2R^{-2}K_2(cR)-y^4c^3R^{-3}K_3(cR)\big)\,ds =$$

$$= V(x,y)+W(x,y),$$

where

$$V(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\frac{f_1(s)-f_1(x)}{s-x}-f_1'(x_0)\Big)\big(3y^2c^2R^{-2}(s-x)K_2(cR)-y^4c^3R^{-3}(s-x)K_3(cR)\big)\,ds,$$

$$W(x,y)=\frac{1}{\pi}f_1'(x_0)\int_{-\infty}^{\infty}\big(3y^2c^2R^{-2}(s-x)K_2(cR)-y^4c^3R^{-3}(s-x)K_3(cR)\big)\,ds.$$

Introducing the transformation of the variable $s = y + t$, we get

$$W(x,y)=\frac{1}{\pi}f_1'(x_0)\int_{-\infty}^{\infty}\big(3y^2c^2R_1^{-2}tK_2(cR)-y^4c^3R_1^{-3}tK_3(cR_1)\big)\,ds = 0$$

because the function under the integral sign is odd and from (10), (11) follows that the integral

$$\int_{-\infty}^{\infty}\big(3y^2c^2R_1^{-2}tK_2(cR_1)-y^4c^3R_1^{-3}tK_3(cR_1)\big)\,dt$$

is convergent for every $y > 0$.

From the continuity of the function f_1' at the point x_0 follows that there exist a number $\delta > 0$ such that

$$\left|\frac{f_1(s)-f_1(x)}{s-x}-f_1(x_0)\right| < \frac{\varepsilon}{16c}$$

for $|s-x_0| < \delta$ and $|x-x_0| < \delta$.

Let $V(x,y) = V_1(x,y) + V_2(x,y)$, where

$$V_1(x,y) = \frac{1}{\pi}\int\limits_{|s-x_0|<\delta}\left[\frac{f_1(s)-f_1(x)}{s-x} - f_1'(x_0)\right]\left[3y^2c^2R^{-2}(s-x)K_2(cR) + \right.$$
$$\left. - y^4c^3R^{-3}(s-x)K_3(cR)\right]ds,$$

$$V_2(x,y) = \frac{1}{\pi}\int\limits_{|s-x_0|<\delta}\left[\frac{f_1(s)-f_1(x)}{s-x} - f_1'(x_0)\right]\left[3y^2c^2R^{-2}(s-x)K_2(cR) + \right.$$
$$- y^4c^3R^{-3}(s-x)K_3(cR)\, ds.$$

From (10) and (11), we obtain

$$|V_1(x,y)| < \frac{\varepsilon}{20c}\int\limits_{-\infty}^{\infty} 3y^2c^2R_1^{-2}|t|K_2(cR_1) + y^4c^3R_1^{-3}|t|K_3(cR_1)\,dt \leqslant$$

$$\leqslant \frac{\varepsilon}{10c}\left[\int\limits_0^{\infty} 3y^2c^2R_1^{-2}|t|K_2(cR)\,dt + \int\limits_0^{\infty} y^4c^3R_1^{-3}|t|K_3(cR_1)\,dt\right] \leqslant$$

$$\leqslant \frac{\varepsilon}{10c}\left(3yc^2K_1(cy) + c^3y^2K_2(cy)\right)$$

for $|x - x_0| < \delta$ and arbitrary $y > 0$.

From the asymptotic properties of the Mac Donald functions ([4], p. 146) follows that

$$\lim_{y\to 0^+} 3yc^2K_1(cy) = 3c,$$

$$\lim_{y\to 0^+} c^3y^2K_2(cy) = 2c,$$

hence $|V_1(x,y)| \leqslant \frac{\varepsilon}{2}$ for $|x-x_0| < \delta$ and $0 < y < \lambda_1$, where λ_1 is convenient positive number.

If $|x-x_0| < \frac{\delta}{2}$ and $|s-x_0| \geqslant \delta$ then $|s-x| > \frac{\delta}{2}$. From (12) follows that

$$|V_2(x,y)| \leqslant \frac{1}{\pi}\int\limits_{|s-x|\geqslant\frac{\delta}{2}}\left|\frac{f_1(s)-f_1(x)}{s-x}\right|\left|3y^2c^2R^{-2}(s-x)K_2(cR) - y^4c^3R^{-3}(s-x)K_3(cR)\right|ds +$$

$$+ \frac{1}{\pi}|f_1'(x_0)|\int\limits_{|s-x|\geqslant\frac{\delta}{2}}\left|3y^2c^2R^{-2}(s-x)K_2(cR) - y^4c^3R^{-3}(s-x)K_3(cR)\right|ds \leqslant$$

$$\leqslant \frac{1}{\pi}\sup_{|s-x|\geqslant\frac{\delta}{2}}\left[3y^2c^2R^{-2}|s-x|K_2(cR) + y^4c^3R^{-3}|s-x|K_3(cR)\right]\int\limits_{-\infty}^{\infty}|f_1'(s)|\,ds +$$

$$+ k_1|f_1'(x_0)|\int\limits_{|s-x|\geqslant\frac{\delta}{2}} yR^{-2}\,ds.$$

Similarly like in the proof of lemma 5 we get $|V_2(x,y)| < \frac{\varepsilon}{2}$ for $|x-x_0| < \frac{\delta}{2}$ and $0<y<\lambda_2$, where λ_2 is a convenient positive number. Therefore $|V(x,y)|<\varepsilon$ for $|x-x_0|<\frac{\delta}{2}$ and $0<y<\min(\lambda_1, \lambda_2)$ which finishes the proof of lemma 6.

Lemmas 4, 5 and 6 imply

Theorem 2. If the functions f_1, f_2, f_1' are absolutely integrable in the interval $(-\infty,\infty)$ and continuous at the point x_0, then the solution of Lauricelli problem in half-plane $y>0$ for equation (1) with the boundary conditions (2) is the function defined by (7).

B i b l i o g r a p h y

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