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ON CERTAIN BOUNDARY-VALUE PROBLEM FOR THE EQUATION $(\Delta - C^2)^2 u(x,y,z) = 0$

1. In this paper we shall solve the boundary value problem for the equation

$$(1) \qquad (\Delta - C^2)^2 u(x,y,z) = \Delta^2 u(X) - 2C^2 \Delta u(X) + C^4 u(X) = 0,$$

$C > 0$, C-constant in the half-space E_3^+ $(z > 0)$ with boundary conditions

$$(2) \qquad \lim_{(x,y,z)\to(x_0,y_0,0+)} u(x,y,z) = f_1(x_0,y_0),$$

$$(2a) \qquad \lim_{(x,y,z)\to(x_0,y_0,0+)} \Delta u(x,y,z) = f_2(x_0,y_0),$$

f_1, f_2 being given functions. This problem will briefly be called (R-H) problem. In the paper [1] the analogous problem for the half-plane is solved.

2. To the construction of the solution we shall use the convenient Green function. Let $X(x,y,z)$, $Y(s,t,w)$, $X \subset E_3^+$ denote the arbitrary points is three dimentional space E_3. Let

$$(3) \qquad r^2 = (s-x)^2 + (t-y)^2 + (w-z)^2$$

$$(4) \qquad U(X,Y) = U(r) = e^{-Cr}$$

It is easy to verify that the function $U X,Y$ is a solution ot the equation (1). The function U is called the fundamental solution of the equation (1). Let D denote a bounded domain in the space E and S its boundary. We assume that S is of class C^1. Let $u(x,y,z)$ and $v(x,y,z)$ be the functions of class C^4 in D and of class C^3 in its closure $D \cup S$. Let $K(r,X)$ denote the sphere with radius r and center at a point X. Let n be the inward normal. Let us consider the fundamental formulae for the functions u, v and domain D, [2]:

$$\iiint_D (u\Delta v - v\Delta u)\,dxdydz = -\iint_S (uD_n v - vD_n u)\,dS,$$

$$\iiint_D (\Delta u\,\Delta v - v\Delta^2 u)\,dxdydz = -\iint_S \left[(\Delta u)D_n v - vD_n(\Delta u)\right] dS,$$

$$\iiint_D (u\Delta^2 v - \Delta u\Delta v)\,dxdydz = -\iint_S \left[uD_n(\Delta v) - (\Delta v)D_n u\right] dS.$$

From the above formulae follows the formula

(5) $$\iiint_D \left[u(\Delta - C^2)^2 v - v(\Delta - C^2)^2 u\right] dxdydz =$$

$$= \iint_S \left[vD_n(\Delta u) - 2C^2 vD_n u + 2C^2 uD_n v - (\Delta u)D_n v + \right.$$

$$\left. + (\Delta v)D_n u - uD_n(\Delta v)\right] dS.$$

On the surface $\partial K(r,X)$ of the sphere $K(r,X)$

(6) $$D_n u(r) = Ce^{-Cr},$$

(7) $$\Delta u(r) = C^2 e^{-Cr} - 2r^{-1}Ce^{-Cr},$$

(8) $$D_n\Delta u(r) = -C^3 e^{-Cr} + 2r^{-2}Ce^{-Cr} + 2r^{-1}C^2 e^{-Cr}.$$

We shall prove

<u>Lemma 1</u>. If $u(X)$ is a solution of the equation (1) of class C^4 in D and of class C^3 in $\bar{D}$, $v(Y) = U(X,Y)$, then

(9) $$\iint_S \left[UD_n\Delta u - 2C^2 UD_n u + 2C^2 uD_n U - (\Delta u)D_n U + (\Delta_Y U)D_n u - uD_n\Delta_Y U\right] dS$$

$$= \begin{cases} \dfrac{1}{8\pi C}\, u(X) & \text{for } X \in D, \\ 0 & \text{for } X \in C(\bar{D}). \end{cases}$$

Proof. In view of (5)

(10) $$0 = \iint_{S\cup\partial K(r,X)} \left[U(X,Y)D_n\Delta_Y u(Y) - 2C^2 U(X,Y)D_n u(Y) + 2C^2 u(Y)D_n U(X,Y) - \right.$$

$$\left. -(\Delta_Y u(Y))D_n U(X,Y) + (\Delta_Y U(X,Y))\,D_n u(Y) - u(Y)D_n\Delta_Y U(X,Y)\right] dS.$$

From (10) follows that

$$\iint_S \Big[U(X,Y)D_n\Delta u(Y) - 2C^2U(X,Y)D_nu(Y) + 2C^2u(Y)D_nU(X,Y) - (\Delta u(Y))\,D_nU(X,Y) + (\Delta_Y U(X,Y))\,D_nu(Y) - u(Y)D_n(\Delta_Y U(X,Y))\Big]dS = -\iint_{\partial K(r,X)}\Big[UD_nu + 2C^2uD_nU - 2C^2UD_nu - (\Delta u)D_nU + (\Delta_Y U)D_nu - uD_n(\Delta_Y U)\Big]dS.$$

On the right hand on the above formula we get the integrals

$$J_1 = C_1\iint_{\partial K(r,X)} e^{-Cr}\, D_n\Delta u(Y)dS\,, \qquad J_4 = C_4\iint_{\partial K(r,X)} e^{-Cr}\Delta u(Y)dS\,,$$

$$J_2 = C_2\iint_{\partial K(r,X)} e^{-Cr}\, D_nu(Y)dS\,, \qquad J_5 = C_5\iint_{\partial K(r,X)} e^{-Cr}r^{-1}D_nu(Y)dS\,,$$

$$J_3 = C_3\iint_{\partial K(r,X)} e^{-Cr}u(Y)dS\,, \qquad J_6 = 2C\iint_{\partial K(r,X)} e^{-Cr}r^{-2}u(Y)dS\,.$$

C_i, $i=1,2,3,4,5$, being the convenient constants. Applying the mean value theorem to the integrals J_i, $i=1,2,3,4,5$, for $r\to 0$ we get

$$J_i \longrightarrow 0 \qquad \text{for} \quad i=1,2,3,4,5,$$

and

$$J_6 = 2C4\pi r^2 e^{-Cr}r^{-2}u(Q) \longrightarrow 8\pi Cu(X), \quad Q\in\partial K(r,X),\ X\in D.$$

If $X\in C(\bar{D})$, then by the formula 5 we get the second part of thesis of lemma 1.

3. Let

$$r_1^2 = (s-x)^2 + (t-y)^2 + (z+w)^2,$$

$$R^2 = (s-x)^2 + (t-y)^2 + z^2,$$

$$G(X,Y) = U(r) - U(r_1).$$

We shall prove

Theorem 1. $G(X,Y)$ is the Green function for the equation (1), for the half-space E_3^+ with a pole at a point X, satysfying the boundary conditions

(11) $$G(X,Y)\Big|_{w=0} = 0, \qquad \Delta_Y G(X,Y)\Big|_{w=0} = 0, \quad X \neq Y.$$

Proof. The function $G(X,Y)$ satisfied the equation (1) with respect to the point Y. For $w=0$ $r = r_1 = R$ and in view of (4) and (7) we get the conditions (11). From (7) follows that the point X is a pole for the function $\Delta_Y G(X,Y)$ if $X = Y$. Applying the formula (9) to the function $U(r)$ and $U(r_1)$ at the point $X \in E_3^+$ we get

(12) $$u(X) = \frac{1}{8\pi C} \iint_{-\infty}^{+\infty} \left[f_1(s,t)\left(2C^2 D_w G(X,Y) - D_w(\Delta_Y G(X,Y))\right)\right]\Big|_{w=0} dsdt -$$

$$- \frac{1}{8\pi C} \iint_{-\infty}^{+\infty} f_2(s,t)\, D_w G(X,Y)\Big|_{w=0} dsdt,$$

where

$$D_w G(X,Y) = 2CzR^{-1}e^{-Cr},$$

$$D_w(\Delta_Y G(X,Y))\Big|_{w=0} = 2C^3R^{-1}z\, e^{-CR} - 4zR^{-3}Ce^{-R} - 4zR^{-3}C^2e^{-CR}.$$

4. We shall prove that under the same assumptions concerning $f_1(s,t)$ and $f_2(s,t)$ the function $u(X)$ defined by the formula

(12a) $$u(X) = I_1(X) + I_2(X),$$

where

$$I_1(X) = a_1 \iint_{-\infty}^{+\infty} f_1(s,t)\left(2C^3\, zR^{-1}e^{-CR} + 4(C^2 + C)\, zR^{-3}e^{-CR}\right) dsdt,$$

$$I_2(X) = - a_1 \iint_{-\infty}^{+\infty} f_2(s,t)\, 2CzR^{-1}e^{-CR}\, dsdt,$$

$$a_1 = \left(32\pi(C^2 + C)\right)^{-1}$$

is a solution of the problem (B-H).

We shall prove that the function $u(X)$ defined by the formula (12a) is of class C^4 and satisfies the equation (1) in E_3^+.

Let

$$(13)\qquad a_1\iint_{-\infty}^{+\infty} f_1(s,t)\,D_x{}^i{}_y{}^j{}_z{}^k\left(2C^3\, zR^{-1}\, zR^{-1}e^{-CR} + 4\left(C^2+C\right)\, zR^{-3}e^{-CR}\right)dsdt,$$

$$(14)\qquad a_1\iint_{-\infty}^{+\infty} f_2(s,t)\,D_x{}^i{}_y{}^i{}_z{}^k\left(2CzR^{-1}e^{-CR}\right)dsdt$$

$$(i,j,k = 0,1,2,3,4).$$

Let W denote the set:

$$W = \left\{(x,y,z) : a \leqslant x \leqslant A,\ b \leqslant y \leqslant B,\ h < z < H\right\},$$

where a,A,b,B are arbitrary numbers, h,H arbitrary positive numbers

Lemma 2. If the functions f_1, f_2 are bounded, continuous and absolutely integrable, then the integrals (13) and (14) are uniformly convergent in every set W.

Proof. The integrals

$$C_{ijk}^{(0)}\iint_{-\infty}^{+\infty}|f_1(s,t)|\,dsdt + C_{ijk}^{(1)}\iint_{-\infty}^{+\infty}|f_2(s,t)|\,dsdt + C_{ijk}^{(2)}\iint_{-\infty}^{+\infty}R^{-3}\,dsdt +$$

$$+ C_{ijk}^{(3)}\iint_{-\infty}^{+\infty} zR^{-3}\,dsdt,$$

$C_{ijk}^{(l)}\left(i,j,k = 0,1,2,3,4,\ \ l = 0,1,2,3\right)$ being non negative constants, are majorants for integrals (13) and (14). The two first integrals are convergent by assumption. For the two others, after the change of variables

$$(14a)\qquad s - x = z\varrho\cos\varphi, \qquad t - y = z\varrho\sin\varphi,$$

we get the estimation

$$\iint_{-\infty}^{+\infty} R^{-3}\,dsdt \leqslant k_1h^{-4}\int_0^\infty \frac{\varrho\,d\varrho}{(1+\varrho^2)^3},$$

$$\iint_{-\infty}^{+\infty} zR^{-3}\,dsdt \leqslant k_2h^{-3}\int_0^\infty \frac{\varrho\,d\varrho}{(1+\varrho^2)^3},$$

where k_1, k_2 are convenient constants. Hence the integrals (13), (14) are uniformly convergent in every set W.
Lemma 2 imply

Lemma 3. If the assumptions of 2 are satisfied, then there exist the derivatives $D_{x^i y^j z^k} I_1(X)$, $D_{x^i y^j z^k} I_2(X)$ and

$$(15)\quad D_{x^i y^j z^k} I_1(X)= a_1\iint_{-\infty}^{+\infty} f_1(s,t)D_{x^i y^j z^k}(2C^3 zR^{-1}e^{-CR}+(C^2+C)zR^{-3}e^{-CR})dsdt,$$

$$(16)\quad D_{x^i y^j z^k} I_2(X)= - a_1\iint_{-\infty}^{+\infty} f_2(s,t)D_{x^i y^j z^k}(2Cze^{-CR})\,dsdt.$$

From lemma 3 follows that the function $u(X)$ defined by the formula (12a) is of class C^4 in E_3^+.
Moreover using the symmetricity of the Green function we get

$$(17)\qquad (\Delta_Y - C^2)^2\, G(X,Y) \equiv 0.$$

From (17) it follows that

$$(\Delta_X - C^2)^2u(X)= a_1\iint_{-\infty}^{+\infty}f_1(s,t)(\Delta_X - C^2)^2(2C^3\, zR^{-1}e^{-CR}+(C^2+C)zR^{-3}e^{-CR})dsdt-$$

$$- a_1\iint_{-\infty}^{+\infty}f_2(s,t)(\Delta_X-C^2)^2(R^{-1}2Cze^{-CR})dsdt = a_1\iint_{-\infty}^{+\infty}f_1(s,t)\Big[2C^2D_w(\Delta_X-C^2)^2G(X,Y)-$$

$$-D_w(\Delta_Y(\Delta_X-C^2)^2)G(X,Y)\Big]_{w=0}dsdt- a_1\iint_{-\infty}^{\infty}f_2(s,t)\,D_w\Big[(\Delta_X-C^2)^2G(X,Y)\Big]\Big|_{w=0}dsdt.$$

5. Now we shall prove that the boundary conditions (2) and (2a) are satisfied.
Let $M = \sup(|f_1|, |f_2|)$.

Lemma 4. If the function $f_2(s,t)$ is bounded and continuous, then

$$\iint_{-\infty}^{\infty} f_2(s,t)\, zR^{-1}e^{-CR}\, dsdt \longrightarrow 0 \qquad \text{as}\,(x,y,z)\longrightarrow(x_0,y_0,0+).$$

Proof. In view of the estimation

$$\Big|\iint_{-\infty}^{+\infty} f_2(s,t)zR^{-1}e^{-CR}dsdt\Big| \leqslant M\iint_{-\infty}^{+\infty}ze^{-CR}R^{-1}dsdt \leqslant M\iint_{-\infty}^{+\infty}zR^{-2,2}dsdt$$

after the change of variables (14a) we get

$$\iint_{-\infty}^{+\infty} zR^{-2,2}\, dsdt = z^{0,9}\, 2\pi \int_0^{\infty} \frac{d\varrho}{(1+\varrho^2)^{1,1}} \to 0 \quad \text{as } (x,y,z)\to(x_0,y_0,0+).$$

Lemma 5. If the function $f_1(s,t)$ is absolutely integrable, bounded, continuous and satisfies the condition

$$(18) \quad z\iint_{-\infty}^{+\infty} |f_1(s,t)|\, R^{-3}(1-e^{-CR})\, dsdt \to 0 \quad \text{as } (x,y,z)\to(x_0,y_0,0+),$$

then

$$a_1 \iint_{-\infty}^{+\infty} f_1(s,t)\, 4(C^2+C)\, zR^{-3}e^{-CR}\, dsdt \to f_1(x_0,y_0) \quad \text{as } (x,y,z)\to(x_0,y_0,0+).$$

Proof. From the identity

$$a_1 \iint_{-\infty}^{+\infty} f_1(s,t)\, 4(C^2+C)\, zR^{-3}e^{-CR}\, dsdt = \frac{1}{8\pi}\iint_{-\infty}^{+\infty} f_1(s,t)\, zR^{-3}\, dsdt +$$

$$+\frac{z}{8\pi}\iint_{-\infty}^{+\infty} f_1(s,t)\, R^{-3}(1-e^{-CR})\, dsdt,$$

in view of (18) and [3] we get the thesis of lemma 5.
The boundary condition (2) follows from lemmas 4 and 5.

Lemma 6. If the functions f_1, f_2 satisfies assumptions of lemmas 3,4,5, then

$$\Delta u(X) \to f_2(x_0,y_0) \quad \text{as } (x,y,z)\to(x_0,y_0,0+).$$

Proof. Since the Green function is symmetric applying lemma 3 we get

$$\Delta u(X) = K_1(X) + K_2(X),$$

where

$$K_1(X) = a_1 \iint_{-\infty}^{+\infty} f_1(s,t)\left(2C^2 D_w \Delta_Y G - D_w(\Delta_Y^2 G)\right)\Big|_{w=0} dsdt,$$

$$K_2(X) = -a_1 \iint_{-\infty}^{+\infty} f_2(s,t)\, D_w(\Delta_Y G)\Big|_{w=0} dsdt.$$

In virtue of lemma 4

$$K_2(X) \to f_2(x_0,y_0) \quad \text{as } (x,y,z)\to(x_0,y_0,0+)$$

and

$$K_1 = a_1 \iint_{-\infty}^{+\infty} f_1(s,t)\, C^4\, D_w G \Big|_{w=0} \, dsdt \longrightarrow 0 \quad \text{as} \quad (x,y,z) \longrightarrow (x_o, y_o, 0+).$$

From the lemmas 4,5,6 follows

Theorem 4. If the functions f_1, f_2 satisfies the assumptions of the lemmas 4,5,6, then the function $u(X)$ defined by the formula (12a) is a solution of the problem (R-H).

B i b l i o g r a p h y

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