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ON CERTAIN BOUNDARY VALUE PROBLEMS AND THE ORLICZ SPACE

1. In the paper we shall give the solution of the equations $\Delta u - c^2 u = 0$ and $\Delta^2 u - c^2 \Delta u = 0$ in the circle $D: x^2 + y^2 < R^2$ with convenient boundary functions continuous and boundary functions belonging to the Orlicz space. In the paper [1] the similar problem was solved for harmonic functions.

Let C denotes the set of periodic functions, continuous in $[0,2\pi]$ and $L_M^{\Delta_2}$ the Orlicz space of the functions defined in $[0,2\pi]$ satisfying Δ_2 condition [2]. Let $|u|_C$ and $|u|_M$ denotes the convenient norms in these spaces.

2. We shall construct the function $u(x,y)$ of class C^2 in D satisfying the equation

(1) $$\Delta u(x,y) - c^2 u(x,y) = 0, \; c > 0, \; c \text{ constant}$$

and the boundary condition

(2) $$u(x,y) \longrightarrow \varphi(x_0,y_0) \quad \text{as} \quad (x,y) \longrightarrow (x_0,y_0), \; (x,y) \in D,$$

$(x_0,y_0) \in \partial D$.

To the construction of the function $u(x,y)$ we shall use the method of separating the variables with the help of polar coordinates (r,t). Let

$$x = r\cos t, \; y = r\sin t, \; 0 < r < R, \; 0 \leqslant t \leqslant 2\pi,$$

$$v(r,t) = u(r\cos t, r\sin t), \quad f(t) = \varphi(R\cos t, R\sin t).$$

If the function $u(x,y)$ satisfies (1), (2), then the function $v(r,t)$ satisfies the equation

(3) $$r^{-1}D_r(rD_r v) + r^{-2}D_{t^2} v - c^2 v = 0, \; r > 0,$$

(4) $$v(r,t) \longrightarrow v(R,t_0) \quad \text{as} \quad (r,t) \longrightarrow (R,t_0)$$

and

$$v(r,t) \longrightarrow A, A = \text{constant}, \text{ as } (r,t) \longrightarrow (0,t_o).$$

Let

(5) $$v(r,t) = W(r)F(t).$$

Using 3, after the separation of variables we get a system of ordinary equations

(6) $$\ddot{F}(t) + n^2F(t) = 0, \ n \text{ constant},$$

(7). $$r^2W''(r) + rW'(r) - (c^2r^2 + n^2)W(r) = 0.$$

From continuity of $v(r,t)$ follows that n is positive integer or $n=0$ and we get

$$F_o(t) = \cos nt, \text{ for } n = 0,$$

$$F_n(t) = p_n \cos nt + q_n \sin nt, \ n = 1,2,3,\ldots$$

and

$$W_n(r) = A_n I_n(cr) + B_n K_n(cr), \ n = 0,1,2,\ldots$$

where p_n, q_n, A_n, B_n are constants and $I_n(cr)$, $K_n(cr)$ the convenient Bessel functions [3]. Since $K_n(cr) \to \infty$ for $r \to 0$ and $v(r,t)$ is continuous, then $B_n = 0$ and the functions

$$v_n(r,t) = I_n(cr)(\alpha_n \cos nt + \beta_n \sin nt), \ \alpha_n, \beta_n \text{ constants},$$

satisfies the equation (3) for $r > 0$.

Let

(8) $$v(r,t) = \frac{a_o I_o(cr)}{2I_o(cR)} + \sum_{n=1}^{\infty} \frac{I_n(cr)}{I_n(cR)} (a_n \cos nt + b_n \sin nt)$$

and

(9) $$v(R,t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = f(t),$$

a_n, b_n being constants.

3. Now we shall prove some lemmas:

<u>Lemma 1</u>. If the series (9) is uniformly convergent to the continuous function f in the intervall $[0,2\pi]$, then the series (8) converges to f uniformly with respect to t if $r \to R$.

Proof. Since

$$0 < I_n(cr) < I_n(cR) \quad \text{for} \quad 0 < r < R, \; n = 0,1,2,\ldots$$

and

$$\lim_{r \to R} \frac{I_n(cr)}{I_n(cR)} = 1, \quad n = 0,1,2,\ldots$$

thus the series (8) is uniformly convergent for $0 < r_0 \leqslant r \leqslant R$ and

$$\lim_{r \to R} v(r,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \lim_{r \to R} \frac{I_n(cr)}{I_n(cR)} (a_n \cos nt + b_n \sin nt) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = f(t).$$

Let

$$z = x + iy = r\cos t + ir\sin t, \quad r^n \cos nt = \mathrm{Re}(x+iy)^n = S_n(x,y),$$

$$r^n \sin nt = \mathrm{Im}(x + iy)^n = T_n(x,y),$$

where $S_n(x,y)$ and $T_n(x,y)$ are convenient polynomials. Let

(10) $$r = \sqrt{x^2 + y^2}, \quad t = \arccos \frac{x}{r}, \quad t = \arcsin \frac{y}{r}, \quad 0 \leqslant t \leqslant 2\pi ,$$

Lemma 2. If $x^2 + y^2 < R^2$ and a_n, b_n are uniformly bounded, then for r and t satisfying (10) the series

$$\frac{a_0 I_0(cr)}{2\, I_0(cR)} + \sum_{n=1}^{\infty} \frac{I_n(cr)}{I_n(cR)} (a_n \cos nr + b_n \sin nt)$$

a power series.

Proof. Using (10) we get

$$I_0(cr) = \sum_{k=0}^{\infty} \left(\frac{c}{2}\right)^{2k} (x^2 + y^2)^k \cdot \frac{1}{k!\, k!} ,$$

$$I_n(cr)\, a_n \cos nt + I_n(cr)\, b_n \sin nt =$$

$$= a_n \left(\frac{c}{2}\right)^n (r^n \cos nt) \sum_{k=0}^{\infty} \left(\frac{c}{2}\right)^{2k} r^{2k} \frac{1}{k!\,(n+k)!} +$$

$$+ b_n\left(\frac{c}{2}\right)^n (r^n \sin nt) \sum_{k=0}^{\infty} \left(\frac{c}{2}\right)^{2k} r^{2k} \frac{1}{k!\,(n+k)!} =$$

$$= a_n\left(\frac{c}{2}\right)^n S_n(x,y) \sum_{k=0}^{\infty} \left(\frac{c}{2}\right)^{2k} (x^2 + y^2)^k \frac{1}{k!\,(n+k)!} +$$

$$+ b_n\left(\frac{c}{2}\right)^n T_n(x,y) \sum_{k=0}^{\infty} \left(\frac{c}{2}\right)^{2k} (x^2 + y^2)^k \frac{1}{k!\,(n+k)!} = \sum_{n=0}^{\infty} P_n^0(x,y).$$

$P_n^0(x,y)$ being convenient homogenous polynomials of degree n.

Finally we get

(11) $$v(r,t) = u(x,y) = \sum_{n=0}^{\infty} P_n(x,y), \quad v(0,t) = P_0(0,0),$$

where $P_n(x,y)$ being convenient homogenous polynomials of degree n.

4. Now we shall prove that the function $u\,x,y$ defined by the formula (11) is of class C^2 in D and satisfies the equation (1). In order to prove this theorem we shall use.

Lemma 3. If a_n and b_n are uniformly bounded, then the series (11) is absolutely and almost uniformly convergent in D.

Proof. Since

$$I_n(cr) = \left(\frac{cr}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{cr}{2}\right)^{2k} \frac{1}{k!(n+k)!}, \quad n = 0,1,2,\ldots$$

then

$$\frac{I_n(cr)}{I_n(cR)} < \left(\frac{r}{R}\right)^n.$$

Consequently the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n$$

is a majorant of the series (11). The series

$$\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n$$

is convergent for $r < R$ and the series (11) for $x^2 + y^2 < R^2$.

Lemma 4. If a_n and b_n are uniformly bounded then the series

$$(12) \qquad \sum_{n=0}^{\infty} D_{x^i y^j} P_n(x,y), \qquad i,j = 0,1,2,$$

are absolutely and almost uniformly convergent in D.

The lemma follows from the convenient theorems concerning the radio of convergence of derivatives of the power series [4].

Lemma 5. If a_n and b_n are uniformly bounded then the series

$$(13) \qquad \sum_{n=1}^{\infty} D_{r^i t^j} \frac{I_n(cr)}{I_n(cR)} (a_n \cos nt + b_n \sin nt), \quad i,j = 0,1,2,$$

are absolutely and almost uniformly convergent for $0 \leqslant r < R$, $0 \leqslant t \leqslant 2\pi$.

Proof. The series

$$k_1 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n, \quad k_2 \sum_{n=1}^{\infty} n\left(\frac{r}{R}\right)^n, \quad k_3 \sum_{n=1}^{\infty} n^2 \left(\frac{r}{R}\right)^n,$$

k_1, k_2, k_3 being positive constants, are majorants of the series (13). These series are convergent for $r < R$, $0 \leqslant t \leqslant 2\pi$.

From lemmas 4 and 5 follows that the function $u(x,y)$ defined by formula (11) is of class C^2 in D and satisfies the equation (1) in D if $u(0,0) = P_0(0,0)$, and

$$D_{x^i y^j} u(x,y)\Big|_{(0,0)} = \sum_{n=0}^{\infty} D_{x^i y^j} P_n(x,y)\Big|_{(0,0)}.$$

5. Now we shall prove that the function $u(x,y)$ satisfies the boundary condition (2). Let $f(s)$ denote the periodic, continuous function defined in $[0,2\pi]$ and let

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos ns \, ds, \quad n = 0,1,2,\ldots,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin ns \, ds, \quad n = 1,2,3,\ldots$$

and

(14) $$v(r,t)=\frac{I_o(cr)}{2I_o(cR)\pi}\int_0^{2\pi} f(s)ds+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{I_n(cr)}{I_n(cR)}\int_0^{2\pi} f(s)\cos n(s-t)ds,$$

(15) $$P_r(t)=\frac{I_o(cr)}{I_o(cR)}+2\sum_{n=1}^{\infty}\frac{I_n(cr)}{I_n(cR)}\cos nt.$$

Since for every $r\in(0,R)$ and $t\in[0,2\pi]$ the series

$$\frac{I_o(cr)}{2\pi I_o(cR)}f(s)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{I_n(cr)}{I_n(cR)}f(s)\cos n(s-t)$$

is uniformly convergent with respect to $s\in[0,2\pi]$ we get

(16) $$v(r,t)=\frac{1}{2\pi}\int_0^{2\pi} f(s)\,P_r(s-t)dt.$$

Lemma 7. If $f_n(t)$ is a sequence of trigonometric polynomials and $f_n(t)$ is uniformly convergent to $f(t)$ for $t\in[0,2\pi]$ then the functions of the sequence

(17) $$v_n(r,t)=\frac{1}{2\pi}\int_0^{2\pi} f_n(s)P_r(s-t)ds$$

satisfies the equation (3) for $0<r<R$, and if $r\to R$, $v_n(r,t)$ is uniformly convergent to $f(t)$ with respect to $t\in[0,2\pi]$.

Proof. Using the Harnack theorem [4] we conclude that

$$v_n(r,t)\longrightarrow v(r,t),\ \text{as } n\to\infty,$$

$V(r,t)$ satisfies the equation (3) and from lemma 1 it follows that

$$V(r,t)\longrightarrow f(t)\ \text{as } r\to R$$

uniformly for $t\in[0,2\pi]$. Since the solution of the Dirichlet problem is unique, the function $V(r,t)$ is identique with the function $v(r,t)$ defined by (8).

Theorem 1. If the function f satisfies the assumptions of lemma 1, then the function $u(x,y)=v(r,t)$ defined by formula (11) satisfies boundary condition (2).

Proof. It is sufficient to prove that

$$v(r,t)\longrightarrow f(t_o)\quad \text{as } (r,t)\longrightarrow(R,t_o).$$

Let $\varepsilon > 0$ denotes arbitrary positive number. From continuity of the function f, follows that there exists a number $\delta(\varepsilon) = \delta$ such that if $|t - t_o| < \delta$, then $|f(t) - f(t_o)| < \frac{\varepsilon}{2}$ and from uniformly convergence of $v(r,t)$ to $f(t)$ follows that if $|r - R| < \delta$ then $|v(r,t) - f(t)| < \frac{\varepsilon}{2}$ for every $t \in [0,2\pi]$ and

$|v(r,t) - f(t_o)| \leq |v(r,t) - f(t)| + |f(t) - f(t_o)| < \varepsilon$ if $|t - t_o| < \delta$, $|r - R| < \delta$.

6. Let us suppose that the function $f \in L_M^{\Delta_2}$, and let

(17)
$$J_r(f,s) = \int_0^{2\pi} f(t) K_r(s,t)\, dt.$$

Lemma 8. ([5]). If $K_r(s,t)$ is a set of mesurable functions in the square $P:\{0 \leqslant s \leqslant 2\pi,\ 0 \leqslant t \leqslant 2\pi\}$, and $0 < r < R$ and the exists a constant A such that

$$\int_0^{2\pi} |K_r(s,t)|\, dt \leqslant A, \qquad \int_0^{2\pi} |K_r(s,t)|\, ds \leqslant A$$

for almost every s or t and for arbitrary $r \in (0,R)$ then the integrals (17) exist for almost every $s \in [0,2\pi]$ and belong to L_M. Moreover if for every function $f \in H$, H being the everywhere dense set in L_M and

(18)
$$\lim_{\substack{r \to R \\ p \to R}} \int_0^{2\pi} M(|J_r(f,s) - J_p(f,s)|)\, ds = 0$$

then (18) is satisfies for every $f \in L_M$.

If the condition (18) is satisfied, then $J_r(f,s)$ is convergent in the norm $L_M^{\Delta_2}$ and inversely.

The set C is dense in $L_M^{\Delta_2}$ and from lemma 8 follows

Lemma 9. If $K_r(s,t)$ satisfies the assumptions of lemma 8 and for every $f \in C$, the integrals (17) are convergent to f in the norm $L_M^{\Delta_2}$, then integrals (17) are convergent to every $f \in L_M^{\Delta_2}$ in the norm $L_M^{\Delta_2}$.

7. Now we shall prove

Lemma 10. The kernel $P_r(s)$ defined by formula (15) is nonnegative and

(19)
$$\frac{1}{2\pi} \int_0^{2\pi} P_r(s)\, ds = \frac{I_o(cr)}{I_o(cR)}.$$

Proof. Let P, Q denote two points, $P \in D$, $Q \in \bar{D}$, $P \neq Q$ and let $G(P,Q)$ be the Green function for the problem (1), (2) with a pole at the point P. Since $G(P,Q) = 0$ for $Q \in \partial D$ and $G(P,Q) > 0$ for $Q \in D, Q \neq P$, from the theorem of Olejnik [4] follows that

$$\frac{dG(P,Q)}{dn_Q} > 0 \quad \text{for} \quad Q \in \partial D,$$

n_Q being the inward normal. The solution $u(x,y)$ of the Dirichlet problem (1), (2) is given [4] by formula

$$u(x,y) = \frac{1}{2\pi} \int_{\partial D} g(Q) \frac{dG(P,Q)}{dn_Q} \, ds_Q,$$

where $g(Q)$ denotes boundary function. From the theorem 1 and uniqueness of the solution of the problem (1), (2) follows that

$$\int_0^{2\pi} f(s) P_r(s-t)\, ds = \int_0^{2\pi} f(s) \left.\frac{dG(P,Q)}{dn_Q}\right|_{Q = (R\cos s,\, R\sin s)} ds_Q$$

for every function $f \in C$ and that the kernel $P_r(s-t)$ is nonnegative.

Since the serie (25) is uniformly convergent with respect to s for every $r < R$, by integration of the series (15) we get (19).

Theorem 2. If $f \in L_M^{\Delta_2}$ then the function $u(x,y) = v(r,t)$, defined by the formula (16)

1° satisfies the equation (1) in D,

2° $\lim_{r \to R} |v(r,\cdot) - f|_M = 0.$

Proof. The proof of the condition 1° is similar to that of theorem 1. In order to prove the condition 2° at first we shall prove 2° for $f \in C$. Let $f \in C$. From theorem 1 follows that for every $\varepsilon > 0$ there exists a number r_0 such that for every $r \in (0,R)$

$$|v(r,t) - f(t)| < \varepsilon$$

uniformly with respect to $t \in [0, 2\pi]$ and consequently

$$\int_0^{2\pi} M(|v(r,t) - f(t)|)\, dt \leqslant \int_0^{2\pi} M(\varepsilon)\, dt = 2\pi M(\varepsilon)$$

for every $r \in (0,R)$. If $\varepsilon \to 0$ then $M(\varepsilon) \to 0$ and

$$|v(r,\cdot) - f|_M \to 0 \quad \text{for} \quad r \to R.$$

From lemma 10 and from the inequality (19) and monotonicity of the function $I_n(cr)$ ([3]) follows that $P_r(s-t)$ satisfies the assumptions of lemma 8. Using lemma 9 we get the condition 2^o.

8. Let $\varphi_1(x,y)$, $\varphi_2(x,y)$ be the functions defined on ∂D. We shall construct the function $u(x,y)$ satisfying in D the equation

$$\Delta^2 u(x,y) - c^2 \Delta u(x,y) = 0, \quad c>0, \ c \text{ constant} \tag{20}$$

and the boundary conditions

$$u(x,y) \longrightarrow \varphi_1(x_0,y_0), \tag{21}$$

$$u(x,y) \longrightarrow \varphi_2(x_0,y_0)$$

if $(x,y) \in D$, $(x_0,y_0) \in \partial D$, $(x,y) \to (x_0,y_0)$.

The problem (20), (21) is called the Riquier problem.

Using the known theorem [6] we assume that

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

where $u_1(x,y)$ is a harmonic function and $u_2(x,y)$ a function satisfying the equation (1). In order to solve the Riquier problem we shall construct the functions $u_1(x,y)$ and $u_2(x,y)$ such that

$$\begin{cases} \Delta u_1(x,y) = 0 \quad \text{in } D, \\ u(x,y) \to \varphi_1(x_0,y_0) - \frac{1}{c^2}\varphi_2(x_0,y_0) \quad \text{as } (x,y)\to(x_0,y_0), \end{cases} \tag{22}$$

$$\begin{cases} \Delta u_2(x,y) - c^2 u_2(x,y) = 0 \quad \text{in } D, \\ u_2(x,y) \to \frac{1}{c^2}\varphi_2(x_0,y_0) \quad \text{as } (x,y)\to(x_0,y_0). \end{cases} \tag{23}$$

Let

$$u(x,y) = v_1(r,t) + v_2(r,t), \tag{24}$$

where

$$v_1(r,t) = \frac{1}{2\pi}\int_0^{2\pi}\left[f_1(s) - \frac{1}{c^2}f_2(s)\right]\frac{R^2 - r^2}{R^2 - 2Rr\cos(s-t) + r^2}\,ds, \tag{25}$$

$$v_2(r,t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{c^2}f_2(s)\,P_r(s-t)\,ds \tag{26}$$

and $\quad f_1(s) = \varphi_1(R\cos s,\ R\sin s),\ f_2(s) = \varphi_2(R\cos s,\ R\sin s).$

Theorem 3. If the functions φ_1, φ_2 are continuous on ∂D, then the function $u(x,y)$ defined by the formula (24) is of class C^4 in D, satisfies the equation (20) in D and the boundary condition (21).

Proof. The function $v_1(r,t)$ is the Poisson integral, is analytic and satisfies (22). From theorem 1 it follows that the function $u_2(x,y) = v_2(r,t)$ is analytic too and satisfies the conditions (23).

Let $V(r,t)$ denote $\Delta u(x,y)$ in polar coordinates. We shall prove

Theorem 4. If $f_1 \in L_M^{\Delta 2}$, $f_2 \in L_M^{\Delta 2}$ then the function $u(x,y)$ defined by the formula (24) satisfies the equation (20) in D and the boundary conditions

(27) $$|v(r,\cdot) - f_1|_M \longrightarrow 0 \quad \text{if} \quad r \to R,$$

(28) $$|V(r,\cdot) - f_2|_M \longrightarrow 0 \quad \text{if} \quad r \to R,$$

Proof. Using the convenient theorem 1 we get

$$|v_1(r,\cdot) - f_1 + \frac{1}{c^2} f_2|_M \longrightarrow 0 \quad \text{if} \quad r \to R.$$

From theorem 2 follows that the function $u_2(x,y)$ defined by the formula (26) satisfies the equation (20) in D and

$$|v_2(r,\cdot) - \frac{1}{c^2} f_2|_M \to 0 \quad \text{if} \quad r \to R.$$

From the triangle inequality follows that

$$|v(r,\cdot) - f_1|_M \leqslant |v_1(r,\cdot) - f_1 + \frac{1}{c^2} f_2|_M + |v_2(r,\cdot) - \frac{1}{c^2} f_2|_M$$

and consequently (27). The validity of (28) follows from the formula:

$$\Delta u(x,y) = \Delta u_2(x,y) = c^2 u_2(x,y).$$

Theorem 5. If the function $u(x,y)$ satisfies (20) and is of class C^3 in D and of class C^4 in D, then $u(x,y)$ is unique.

Proof. Let

$$u(x,y) = u_1(x,y) + u_2(x,y),$$
$$w(x,y) = w_1(x,y) + w_2(x,y)$$

denote the two solution of (20), u_1, w_1, u_2, w_2 being the convenient solutions of (22) and (23). From the uniqueness of the problems (22), (23) follows that $u_1 = w_1$, $u_2 = w_2$ in D and $u(x,y) = w(x,y)$ in D.

Remark. If

$$v(r,t) \to \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad \text{as} \quad r \to R,$$

then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

is called (B) limes of the series (8). A question arises about the connexion between the Abel-Poisson method and that of (B) summability.

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