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ON A FUNCTIONAL EQUATION OF INVARIANT CURVES

In the present paper we shall consider the functional equation

$$\varphi\{F[x, \varphi(x)]\} = s\varphi(x), \tag{1}$$

where φ is an unknown real-valued function of real variable, F is a given real-valued function of two variables, and s is a given positive number. The solutions φ of equation (1) are invariant curves under transform

$$x' = F(x,y), \quad y' = sy \tag{2}$$

of real plane into itself. The equation of invariant curves has been investigated by many authors /see [4] /, but mostly they were the investigations of the local solutions. In the case where $s = 1$, the general continuous solution of equation (1) has been given in the papers [1] and [2] /and also in [6] and [7] in the special case where $F(x,y) = x+y$/. Thus we may restrict our considerations to the case where $s \neq 1$.

We know /see [4] / that in general the equation (1) has in the neighbourhood of a fixed point of transform (2) a continuous solution depending on an arbitrary function. However, under the assumption that these solutions have a limit /finite or not/ at infinity, we can show that global continuous solutions form the one - or two-parameter family /see [1], [2], [6] and [7]/ in case where $s = 1$.

In the present paper we shall consider continuous solutions of equation (1) under the stronger condition when the limit at infinity is zero.

We shall assume the following hypotheses:

(H_1) The function F is defined and continuous on the real plane and its range is the real axis.

(H_2) The function F is strictly increasing with respect to the first variable and it is strictly decreasing with respect to the other.

(H_3) The function F is strictly increasing with respect to both variables.

We shall study the solutions of (1) continuous on the real axis, and satisfying the condition:

$$(3) \qquad \lim_{x \to \infty} \varphi(x) = \lim_{x \to -\infty} \varphi(x) = 0.$$

In the sequel it will be very convenient for us to deal with the function Φ, defined by the following:

Definition 1. If φ is a solution of equation (1), we denote

$$(4) \qquad \Phi(x) = F[x, \varphi(x)],$$

$$(5) \qquad \Phi^0(x) = x, \quad \Phi^{n+1}(x) = \Phi[\Phi^n(x)], \quad \text{for } n = 0,1,\ldots .$$

The function Φ depends on the function φ. The properties of Φ and Φ^n are given by the following

Lemma 1. If φ is a continuous solution of equation (1), then Φ^n is a continuous function for $n = 0,1,\ldots$ and

$$(6) \qquad \Phi[\Phi^n(x)] = s^n \varphi(x), \quad \text{for } x \in (-\infty, \infty), \quad n = 0,1,\ldots .$$

Proof. The continuity of the functions Φ^n is a consequence the of continuity of functions φ and F. The formula (6) results from (5), (4) and (1) by simple induction.

In the sequel we shall deal with the positive solutions of equation (1) and with the case where

$$(7) \qquad 0 < s < 1.$$

Lemma 2. Let the hypotheses $(H_1), (H_2)$ and the condition (7) be fulfilled. Let φ be a continuous positive solution of equation (1).

If a is such a point that the inequality

$$(8) \qquad \Phi(a) > a$$

holds, then the sequence $\Phi^n(a)$ is strictly increasing and

$$(9) \qquad \Phi^n(a) \leq \Phi^n(x) \leq \Phi^{n+1}(a), \quad \text{for } x \in [a, \Phi(a)], \; n = 0,1,\ldots$$

Proof. If $n = 0$, the condition (9) means that $x \in [a, \Phi(a)]$. We are going to prove (9), for $n = 1$.

Let us assume that there exists such a point $t \in [a, \Phi(a)]$, that (9) does not hold at t. It is obvious that it has to be

$$(10) \qquad a < t < \Phi(a).$$

There are two possible cases:

$$(11) \qquad \Phi(t) < \Phi(a)$$

or

(12) $\Phi^2(a) < \Phi(t)$.

Let us assume that the inequality (11) holds. As the function Φ is continuous, by virtue of lemma 1, there exists such a point c that

(13) $c \in (t, \Phi(a))$

and

(14) $\Phi(c) = \Phi(a)$.

It follows from (1), (4) and (14) that

$$s\varphi(c) = \varphi\{F[c, \varphi(c)]\} = \varphi[\Phi(c)] = \varphi[\Phi(a)] = \varphi\{F[a, \varphi(a)]\} = s\varphi(a),$$

whence

$$\varphi(c) = \varphi(a).$$

Thus, we obtain, by virtue of (4) and (14),

$$F[c, \varphi(c)] = \Phi(c) = \Phi(a) = F[a, \varphi(a)] = F[a, \varphi(c)].$$

Since the function F is strictly increasing with respect to the first variable, then by virtue of hypothesis (H_2), we have a = c, which contradicts (13) and (10).

Now, let us assume that the inequality (12) holds. Then, there exists such a point c that

(15) $c \in (a,t)$

and

(16) $\Phi(c) = \Phi^2(a)$.

It follows from (1), (4), (16) and (5) that

$$s\varphi(c) = \varphi\{F[c, \varphi(c)]\} = \varphi[\Phi(c)] = \varphi[\Phi^2(a)] =$$

$$= \varphi(F\{F[a, \varphi(a)], \varphi(F[a, \varphi(a)])\}) = s^2 \varphi(a).$$

Thus, we have

$$\varphi(c) = s\varphi(a),$$

whence

$$F[c, \varphi(c)] = \Phi(c) = \Phi^2(a) = F\{F[a, \varphi(a)], \varphi(F[a, \varphi(a)])\} =$$

$$= F\{F[a, \varphi(a)], s\varphi(a)\} = F\{F[a, \varphi(a)], \varphi(c)\},$$

by virtue of (4), (16) and (1). As the function F is strictly increasing with respect to the first variable, we obtain

$$c = \Phi(a),$$

which contradicts (10) and (15).

The condition (9) has been proved for $n = 1$. Now, let us assume that (9) holds for an $n \geqslant 0$. Putting a for $\Phi^n(a)$ and x for $\Phi^n(x)$, we can see that the condition (9) holds for $n + 1$, by virtue of (9), already proved for $n = 1$. The lemma is proved then by induction.

In an analogous way as for lemma 2, we can prove the following

Lemma 3. Let the hypotheses (H_1), (H_3) and the condition (7) be fulfilled. Let φ be a continuous negative solution of equation (1). If a is such a point that the inequality (8) holds, then the sequence $\Phi^n(a)$ is strictly increasing and tle condition (9) is satisfied.

Lemma 4. Let the hypothesis (H_1) and the condition (7) be fulfilled. Moreover, let either

1^o the hypothesis (H_2) be fulfilled and φ be a continuous negative solution of equation (1)

or

2^o the hypothesis (H_3) be fulfilled and φ be a continuous positive solution of equation (1).

If a is such a point that the inequality

(17) $$\Phi(a) < a,$$

holds, then the sequence $\Phi^n(a)$ is strictly decreasing and

(18) $$\Phi^n(a) \geqslant \Phi^n(x) \geqslant \Phi^{n+1}(a), \quad \text{for } x \in [\Phi(a), a], \quad n = 0,1,\dots.$$

Lemma 5. Let the hypothesis (H_1) and the condition (7) be fulfilled. Then

a/ if there exists such a point a that the inequality (8) holds, the sequence $\Phi^n(a)$ is strictly increasing and either

1^o the hypothesis (H_2) is fulfilled and φ is a continuous negative solution of equation (1)

or

2^o the hypothesis (H_3) is fulfilled and φ is a continuous positive solution of equation (1),

then the condition (9) is fulfilled.

b/ If there exists such a point a that the inequality (17) holds, the sequence $\Phi^n(a)$ is strictly decreasing and either

1^o the hypothesis (H_2) is fulfilled and φ is a continuous positive solution of equation (1)

or

2^o the hypothesis (H_3) is fulfilled and φ is a continuous negative solution of equation (1),

then the condition (18) is fulfilled.

We can prove the lemma 5 in a similar way as the lemma 2. Let us observe that the conditions of the lemma 5 need not imply the monotonicity of the sequence $\Phi^n(a)$. However, we can prove the following

Lemma 6. Under the conditions of lemma 5

$$\Phi^{n+1}(a) \neq \Phi^n(a), \quad \text{for} \quad n = 0,1,\dots \tag{19}$$

Proof. We are going to prove the lemma in the case where the inequality (8) holds. In the other case the proof will be similar. Let us suppose that the inequality (19) does not hold, i.e., there exists such a number p that

$$\Phi^{p+1}(a) = \Phi^p(a). \tag{20}$$

It follows from the inequality (8) that $p > 0$. We have, from (1), (4) and (20)

$$s\,\varphi[\Phi^p(a)] = \varphi\{F(\Phi^p(a),\ \varphi[\Phi^p(a)])\} = \varphi[\Phi^{p+1}(a)] = \varphi[\Phi^p(a)],$$

whence

$$s^{p+1}\varphi(a) = s^p\varphi(a),$$

by virtue of (6). Since s satisfies the condition (7), the last equality implies that $\varphi(a) = 0$. We obtain from (4) that

$$\Phi(a) = F(a,0) \quad \text{and by (5)} \quad \Phi^n(a) = F[\Phi^{n-1}(a),\ 0] > \Phi^{n-1}(a), \quad \text{for}$$

$n = 0,1,\dots,$ because the function F is strictly increasing with respect to the first variable, and by virtue of (8). But it contradicts (20). This ends the proof.

As a simple consequence of the foregoing lemmas we obtain the following

Lemma 7. Let the condition (7), the hypotheses (H_1) and either (H_2) or (H_3) be fulfilled. If φ is a continuous solution of equation (1) and φ is either positive or negative, then for each a satisfying the inequality

$$\Phi^n(a) \neq a$$

there exists such a positive integer k that the suquence $\Phi^n(a)$ is strictly monotonic for $n > k$.

Proof. We are going to prove the lemma in the case where the hypothesis (H_2) is fulfilled and the function φ is positive. In the other cases the proof is similar - we have to apply either lemma 3 or 4, instead of the lemma 2.

If the inequality (8) holds, then the lemma follows from the lemma 2. If the inequality (17) holds, then either the sequence $\Phi^n(a)$ is strictly decreasing, or there exists such a positive integer k that the sequence $\Phi^n(a)$ is strictly increasing for $n \geqslant k$, by virtue of lemmas 6 and 2. This ends the proof.

Let φ be an arbitrarily chosen solution of equation (1). We are going to introduce the following

Definition 2. Let φ be a continuous solution of equation (1). If an a satisfies the inequality (8), we denote

$$d^a = \sup\{c : F[x, \varphi(x)] > x, x \in (a,c)\},$$

$$d_a = \inf\{c : F[x, \varphi(x)] > x, x \in (c,a)\}.$$

If an a satisfies the inequality (17), we denote

$$d^a = \sup\{c : F[x, \varphi(x)] < x, x \in (a,c)\},$$

$$d_a = \inf\{c : F[x, \varphi(x)] < x, x \in (c,a)\}.$$

If $F[a, \varphi(a)] = a$, then we put $d^a = d_a = a$.

Of course, both d_a and d^a can be infinite.

Lemma 8. Let the hypotheses (H_1), (H_2) and the condition (7) be fulfilled. If φ is a positive continuous solution of equation (1), satisfying the condition (3), then

$$\lim_{x \to d_a} \varphi(x) = \lim_{x \to d^a} \varphi(x) = 0. \tag{21}$$

Proof. If both d_a and d^a infinite, then the condition (21) follows from the condition (3). If $F[a, \varphi(a)] = a$, we have from the definition 2 and from (1) that

$$s\,\varphi(a) = \varphi\{F[a, \varphi(a)]\} = \varphi(a),$$

then $\varphi(a) = 0$, because of (7). In this case the condition (21) follows from the continuity of the function φ. Let us consider the case where $F[a, \varphi(a)] \neq a$ and either d_a or d^a is finite. Let us suppose that d_a is finite. Then we have, by virtue of definition 2, that $F[d_a, \varphi(d_a)] = d_a$, whence $\varphi(d_a) = s\,\varphi(d_a)$, by virtue of (1) and then $\varphi(d_a) = 0$, because of (7). In an analogical way we obtain that $\varphi(d^a) = 0$ if d^a is finite. Then, the condition (21) follows from the continuity of the function φ.

Lemma 9. Let the hypotheses (H_1), and either (H_2) or (H_3) and the condition (7) be fulfilled. If φ is a positive continuous solution of equation (1) satisfying the condition (3), then for each a

$$\text{either } \lim_{n\to\infty} \Phi^n(a) = d_a \quad \text{or} \quad \lim_{n\to\infty} \Phi^n(a) = d^a. \tag{22}$$

Proof. The condition (22) is obvious if $F[a, \varphi(a)] = a$, because of the definition 2. If $F[a, \varphi(a)] \neq a$, we are going to prove the lemma in the case where φ is positive and the hypothesis (H_2) is fulfilled. In the other cases the proof will be similar.

Let us suppose that the inequality (8) holds, and $d^a < \infty$ Then, by virtue of lemma 2, there exists the limit $\lim_{n\to\infty} \Phi^n(a)$ /finite or not/. It follows from (8), from the definition 2 and from the hypothesis (H_2) that

$$a < F[d^a, \varphi(a)] < F[d^a, \varphi(a)] \leqslant F(d^a, 0) = d^a.$$

We can prove by induction, that

$$a < \Phi^n(a) \leqslant d^a, \quad \text{for} \quad n = 0, 1, \ldots$$

It implies that

$$c = \lim_{n\to\infty} \Phi^n(a) \leqslant d^a \quad \text{and} \quad c \geqslant a.$$

Let us suppose that $c < d^a$. As φ is a continuous function, we have, by virtue of lemma 6 and (1), that

$$c = \lim_{n\to\infty} \varphi[\Phi^n(a)] = \lim_{n\to\infty} s^n \varphi(a) = 0,$$

because of the condition (7). Then we have, by virtue of definition 1,

$$0 = \varphi(c) = \lim_{n\to\infty} \Phi^{n+1}(a) = \lim_{n\to\infty} F\left[\Phi^n(a), \varphi(\Phi^n(a))\right] = F(c, 0),$$

which is impossible, because of the definition 2.

If

(23) $\quad d^a = \infty,$

we can show in a similar way that if $c = \lim_{n\to\infty} \varphi[\Phi^n(a)] < \infty$,

then $F[c, \varphi(c)] = c$,

in spite of (23).

In an analogical way we can prove our lemma in the case where $d_a > -\infty$.

Theorem 1. Let the hypothesis (H_1), the condition (7) and either the hypothesis (H_2) or (H_3) be fulfilled. If φ is a continuous solution of equation (1) satisfying the condition (3), then

(24) $\quad \varphi(x) = 0, \quad \text{for} \quad x \in (-\infty, \infty).$

Proof. Let us assume that there exists such a point a that

(25) $\quad \varphi(a) > 0.$

It follows from the lemma 9 and from the condition (3) that there exists such a point $c \in [d_a, a]$ that $\varphi(c) = s\,\varphi(a)$. Let us put

(26) $\quad t_1 = \inf\{c : \varphi(x) \geqslant s\,\varphi(a),\ c < x < a\}.$

We have from (26) that

$$t_1 \in (d_a, a).$$

Similarly, it follows from the lemma 9 that there exists such a point $t_2 \in [d_a, t_2]$ that $\varphi(x) \leq s\varphi(a)$, for $x \in [d_a, t_2]$. As the functions φ and F are continuous, we obtain from the lemma 9 that there exists such a point $b \in [d_a, t_2]$, that $F[b, \varphi(b)] \in [t_1, a]$. But it contradicts (25) and (26), because

$$F[b, \varphi(b)] = s\varphi(b) \leq s^2\varphi(a),$$

by virtue of (1).

If there exists such a point a that $\varphi(a) > 0$, we obtain the contradiction in an analogical way. This ends the proof.

If the transform (2) is a homeomorphism of the real plane into itself, then the inverse transform takes the form

(27) $$x' = f(x,y), \; y' = \frac{1}{s}y.$$

The transform (27) satisfies the hypothesis (H_1) and if the transform (2) satisfies the hypothesis (H_2), then the transform (27) satisfies the hypothesis (H_3) and vice versa.As each invariant curve under transform (2) is also invariant under inverse transform (27), then it has to satisfy the following equation

(28) $$\varphi\{f[x, \varphi(x)]\} = \frac{1}{s}\varphi(x).$$

Thus, if s satisfies the condition $s > 1$ instead of the condition (7), then the conditions of the theorem 1 are fulfilled for the equation (28). Then we obtain the following

Theorem 2. Let the hypotheses (H_1) and either (H_2) or (H_3) be fulfilled. If the transform (2) is a homeomorphism of the real plane into itself, $s > 1$ and φ is a continuous solution of equation (1) satisfying the condition (3), then

$$\varphi(x) = 0, \quad \text{for} \quad x \in (-\infty, \infty).$$

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