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ON THE EQUIVALENCE OF TWO DEFINITIONS OF THE FULFILMENT OF THE TRANSLATION EQUATION AND ON THE EXTENSIONS OF THE SOLUTIONS OF THIS EQUATION

1. Introduction

In mathematical theories there appear different definitions of the fulfilment of the translation equation. In this paper we shall consider two definitions - one of them occurs in the theory of the algebraic objects ([3], p.68/), the other one is a natural generalization of the definition occurring in the theory of abstract machines ([1],p.48). These two definitions are not equivalent. In the present paper we shall give some necessary and some sufficient conditions of the equivalence of these definitions. Besides we shall give some conditions sufficient for the extensibility of the solutions of the translation equation.

2. Basic notations

By $f : A \multimap B$ we shall denote the function /called partial function/ the domain of which is contained in the set A and the range of which is contained in the set B. The domain of the function f will be denoted by D_f and the range will be denoted by C_f.

If F is a function of the form:

$$F : A \times B \multimap C,$$

then we shall denote:

$$D_F^1 := \left\{x \in A : \bigvee_{\alpha \in B} \left((x,\alpha) \in D_F\right)\right\},$$

$$D_F^2 := \left\{\alpha \in B : \bigvee_{x \in A} \left((x,\alpha) \in D_F\right)\right\}.$$

Let G be an arbitrary non-empty set and let "$\cdot$" be an arbitrary partial mapping of the set $G \times G$ in the set G. We shall call the pair $(G, \cdot)$ a multiplicative system.

We shall consider two following definitions:

Definition 1 ([3], p.68). The function $F : X \times G \multimap X$, where X is an arbitrary set and $(G, \cdot)$ is an arbitrary multiplicative system, will be called the solution of the translation equation if the following condition is fulfilled:

If $F(x, \alpha)$ and $\alpha \cdot \beta$ are defined then $F(x, \alpha \cdot \beta)$ and $F[F(x, \alpha), \beta]$ are defined and the following equality holds:

$$F[F(x, \alpha), \beta] = F(x, \alpha\beta) \quad . \tag{1}$$

Definition 2. The function $F : X \times G \multimap X$, where X is an arbitrary set and $(G, \cdot)$ is an arbitrary multiplicative system, will be called the solution of the translation equation if the following conditions are fulfilled:

(a) If $\alpha\beta$ is defined then $F(x, \alpha \cdot \beta)$ is defined iff $F[F(x, \alpha), \beta]$ is defined,

(b) If $F(x, \alpha\beta)$ is defined then equality (1) holds.

3. Equivalence of Definitions 1 and 2

Definitions 1 and 2 are not equivalent and none of them is implicated by the other. It is illustrated by the following

Example 1[x]/. Let X be the set of non-negative integers and $(G, \cdot)$ the semigroup of non-negative integers with the multiplication. Let us put:

$$F_1(x, \alpha) = x\alpha \ ,$$

where $F_1(x, \alpha)$ is defined iff $x\alpha$ is an even number; and

$$F_2(x, \alpha) = x\alpha \qquad \text{for} \quad \alpha \neq 0.$$

It is easy to verify that F_1 satisfies definition 1 and does not satisfy definition 2 and on the other hand F_2 satisfies definition 2 and does not satisfy definition 1.

We shall prove that if the multiplicative system $(G, \cdot)$ satisfies some additional assumptions then definitions 1 and 2 are equivalent.

Theorem 1.

If the multiplicative system $(G, \cdot)$ satisfies the following condition:

$$\bigwedge_{(\alpha, \beta) \in D} \ \bigvee_{\gamma \in G} \left((\alpha \cdot \beta) \cdot \gamma = \alpha \right), \tag{2}$$

x/ This example was given by Z. Moszner.

then definitions 1 and 2 are equivalent, i.e. for an arbitrary set X a function $F : X \times G \multimap X$ satisfies definition 1 iff it satisfies definition 2.

Proof. Let us consider an arbitrary set X and an arbitrary multiplicative system $(G,\cdot)$ satisfying condition (2) and let $F : X \times G \multimap X$ be a function satisfying definition 1. Let (α, β) belong to the set D. If $F[F(x,\alpha), \beta]$ is defined, then using definition 1 we obtain that $F(x, \alpha\cdot\beta)$ is defined and equality (1) holds.

Now let us assume that $F(x, \alpha\cdot\beta)$ is defined. From condition (2) it follows that there exists an element $\gamma \in G$ such that $(\alpha\cdot\beta)\cdot\gamma = \alpha$. Using definition 1 we obtain that $F[x, (\alpha\cdot\beta)\cdot\gamma]$ is defined, i. e. $F(x, \alpha)$ is defined. From definition 1 it follows that $F[F(x,\alpha), \beta]$ is defined and equality (1) holds. We have proved that definition 1 implicates definition (2).

Now let us consider the function $F : X \times G \multimap X$ satisfying definition 2. Let us assume that $F(x,\alpha)$ is defined and $(\alpha, \beta) \in D$. According to (2) there exists $\gamma \in G$ such that $F[x,(\alpha\cdot\beta)\cdot\gamma]$ is defined and hence by definition 2 $F(x, \alpha\cdot\beta)$ is defined. Then, from definition 2 it follows that $F(x,\alpha)$ is defined and equality (1) holds. In this way we have proved that function F satisfies definition 1 what completes the proof of theorem 1.

The theorem partially inversed to theorem 1 is the following

Theorem 2. Let X be an arbitrary set such that $\overline{\overline{X}} \geqslant 2$ and let $(G,\cdot)$ be an arbitrary associative multiplicative system. We assume besides that every function $F : X \times G \multimap X$ satisfying definition 1 satisfies definition 2. Then condition (2) is fulfilled in the multiplicative system $(G,\cdot)$.

Proof. Suppose that the thesis of theorem 2 is not fulfilled, i.e. the multiplicative system $(G,\cdot)$ does not possess property (2). It means that there exists a pair $(\alpha_0, \beta_0) \in D$. such that for every $\gamma \in G$ the following condition

$$(3) \qquad (\alpha_0\beta_0, \gamma) \notin D. \quad \text{or} \quad (\alpha_0\cdot\beta_0)\,\gamma \neq \alpha_0$$

is satisfied.

Let us denote:

$$G_0 := \left\{ \alpha \in G : \bigvee_{\gamma \in G} \left(\alpha = (\alpha_0\cdot\beta_0)\cdot\gamma\right)\right\}.$$

From (3) it follows that

$$(4) \qquad \alpha_0 \notin G_0 .$$

Because the multiplicative system $(G, \cdot)$ is associative, so if $(\alpha, \beta) \in D$. and $\alpha_0 \in G$ then $\alpha \cdot \beta \in G_0$.

Let us consider two cases:

a/ $G_0 = \emptyset$,

b/ $G_0 \neq \emptyset$.

In case a/ we put:

(5) $$F(x, \alpha) = x \quad \text{for} \quad x \in X, \ \alpha = \alpha_0 \cdot \beta_0 .$$

It is easy to see that function F defined in this way satisfies definition 1.

Function F does not satisfy definition 2. Otherwise from the fact that $F(x, \alpha_0 \cdot \beta_0)$ is defined, we would obtain that $F(x, \alpha_0)$ is defined, too. From this and from (5) we have:

$$\alpha_0 = \alpha_0 \cdot \beta_0 ,$$

and consequently

$$\alpha_0 = (\alpha_0 \cdot \beta_0) \cdot \beta_0 .$$

Thus $\alpha_0 \in G_0$, which is contrary to (4).

Now let us consider case b/. Let a and b be two different fixed elements of the set X /the existence of such elements is guaranteed by the assumption $\bar{\bar{X}} \geqslant 2$/.

We put:

$$F(x, \alpha) = \begin{cases} a & \text{for} \quad x = a, \ \alpha \in G, \\ a & \text{for} \quad x = b, \ \alpha \in G_0. \end{cases}$$

Function F defined in this manner satisfies definition 1. In the case when $x = a$, for an arbitrary element $\alpha \in G$ both sides of equation (1) are defined and the equality holds.

Now let us consider $x = b$ and let us assume that $(\alpha, \beta) \in D$. and $F(x, \alpha)$ is defined. It means that $\alpha \in G_0$. Thus $\alpha \cdot \beta \in G_0$ and hence $F(b, \alpha \cdot \beta)$ is defined. $F[F(b, \alpha), \beta]$ is defined because $F(b, \alpha) = a$. It is easy to see that equality (1) holds. Thus we have proved that function F satisfies definition 1. Let us suppose that function F satisfies also definition 2. Let us consider an arbitrary element $\gamma \in G$ such that $(\alpha_0 \cdot \beta_0, \gamma) \in D$. ($G_0 \neq \emptyset$ and hence there exists such a γ). Then $F(b, (\alpha_0 \cdot \beta_0) \cdot \gamma)$ is defined. Thus $F(b, \alpha_0 \cdot \beta_0)$ is defined and in consequence $F(b, \alpha_0)$ is defined, too. It follows that $\alpha_0 \in G_0$ which is contrary to (4). It completes the proof of the theorem.

Assumption $\bar{\bar{X}} \geqslant 2$ is essential for theorem 2. It shows the following

Example 2. Let us put: $X = \{a\}$, $G = \{0,1\}$. We define the operation " . " as follows:

$\cdot$	0	1
0	0	0
1	0	0

Let us consider all functions F of the form:

$$F : X \times G \multimap X .$$

There are four of these functions /one of them is empty/. The non-empty functions are defined as follows:

$$F_1(a, 0) = a,$$

$$F_2(a, 1) = a,$$

$$F_3(a, i) = a \quad \text{for} \quad i = 0, 1.$$

It is easy to see that functions F_1 and F_2 do not satisfy either of the definitions 1 and 2, whereas function F_3 /and obviously the empty function/ satisfies both. Thus each of the considered functions either satisfies definitions 1 and 2 or does not satisfy any of them.

It is obvious that in the multiplicative system $(G,\cdot)$ condition (2) is not fulfilled /e.g. for $\alpha = 1$, $\beta = 0$/. It means that condition (2) is an essential condition for theorem 2.

From theorems 1 and 2 we immediately obtain the following

Corollary 1. If the multiplicative system $(G,\cdot)$ is associative then definitions 1 and 2 are equivalent iff condition (2) is fulfilled.

Condition (2) holds in every groupoid*/, because for an arbitrary pair $(\alpha, \beta) \in D$. we have:

$$(\alpha \cdot \beta) \cdot \beta^{-1} = \alpha \cdot (\beta \cdot \beta^{-1}) = \alpha .$$

Hence, from theorem 1 we obtain:

Corollary 2. If the multiplicative system $(G,\cdot)$ is a groupoid then definitions 1 and 2 are equivalent.

4. Extensions of the solutions of the translation equation

Theorem 3.

If a function $F : X \times G \multimap X$ is a solution of the translation equation in the sense of definition 2 and

(6) $\qquad D_F^1 \neq X$,

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*/ We use the term "groupoid" after W. Waliszewski in Note [2] .

$$(7) \qquad \bigwedge_{\alpha,\beta \in D_F^2} \left((\alpha,\beta) \in D. \Rightarrow \alpha\cdot\beta \in D_F^2 \right),$$

then this solution can be extended[x/] on $X \times D_F^2$.

Proof. Let us assume that function $F : X \times G \longrightarrow X$ satisfies the translation equation in the sense of definition 2 and F fulfils (6) and (7). Let a be an arbitrary fixed element of the set $X \setminus D_F^1$. We are going to prove that the function $\bar{F}$ defined as follows:

$$\bar{F}(x,\alpha) := \begin{cases} F(x,\alpha) & \text{for } (x,\alpha) \in D_F, \\ a & \text{for } (x,\alpha) \in (X \times D_F^2) \setminus D_F \end{cases}$$

is an extension on the set $X \times D_F^2$ of the solution F.

From the definition of the function $\bar{F}$ it follows that $D_{\bar{F}} = X \times D_F^2$. Hence, by (7) the conditions:

$$\bar{F}(x,\alpha\cdot\beta) \text{ is defined,}$$

$$\bar{F}\left[\bar{F}(x,\alpha),\beta\right] \text{ is defined,}$$

are equivalent for an arbitrary $x \in X$ and arbitrary $\alpha,\beta \in D_F^2$ such that $(\alpha,\beta) \in D$. In the case when $F(x,\alpha\cdot\beta)$ is defined /because F satisfies definition 2/ we have:

$$\bar{F}\left[\bar{F}(x,\alpha),\beta\right] = F\left[F(x,\alpha),\beta\right] = F(x,\alpha\cdot\beta) = \bar{F}(x,\alpha\cdot\beta).$$

Let now

$$(8) \qquad (x,\alpha\cdot\beta) \in (X \times D_F^2) \setminus D_F .$$

Then $\qquad \bar{F}(x,\alpha\cdot\beta) = a.$

Let us suppose that

$$\bar{F}\left[\bar{F}(x,\alpha),\beta\right] \neq a.$$

Then, from the definition of the function $\bar{F}$ we have:

$$(9) \qquad F(x,\alpha\cdot\beta) \in D_F .$$

From (9) and from the definition of the function $\bar{F}$ we have:

$$\bar{F}(x,\alpha) = a,$$

and hence

$$(10) \qquad (x,\alpha) \in D_F \qquad \text{and} \qquad \bar{F}(x,\alpha) = F(x,\alpha),$$

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x/ By "an extension of the solution of the translation equation" we understand such an extension that satisfies the translation equation in the same sense as the extending solution.

Thus from (9) and (10) $(x,\alpha\cdot\beta) \in D_F$ which is contrary to (8). It completes the proof.

From theorem 3 follows immediately the following

Corollary 3. Let us assume that the function $F : X \times G \multimap X$ satisfies the translation equation in the sense of definition 2. If F satisfies (8) and $D_F^2 = G$ or if F satisfies the following condition:

$$D_F = D_F^1 \times G, \tag{11}$$

then F can be extended to a solution of the translation equation on the set $X \times G$.

For the function $F : X \times G \multimap X$ we shall denote by G_x the following set:

$$G_x := \{\alpha : (x,\alpha) \in D_F\}.$$

Theorem 4.

If a function $F : X \times G \multimap X$ satisfies the translation equation in the sense of definition 2 and

$$\bigwedge_{x \in D_F^1} \bigwedge_{\alpha \in G} \bigvee_{\beta \in G} (\alpha\beta \in G_x), \tag{12}$$

then

(a) $D_F = D_F^1 \times G$,

(b) $C_F \subset D_F^1$.

Proof. Let $F : X \times G \multimap X$ be a solution of the translation equation in the sense of definition 2, satisfying (12). Let $x \in D_F^1$, $\alpha \in G$. It follows from (12) that there exists a $\beta \in G$ such that $\alpha\cdot\beta \in G_x$. It means that $(x, \alpha\beta) \in D_F$. Considering the fact that F satisfies definition 2 we have:

$$(x,\alpha) \in D_F ,$$

and hence for the function F condition (a) is held.

Let now $F(x,\alpha)$ be defined. It follows from (12) that there exists such an element $\beta \in G$ that $(x, \alpha\cdot\beta) \in D_F$. F satisfies definition 2 and therefore $F[F(x,\alpha),\beta] \in D_F$ and hence $F(x,\alpha) \in D_F^1$. Thus condition (b) is fulfilled and this completes the proof.

From theorem 4 and corollary 3 follows

Corollary 4. If $F : X \times G \multimap X$ is a solution of the translation equation in the sense of definition 2 and condition (12) holds then this solution can be extended on the set $X \times G$.

Condition (12) is an essential assumption for theorem 4. It is illustrated by the following

Example 3[*]. Let X be the set of the real numbers different from zero and let $(G,\cdot)$ be a semi-group of real numbers with multiplication.

We put:

$$F(x,\alpha) := x\alpha \quad \text{for} \quad x \in X,\ \alpha \neq 0.$$

It is easy to prove that F satisfies the translation equation in the sense of definition 2. It is easy to see that

$$D_F \neq D_F^1 \times G.$$

We shall show that solution F cannot be extended on the set X x G.

Let us suppose that $\bar{F}$ is an extension of the solution F on the set X x G. Then for arbitrary $\alpha \neq 0$, $x \in X$

(13) $$\bar{F}(x,0) = \bar{F}(x, 0\cdot\alpha) = \bar{F}[\bar{F}(x, 0),\alpha] = \bar{F}(x, 0)\alpha$$

is held.

From the definition of the set X we have:

$$\bar{F}(x, 0) \neq 0,$$

and hence equality (13) does not hold for $\alpha \neq 1$. Thus the solution F cannot be extended on the set X x G. For function F condition (12) is not satisfied.

For an arbitrary $x \in X$ we have:

$$G_x = X \setminus \{0\}$$

and hence for $\alpha = 0$ there does not exist a β such that $\alpha\beta \in G_x$.

We have shown that condition (12) is an essential assumption for theorem 4.

Example 3 shows, too, that not all solutions of the translation equation in the sense of definition 2 satisfying the condition $D_F = X \times D_F^2$ can be extended on the set X x G.

Theorem 5.

If the multiplicative system $(G,\cdot)$ satisfies the following condition:

(14) $$\bigwedge_{\alpha,\beta \in G} \ \bigvee_{\gamma \in G} (\alpha\gamma = \beta),$$

then every solution $F : X \times G \longrightarrow X$ of the translation equation in the sense of definition 1 as well as in the sense of definition 2 can be extended on the set X x G.

Proof. It is easy to see that condition (14) implicates condition (2). From this and from theorem 1 follows that $F : X \times G \longleftrightarrow X$ is a solution of the translation equation in the sense of definition 1 iff F is a solution of the translation equation in the sense of definition 2.

[*] Example 3 was given by Z. Moszner.

Let $F : X \times G \to X$ be a solution of the translation equation in the sense of definition 2. Let x be an arbitrary element of the set D_F^1. For x there exists $\beta \in G$ such that $(x, \beta) \in D_F$. From (14) it follows that for arbitrary $\alpha \in G$ there exists $\gamma \in G$ such that $\alpha \cdot \gamma = \beta$. Then $(x, \alpha \cdot \gamma) \in D_F$ which means that $(x, \alpha) \in D_F$. Hence $D_F = D_F^1 \times G$. From corollary 3 follows that the solution F can be extended to the solution on the set $X \times G$ which completes the proof of the theorem.

Theorem 6.

If the function $F : X \times G \to X$ satisfies the translation equation in the sense of definition 1 and

(15) $$\bigwedge_{\alpha \in G} \bigvee_{\beta \in G} \left((\alpha, \beta) \in D.\right),$$

then

$$C_F \subset D_F^1 .$$

The proof of theorem 6 is obvious.

Theorem 7.

If the function $F : X \times G \to X$ satisfies (11) and

(16) $$C_F \subset D_F^1 ,$$

then F is a solution of the translation equation in the sense of definition 1 iff F is a solution of the translation equation in the sense of definition 2.

Proof. Let $F : X \times G \to X$ be a solution of the translation equation in the sense of definition 1 satisfying (11) and (16). Let $(\alpha, \beta) \in D$. If $(x, \alpha \beta) \in D_F$ then by (11) $(x, \alpha) \in D_F$, and now using (16) we obtain $(F(x, \alpha), \beta) \in D_F$. If $(x, \alpha) \in D_F$ then $(x, \alpha \beta) \in D_F$. Thus *function F satisfies definition 2.*

Let now the function $F : X \times G \to X$ be a solution of the translation equation in the sense of definition 2 satisfying (11) and (16). Let $(\alpha, \beta) \in D$. and $(x, \alpha) \in D_F$. Then $(x, \alpha \beta) \in D_F$ and $(F(x, \alpha), \beta) \in D_F$. Thus function F satisfies definition 2 which completes the proof.

From theorem 7 and corollary 3 follows

Theorem 8.

If the solution $F : X \times G \to X$ of the translation equation in the sense of definition 1 satisfies (11) and (16) then it can be extended on the set $X \times G$.

There exist solutions of the translation equation in the sense of definition 1 such that they cannot be extended to a solution on the set $X \times G$. It is illustrated by the following

Example 4[x/]. Let us put:

$$X := \{a, b, c\},$$

where a, b, c are different elements,

$$G := \{0_1, 0_2, y_1, y_2\}.$$

We define the multiplication " . " in G as follows:

	0_1	0_2	y_1	y_2
0_1	0_1	0_1	0_1	0_1
0_2	0_2	0_2	0_2	0_2
y_1	y_2	y_1	0_2	0_1
y_2	y_1	0_1	0_1	0_2

We put:

$$F(a, x) = a \quad \text{for } x \in G,$$
$$F(b, x) = b \quad \text{for } x \in G,$$
$$F(c, 0_1) = a,$$
$$F(c, 0_2) = b.$$

It is easy to verify that function F defined in such a manner satisfies definition 1. Function F cannot be extended to a solution $\bar{F}$ of the translation equation on the set $\bar{X} \times G$ where $\bar{X} \supset X$.

Let us suppose that such function $\bar{F}$ exists and let us denote:

$$\bar{F}(c, y_1) = d.$$

Then

$$\bar{F}(d, y_1) = \bar{F}[\bar{F}(c, y_1), y_1] = \bar{F}(c, y_1 \cdot y_1) = F(c, 0_2) = b$$

and

$$\bar{F}(d, y_2) = \bar{F}[\bar{F}(c, y_1), y_2] = \bar{F}(c, y_1 \cdot y_2) = F(c, 0_1) = a.$$

From this and from the definition of function F we obtain:

$$a = F(a, y_2) = F[F(d, y_2), y_2] = \bar{F}(d, y_2 \cdot y_2) = \bar{F}(d, 0_2),$$
$$b = F(b, y_1) = F[F(d, y_1), y_1] = \bar{F}(d, y_1 \cdot y_1) = \bar{F}(d, 0_2)$$

which is impossible because $a \neq b$.

x/ Given by Z. Moszner.

B i b l i o g r a p h y

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[3] A. Zajtz: Algebraic Objects, Zeszyty Naukowe UJ, Prace Matematyczne, Zeszyt 12 /1968/, pp.67-72.