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LIMITING DISTRIBUTIONS AND THEIR APPLICATIONS

1. Historical Outline

The results of the researches concerning the problems of random character are often treated as scientific prognoses produced on the base of the mathematical statistics and the calculus of probability.

Beside the classic schema of sampling and discrete functions of distributions three limiting distributions are being used at present with increasing success to study the extreme terms of the distribution series. The theory was initiated in 1927, when M. Fréchet [3] first derived the formula and published in Kraków one of the extreme distributions of the maximum term of the distributing series. This distribution, however changed, is still known, as the II limiting distribution /second kind/ or as the Fréchet's distribution. Furthermore, M. Frechet proved, that the extreme terms of the sample, no matter whether they have or not the different matrix distributions can have mutual limiting distributions; it is only necessary that these distributions should have the same common properties. It is also to his credit, that he introduced the postulate of stability to the theory of extremes according to which the extreme distributions correspond to the initial distributions with the accuracy of the linear transformations. R.A. Fisher and L.H.C. Tippit [2] subsequenty took adventage of the mentioned postulate and in 1928 derived two other extreme distributions, known today as the I and III limiting distributions /the first and the third kind/ or as the Gumbel's and Weinbull's distributions.

The work [2] was a basis for further researches connected with the problems of the extreme values. In the thirthies /1930-1940/ there was less interest in the analysis of extremes, at that time R.V. Mises [8] classified the matrix distributions into three types, depending on three limiting distributions. There were no proper results of the researches confirming the application of this theory in practica, except for sets of the meteorological and hydrogeological phemena, but containing only few data.

However already in 1939 W. Weibull [12] extended the range of application of the theory of extremes to the problems of the strength of materials describing the effect of the scale. The law which he proposed /indepedently from Fisher and Tippet/ in the form of III limiting distribution of minimum to describe the unitary limit od strength of the brittle bodies has been found true and broadly employed. No doubt that this considerably influenced a further and more intensive development of the analysis of the extreme values and showed wide possibilities of its application in practice.

Since the World War II many papers confirming this above mentioned have been published.

Researches conducted by B.W. Gniedenko [4], who formulated necessary and sufficient conditions of covergence for every possible limiting distribution are of particular interest. After the war a fast development of the technics has been obserwed and alongside the increase of interest in the extremes distributions and their practical application.

Among many researches conducted in the late fourth and early fifth decade of this century particular interest are those of E.J. Gumbel [5], concerning first of all the I limiting distribution. At the end of his researches he produced an extensive monography [6] containing the total knowledge of the theory of extremes up to the year 1958. This work has a unique place in the world literature of this kind until now. In the last ten years the methods of probability have been used in almost all branches of science. They have become a more objective research tool in comparison with traditional methods used before. There is no reason to delay the development of these methods considering them to be much time -consuming, because there is a possibility of a common use of computers. The theory of the extreme values, is of great importance in this process although it is not popular in this country. Beside a general description of this theory in this paper [11], J. Murzewski [9],[10] gave much consideration to the application of this theory to the security of building constructions and to the analysis of the measurements results and Z. Kaczmarek to applying it to meteorogical and hydrogeological phenomena.

It seems, that the limiting distributions should be applied more extensively than they were before in studying the nature's phenomena and in solving technical problems, but at any rate not less than any other distributions often used in a wrong manner.

2. Precise distributions of the extreme values

Let us take into consideration a general population characterized by single dimensional random variable X having the distribuant F/x/ and let us sample from this general population a random sample with the numbers n. Let us treat the elements of the sample $x_1, x_2, \dots x_n$ as the sequence of the values of independent random variable $x_1, x_2, \dots x_n$ ordered increasingly or decreasingly. Then, the distribution of the arbitrary ordinal statistics X_k, $k = 1,2,3,\dots n$ may be written down in the following combinatoric form:

$$/1/ \quad \begin{cases} \varphi_k/x/\,dx = n\binom{n-1}{k-1}\left[F/x/\right]^{k-1}\left[1-F/x/\right]^{n-1} \cdot f/x/\,dx, \\ \varphi_k/x/\,dx = n\binom{n-1}{k-1}\left[F/x/\right]^{n-k}\left[1-F/x/\right]^{k-1} \cdot f/x/\,dx \end{cases}$$

adequately to the sample ordered increasingly and decreasingly, where $f/x/ = F'/x/$. The first one of these formulae states a probability that in the sample including n values k-1 values are x, n-k values are x+dx, and exactly one value is contained in the range x,x+dx. An analogical interpretation may be given for the second formula.We are interested in the distributions of the extreme values of x_1 and x_2, which may be taken as particular cases of the formula/1/ when $k = 1$ and $k = n$. So we obtain for the smallest term x_1, and for the biggest x_n and the fallowing formulae for the density functions:

$$/2/ \quad \begin{cases} \varphi_1/x/ = n\left[1 - F/x/\right]^{n-1} \cdot f/x/, \\ \varphi_n/x/ = n\left[F/x/\right]^{n-1} \cdot f/x/ \,. \end{cases}$$

The elementary integration of the above formulae leads to the functions:

$$/3/ \quad \begin{cases} \Phi_1/x/ = 1 - \left[1 - F/x/\right]^{n}, \\ \Phi_n/x/ = \left[F/x/\right]^{n} . \end{cases}$$

These are the accurate distribuants of the minimum and maximum terms of the distributional sequence. Their form depends on the matrix distribution F/x/ of the general population and on the sample numbers n. Therefore for each sample having the determined volume n and the known initial function f/x/ the actual distribution of probability can be given and its characteristics calculated.

Further the formulae for some characteristics of the distribution /3/, being of particular importance in the statistics of extremes are presented. They can be calculated according to these formulae:

- mode $\tilde{x}$,

/4/
$$\begin{cases} \left. \dfrac{f'/x/}{f^2/x/} - \dfrac{n-1}{1-F/x/} \right|_{x=\tilde{x}_1} = 0, \\ \left. \dfrac{f'/x/}{f^2/x/} + \dfrac{n-1}{F/x/} \right|_{x=\tilde{x}_n} = 0, \end{cases}$$

- median $\check{x}$,

/5/
$$1 - F/\check{x}_1/ = F/\check{x}_n/ = \exp\left[-\frac{1}{n} \ln 2 \right],$$

- moments $\bar{x}^{\,l}$ /mean $\bar{x}$ for $l - 1$/,

/6/
$$\begin{cases} \bar{x}_1^{\,l} = n \int_{-\infty}^{\infty} x^l \left[1 - F/x/ \right]^{n-1} d\,F/x/, \\ \bar{x}_n^{\,l} = n \int_{-\infty}^{\infty} x^l \left[F/x/ \right]^{n-1} d\,F/x/, \end{cases}$$

- characteristic extremum $\check{x}$,

/7/
$$\begin{cases} F/\check{x}_1/ = \dfrac{1}{n}, \\ F/\check{x}_n/ = 1 - \dfrac{1}{n}, \end{cases}$$

- extremal intensity $\mu/\check{x}/$,

/8/
$$\begin{cases} \mu/\check{x}_1/ = \alpha_1 = \dfrac{f/\check{x}_1/}{1 - \left[1 - F/\check{x}_1/\right]} = n\,f/\check{x}_1/, \\ \mu/\check{x}_n/ = \alpha_n = \dfrac{f/\check{x}_n/}{1 - F/\check{x}_n/} = n\,f/x_n/, \end{cases}$$

- period of repeatability of the characteristic extreme $T/\check{x}/$,

/9/
$$\begin{cases} T/\check{x}_1/ = \dfrac{1}{1 - \left[1 - F/\check{x}_1/\right]} = n, \\ T/\check{x}_n/ = \dfrac{1}{1 - F/\check{x}_n/} = n, \end{cases}$$

- period of repeatability of the mode $T/\tilde{x}/$,

/10/
$$\begin{cases} T/\tilde{x}_1/ = \dfrac{1}{F/\tilde{x}_1/}, \\ T/\tilde{x}_n/ = \dfrac{1}{1 - F/\tilde{x}_n/}. \end{cases}$$

It is very difficult to use the above formulae in order to calculate the moments /6/, as it is seldom possible to get an effective solution.

For example, let consider the case, when the expression $F/x/=1-e^{-x}$ belongs to the distributions of the exponential type of class 2. In this case the functions:

$$\varphi_n/x/ = n\left[1 - e^{-x}\right]^{n-1} \cdot e^{-x},$$

$$\Phi_n/x/ = \left[1 - e^{-x}\right]^{n}$$

determine the distribution of the maximum term.

The diagram of the function of density for some exemplary values n has been presented in drawing 1, and some characteristics have been snown /we do not present the calculations, as they are rather simple/ by the following formulae:

- mode /characteristic maximum/

$$\tilde{x}_n = \check{x}_n = \ln n,$$

- median

$$\check{x}_n = - \ln /1 - 2^{-\frac{1}{n}}/,$$

- mean

$$\bar{x}_n = \sum_{k=1}^{n} \frac{1}{k},$$

- variance

$$\sigma_n^2 = \sum_{k=1}^{n} \frac{1}{k^2}.$$

The above procedure is to be followed if n is of small order, but when n is great, then asymptotic distribution functions should be used, assuming the accurate form by $n \rightarrow \infty$.

3. Asymptotic distributions of the extreme values

If the sample contains enough numbers and the function F /x/ belongs to one of three determined types of the matrix distributions, then the distributions of the extreme term of the distributing sequence can be calculated by $n \rightarrow \infty$ from the formula /2/ or /3/. Before their form is presented we shall discuss the types of initial distributions using the classification presented by E.J. Gumbel [6] in his paper, although this classification is not a thorough and final one.

According to the above classification the probability functions can be divided into three types:

I. exponential type,

II. Cauchy type,

III. limited type.

To the exponential type belong distributions with all moments and a positive critical ratio Q/x/

$$/11/ \quad Q/x/ = \frac{\mu/x/}{-f'/x/ \cdot f/x/} = \frac{f^2/x/}{f'/x/ \cdot [1 - F/x/]} > 0.$$

Gumbel divided distributions of the exponential type into three classes, depending on the behaviour of the formula

/11/ for $x \to \infty$

$$/12/ \qquad Q/x/ = \begin{cases} 1 + \varepsilon/x/ & \text{for the } 1^{st} \text{ class,} \\ 1 & \text{for the } 2^{nd} \text{ class,} \\ 1 - \varepsilon/x/ & \text{for the } 3^{rd} \text{ class,} \end{cases}$$

where $\lim_{x \to \infty} \varepsilon/x/ = 0$.

The reason of this classification may be also the distributions characteristics /3/, permitting to replace formula /12/ by equivalent equations. Full list of these conditions has been presented in table 1.

T a b l e 1

Three classes of the exponential distribution

Characteristics	Symbol	Class 1	Class 2	Class 3
critical ratio	Q/x/	>1	$=1$	<1
mode	$\tilde{x}_n$	$>\check{x}_n$	$=\check{x}_n$	$<\check{x}_n$
period of mode's repeatability	$T/\tilde{x}_n/$	$>n$	$=n$	$<n$
characteristic maximum	$\check{x}_n$	$<\ln n$	$=\ln n$	$>\ln n$
density value for $\check{x}_n$	$\varphi_n/\check{x}_n/$	increases with the increase of n	constant	decreases with the increase of n
maximum intensity		increases with the increase of n	constant	decreases with the increase of n
intensity -derivative	$\mu'/\check{x}_n/$	>0	$=0$	<0
curvature $\check{x}_n/\ln n/$	$\check{x}_n''/\ln n/$	concave	straight	convex

Using for example criterion /12/ for logistic and logarythmic-normal exponential distributions it is easy to find that they belong to the exponential type of the 1^{st}, 2^{nd}, 3^{rd} class, respectively.

Cauchy type of distributions have not all, or any moment but fulfil the condition:

$$\lim_{x \to \infty} x^k \left[1 - F/x/\right] = A > 0; \quad k > 0, \quad x \geqslant 0 \qquad /13/$$

or both following conditions simultaneously

$$\begin{cases} \lim_{x \to \infty} x^k \left[1 - F/x/\right] = A > 0; \quad k > 0, \\ \lim_{x \to -\infty} \left[-x\right]^{k_1} \cdot F/x/ = A_1 > 0; \quad k_1 > 0. \end{cases} \qquad /14/$$

The distributions fulfilling only condition expressed in formula /13/ or the equivalent equation:

$$\lim_{x \to \infty} x \, \mu/x/ = k > 0; \quad x \geqslant 0 \qquad /13'/$$

are often called Pareto type distributions. The functions of Cause type are also divided into three classes depending on the value of the power index k or k_1, like exponential type distributions:

$$\begin{cases} k > 1 & \text{for class } 1^{st}, \\ k = 1 & \text{for class } 2^{nd}, \\ k < 1 & \text{for class } 2^{rd}. \end{cases} \qquad /15/$$

However, this distribution does not depend on the amount of numbers contained in the sample. Two types of the matrix distributions presented above are unlimited on both sides, or on one side for an this extreme value, whose limiting distributions are being searched. Consequently the initial functions belonging to the I or II type, limited on the right side lead to the limiting minimum distribution, and the functions limited on the left side - to the limiting maximum distributions.

The third limited type of matrix distributions concerns functions limited on the left side, while looking for minimum test of distributing sequence, or on the right side, while studying the distribution of the maximum test. This type of distributions has been called the distribution of the power type /paper [9]/. In another paper [10] by the same author as by paper [9] that ferm includes the Cauchy type. Gumbel does not specify any conditions to be fulfilled by the functions belonging to this type of the matrix distributions. The conditions of B.W. Gniedenko [4]

$$\lim_{n\to\infty} n\left[1 - F/\check{x}_n + \frac{t}{\alpha_n}\right] = e^{-t}, \quad t = \alpha_n/x - \check{x}_n/,$$

/16/ $$\lim_{x\to\infty} \frac{1 - F/x/}{1 - F/cx/} = c^k,$$

$$\lim \frac{1 - F/cx + \omega/}{1 - F/x - \omega/} = c^k,$$

where:

$$\alpha_n > 0, \quad k > 0, \quad c > 0, \quad \varepsilon > 0,$$

$$F/\check{x}_n/ = 1 - \frac{1}{n}, \quad F/\check{x}_n + \frac{1}{\alpha_n}/ = 1 - \frac{1}{nc},$$

$$F/\omega/ = 1, \quad F/\omega - \varepsilon/ < 1,$$

mentioned also in papers [6] and [9], are the conditions necessary and sufficient to determine in turn all above discussed types of the initial functions.

The above discussed types of the matrix distributions, applied to formulae /2/ or /3/ when $n \to \infty$ lead to respective three types of the stable limiting distributions /I, II, III/, while in each on the distribution of the minimum term is connected with the distribution of the maximum term by the principle of symetry.

A detailed discussion of these functions and their application will be presented in the next paper; the present paper gives only the density and distribuants of the mentioned functions and illustrates the density functions on the real numerical values of the parameters of distribution.

I. Limiting distribution according to Gumbel

a/ for minimum

/17/ $$\begin{cases} \varphi_1/x/ = \alpha \exp\left[\alpha/x - \check{x}_1/ - e^{\alpha/x - \check{x}_1/}\right], \\ \Phi_1/x/ = 1 - \exp\left[-e^{\alpha/x - x_1/}\right], \end{cases}$$ /Fig. 2 and 3/

b/ for maximum

/18/ $$\begin{cases} \varphi_n/x/ = \alpha \exp\left[-\alpha/x - \check{x}_n/ - e^{-\alpha/x - \check{x}_n/}\right], \\ \Phi_n/x/ = \exp\left[-e^{-\alpha/x - x_n/}\right], \end{cases}$$ /Fig. 2 and 3/

where $\alpha > 0$, $\check{x}$ parameters of distribution.

II. Limiting distribution according to Frechet

a/ for minimum

/19/
$$\begin{cases} \varphi_1/x/ = \dfrac{k}{\omega - \check{x}_1}\left(\dfrac{\omega - \check{x}_1}{\omega - x}\right)^{k+1} \cdot \exp\left[-\left(\dfrac{\omega - \check{x}_1}{\omega - x}\right)^{k}\right], & \text{/fig. 4 and 5/,} \\ \Phi_1/x/ = 1 - \exp\left[-\left(\dfrac{\omega - \check{x}_1}{\omega - x}\right)^{k}\right], \quad x \leqslant \omega , & \end{cases}$$

b/ for maximum

/20/
$$\begin{cases} \varphi_n/x/ = \dfrac{k}{\check{x}_n - \varepsilon}\left(\dfrac{\check{x}_n - \varepsilon}{x - \varepsilon}\right)^{k+1} \cdot \exp\left[-\left(\dfrac{\check{x}_n - \varepsilon}{x - \varepsilon}\right)^{k}\right], & \text{/fig. 4 and 5/,} \\ \Phi_n/x/ = \exp\left[-\left(\dfrac{\check{x}_n - \varepsilon}{x - \varepsilon}\right)^{k}\right], \quad x \geqslant \varepsilon , & \end{cases}$$

where $k > 0$, $\check{x}$, $0 \leqslant \varepsilon$, $\check{x}_n$, $\omega > \check{x}_1$ parameters of distribution.

III. Limiting distribution after Weinbull

a/ for minimum

/21/
$$\begin{cases} \varphi_1/x/ = \dfrac{k}{\check{x}_1 - \varepsilon}\left(\dfrac{x - \varepsilon}{\check{x}_1 - \varepsilon}\right)^{k-1} \cdot \exp\left[-\left(\dfrac{x - \varepsilon}{\check{x}_1 - \varepsilon}\right)^{k}\right], & \text{/Fig.6 and 7/} \\ \Phi_1/x/ = 1 - \exp\left[-\left(\dfrac{x - \varepsilon}{\check{x}_1 - \varepsilon}\right)^{k}\right], \quad x \geqslant \varepsilon & \end{cases}$$

b/ for maximum

/22/
$$\begin{cases} \varphi_n = \dfrac{k}{\omega - \check{x}_n}\left(\dfrac{\omega - x}{\omega - \check{x}_n}\right) \cdot \exp\left[-\left(\dfrac{\omega - x}{\omega - \check{x}_n}\right)^{k}\right], & \text{/Fig.6 and 7/} \\ \Phi_n = \exp\left[-\left(\dfrac{\omega - x}{\omega - \check{x}_n}\right)^{k}\right], \quad x \leqslant \omega , & \end{cases}$$

where $k > 0$, $\check{x}$, $\check{x}_1 \geqslant \varepsilon \geqslant 0$, $\omega > \check{x}_n$ parameters of distribution.

The distributions of the minima can be obtained from the distributions of maxima and vice versa,using the prinple of symetry

/23/ $$\varphi_1/x/ = \varphi_n/-x/,\ \Phi_1/x/ = 1 - \Phi_n/-x/,\ \check{x}_n = -\check{x}_1,\ \omega = -\varepsilon .$$

Unlike in the ordinary statistics where the dominanting role is played by mean value X in the theory of extreme statistics the most important is parameter x, called the characteristic extreme.

The second important parameter is the extreme intensity, $\mu/\check{x}/=\alpha$. Both values are function of inital distributions and the volume of the sample n. They can be therefore estimated by using equations/7/ and /8/ or using properly prepared statistical data.In connection with this all observed extreme included in the studied distributing sequence should be reliable and obtained in conditions identical from the point of view of statistics. Furthermore the studies should ensure observations to be stochasticly independent, and numerousity big enough. It sometimes happens, that the results of studies do not lead to asymptotic theory of the extreme terms. In this case,however,the matrix distribution of the studies statistic does not fulfil the conditions of B.W. Gniedenko /16/ nor of E.J. Gumbel /12/ and /14/. However, if the initial distribution is not recognized there is no reason to question the application of the asymptotic distributions in practice.

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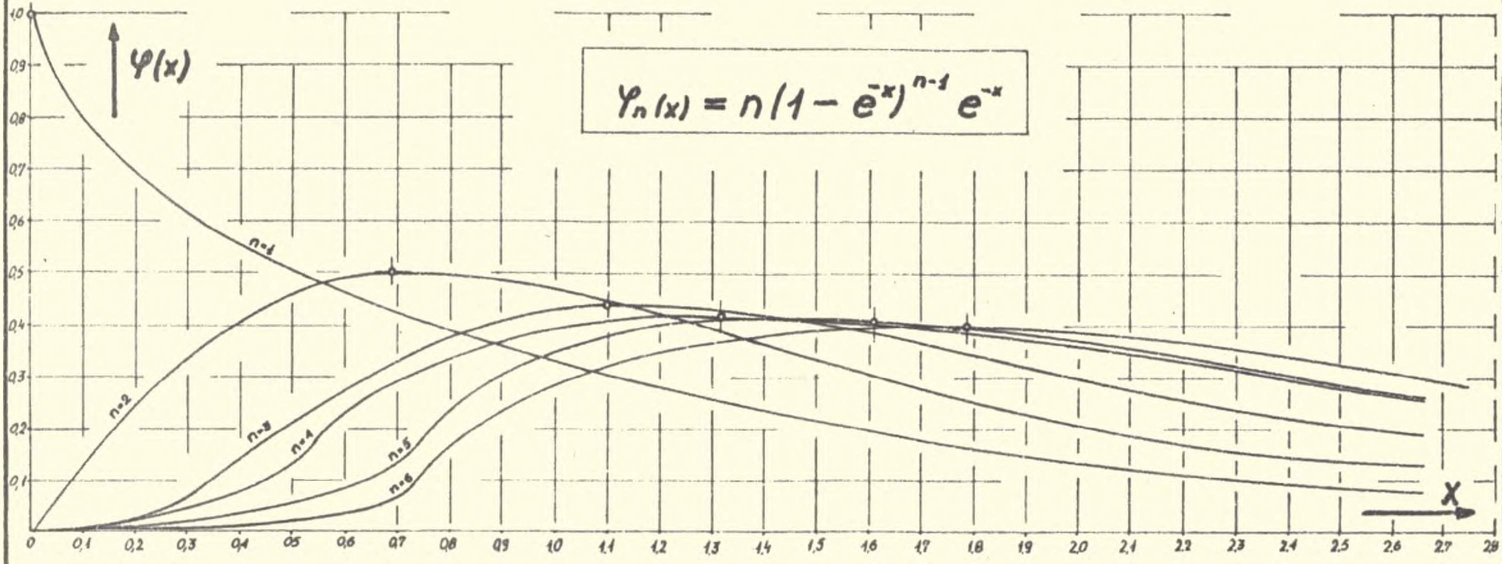

φ(x)
φn (x) = n (1 − e^-x)^n-1 e^-x
n=1
n=2
n=3
n=4
n=5
n=6
X
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
1,1
1,2
1,3
1,4
1,5
1,6
1,7
1,8
1,9
2,0
2,1
2,2
2,3
2,4
2,5
2,6
2,7
2,8

Fig. Nr 1.

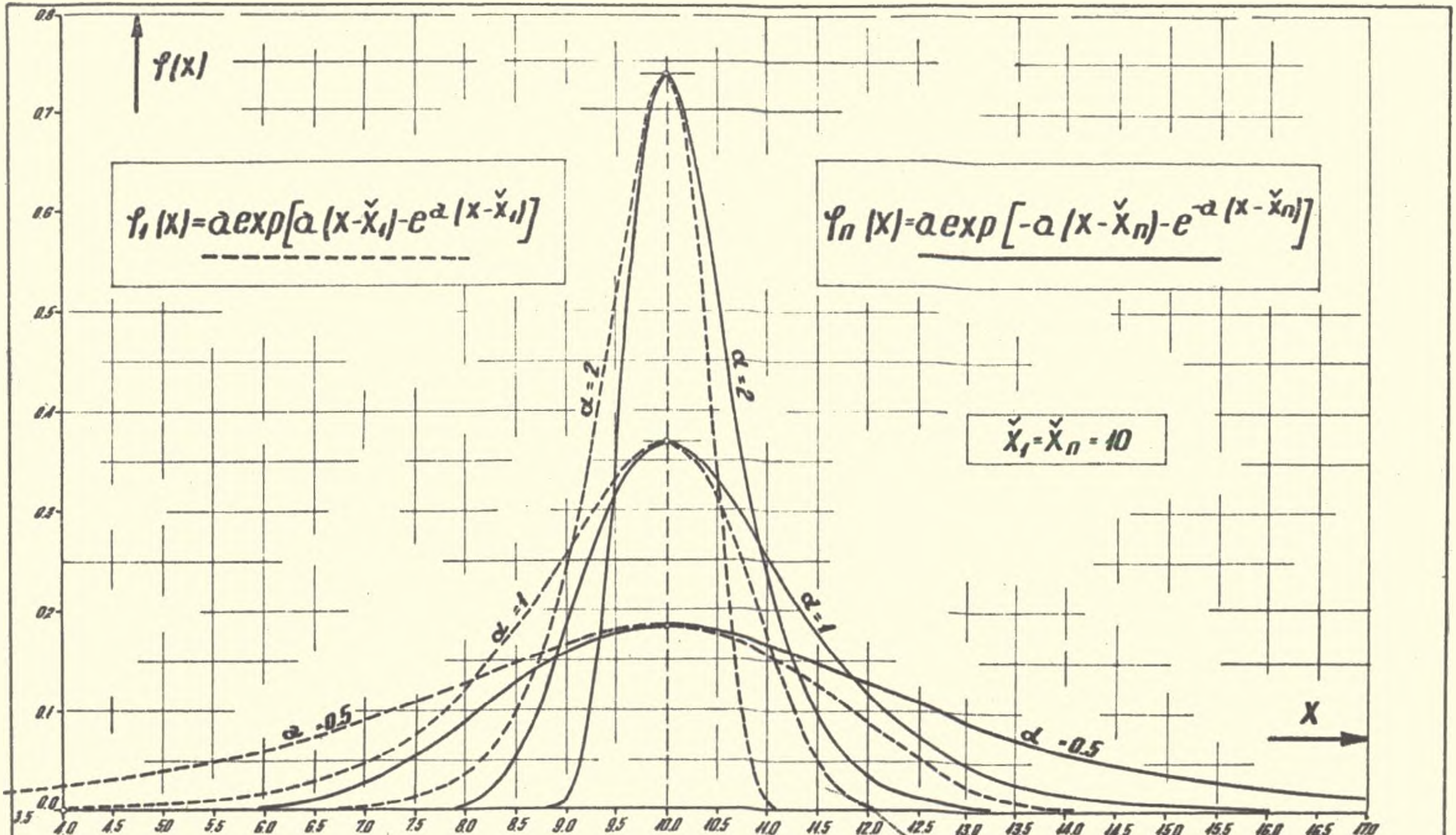
φ(x)
φ₁(x)=αexp[α(x-x̌₁)-e^{α(x-x̌₁)}]
φₙ(x)=αexp[-α(x-x̌ₙ)-e^{-α(x-x̌ₙ)}]
x̌₁=x̌ₙ=10
α=2
α=2
α=1
α=1
α -0.5
α -0.5
X
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
8.0
8.5
9.0
9.5
10.0
10.5
11.0
11.5
12.0
12.5
13.0
13.5
14.0
14.5
15.0
15.5
16.0
16.5
17.0

Fig.Nr 2.

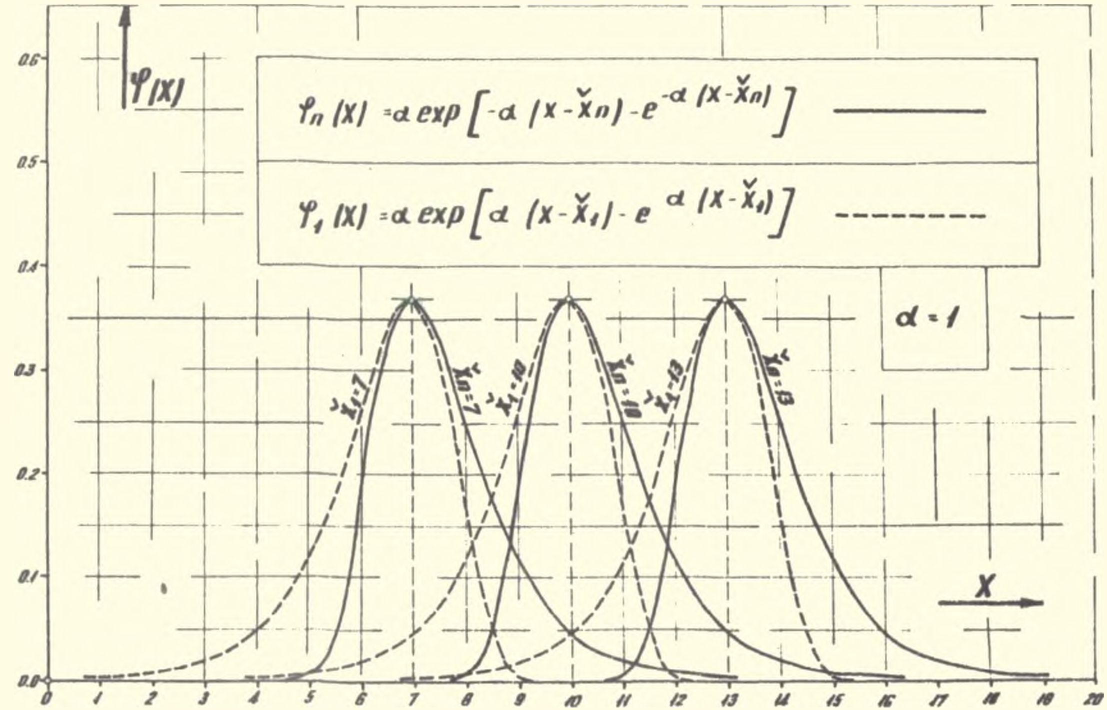
φ(X)
φn (X) = α exp [-α (X - X̌n) - e^-α (X - X̌n)]
φ1 (X) = α exp [α (X - X̌1) - e^α (X - X̌1)]
α = 1
X̌1=7
X̌n=7
X̌1=10
X̌n=10
X̌1=13
X̌n=13
X
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

Fig. Nr 3.

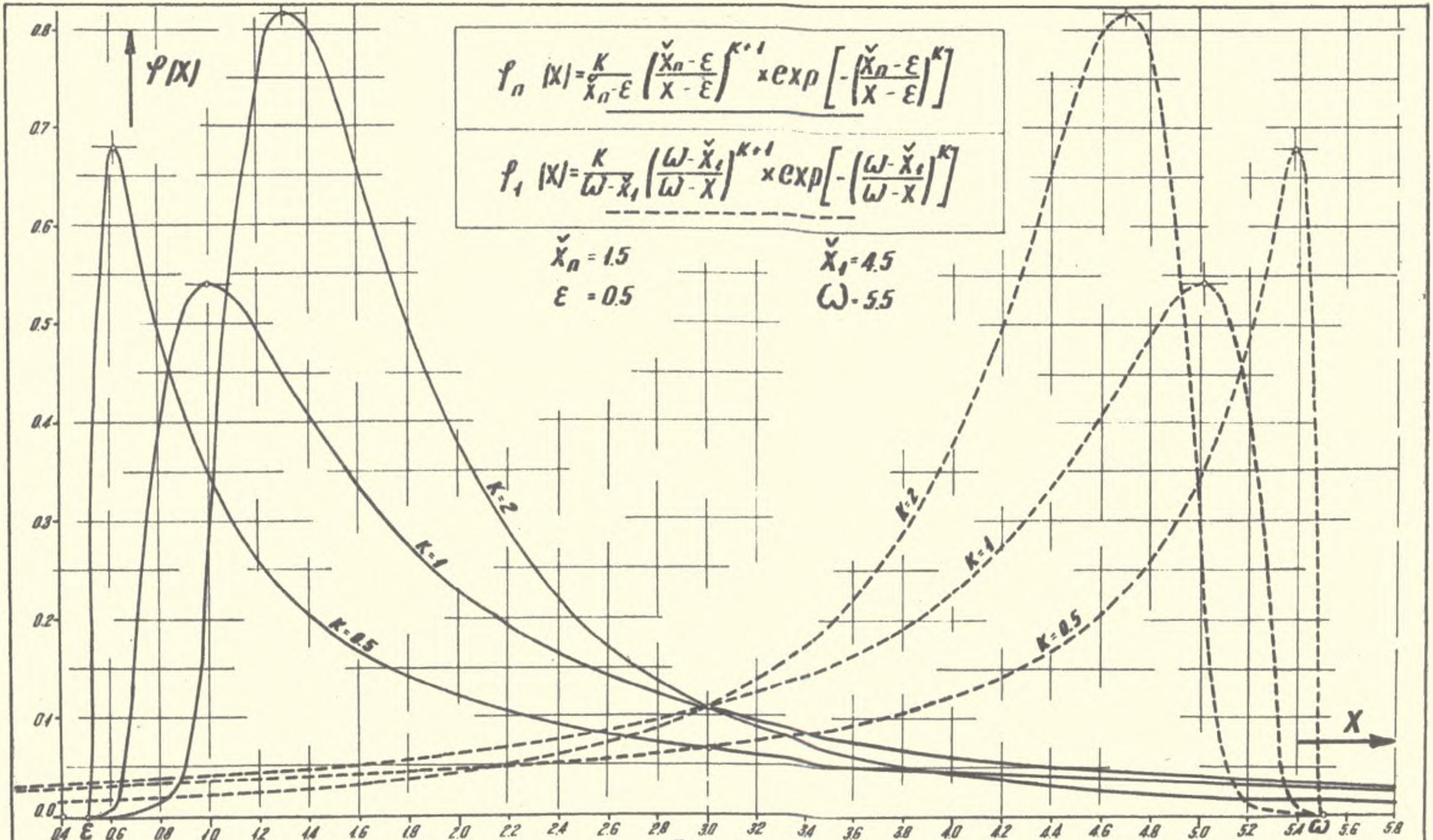
$f_n(X) = \frac{K}{\check{X}_n - \varepsilon} \left(\frac{\check{X}_n - \varepsilon}{X - \varepsilon}\right)^{K+1} \times \exp\left[-\left(\frac{\check{X}_n - \varepsilon}{X - \varepsilon}\right)^K\right]$
$f_1(X) = \frac{K}{\omega - \check{X}_1} \left(\frac{\omega - \check{X}_1}{\omega - X}\right)^{K+1} \times \exp\left[-\left(\frac{\omega - \check{X}_1}{\omega - X}\right)^K\right]$
$\check{X}_n = 1.5$
$\varepsilon = 0.5$
$\check{X}_1 = 4.5$
$\omega = 5.5$
$\varphi(X)$
X
K=2
K=1
K=0.5
K=2
K=1
K=0.5
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.4 ε 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8 3.0 3.2 3.4 3.6 3.8 4.0 4.2 4.4 4.6 4.8 5.0 5.2 5.4 ω 5.6 5.8

Fig. Nr 4.

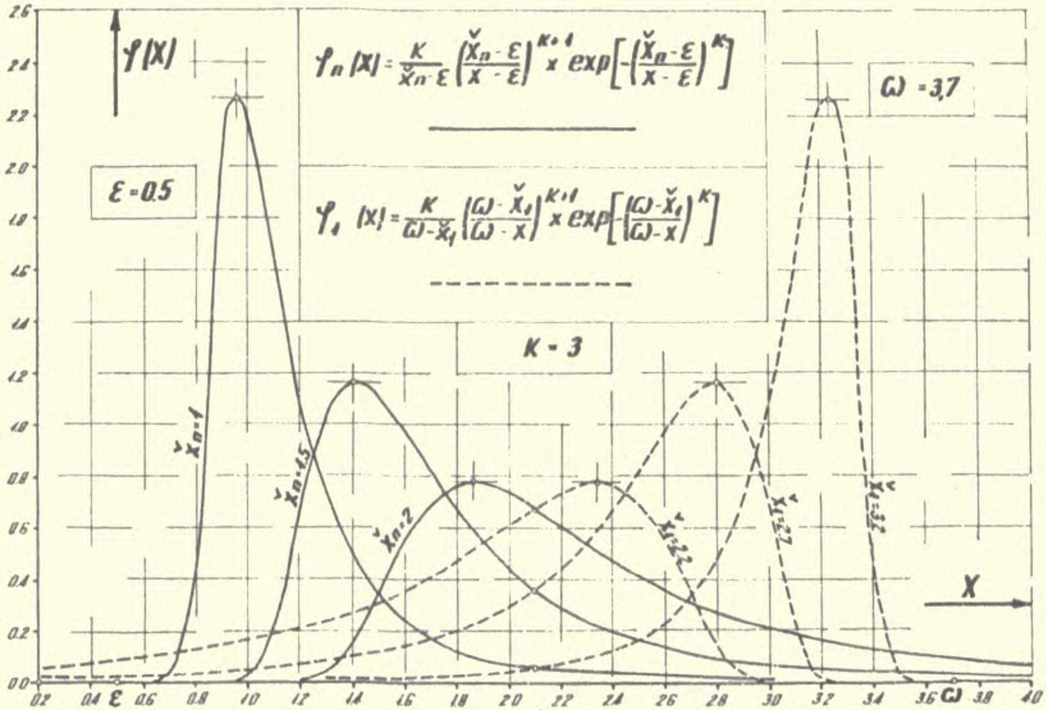
φ(X)
φn (X) = K/(X̌n - ε) ((X̌n - ε)/(X - ε))^(K+1) x exp[-((X̌n - ε)/(X - ε))^K]
φ1 (X) = K/(ω - X̌1) ((ω - X̌1)/(ω - X))^(K+1) x exp[-((ω - X̌1)/(ω - X))^K]
ε = 0.5
ω = 3,7
K = 3
X̌n = 1
X̌n = 1.5
X̌n = 2
X̌1 = 2.2
X̌1 = 2.7
X̌1 = 3.2
X

Fig. Nr 5.

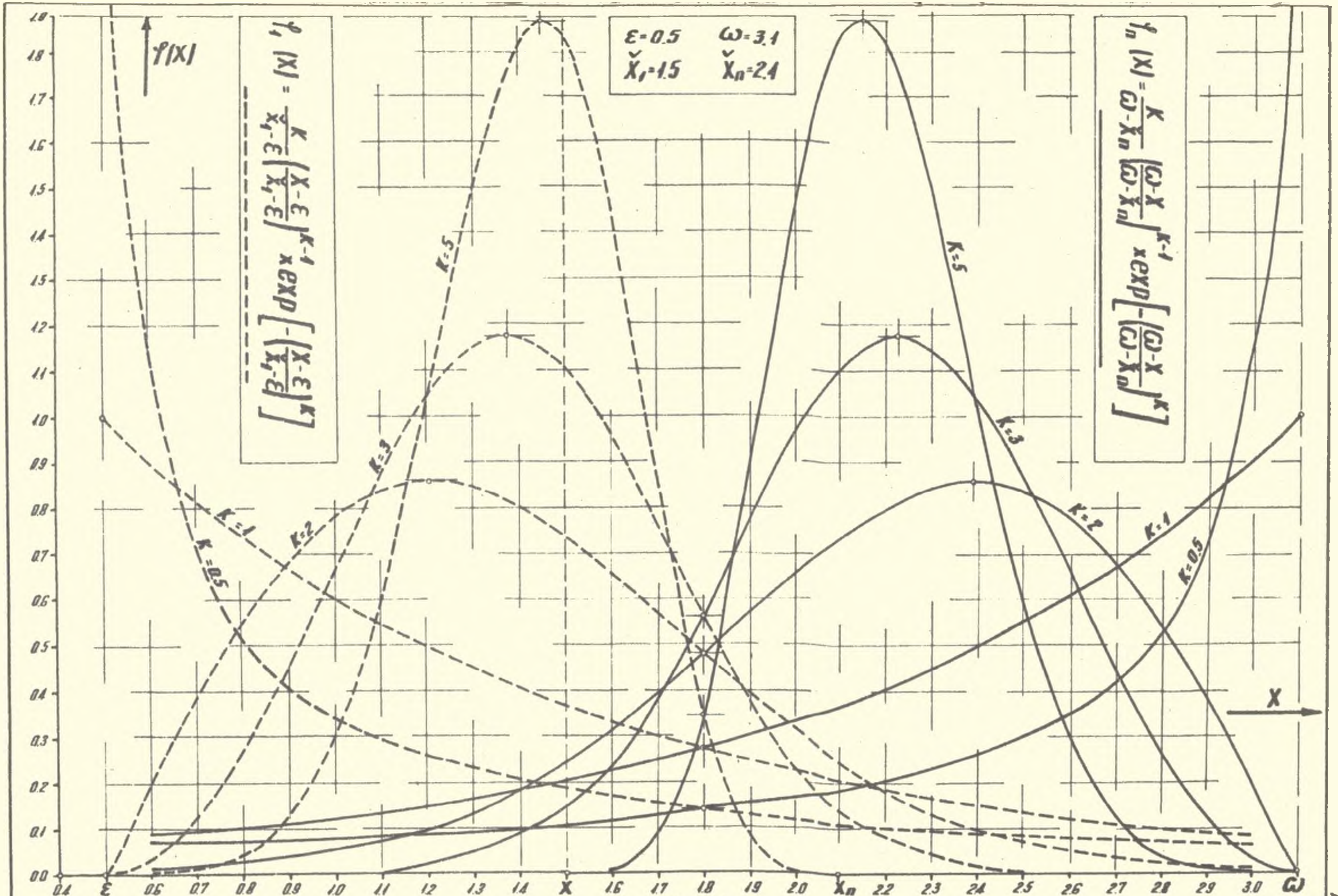
φ(X)
φ₁(X) = K/(X̌₁−ε) · ((X−ε)/(X̌₁−ε))^(K−1) × exp[−((X−ε)/(X̌₁−ε))^K]
ε=0.5 ω=3.1
X̌₁=1.5 X̌n=2.4
φn(X) = K/(ω−X̌n) · ((ω−X)/(ω−X̌n))^(K−1) × exp[−((ω−X)/(ω−X̌n))^K]
K=5
K=3
K=2
K=1
K=0.5
X
ε
ω
Xn

Fig. Nr 6.

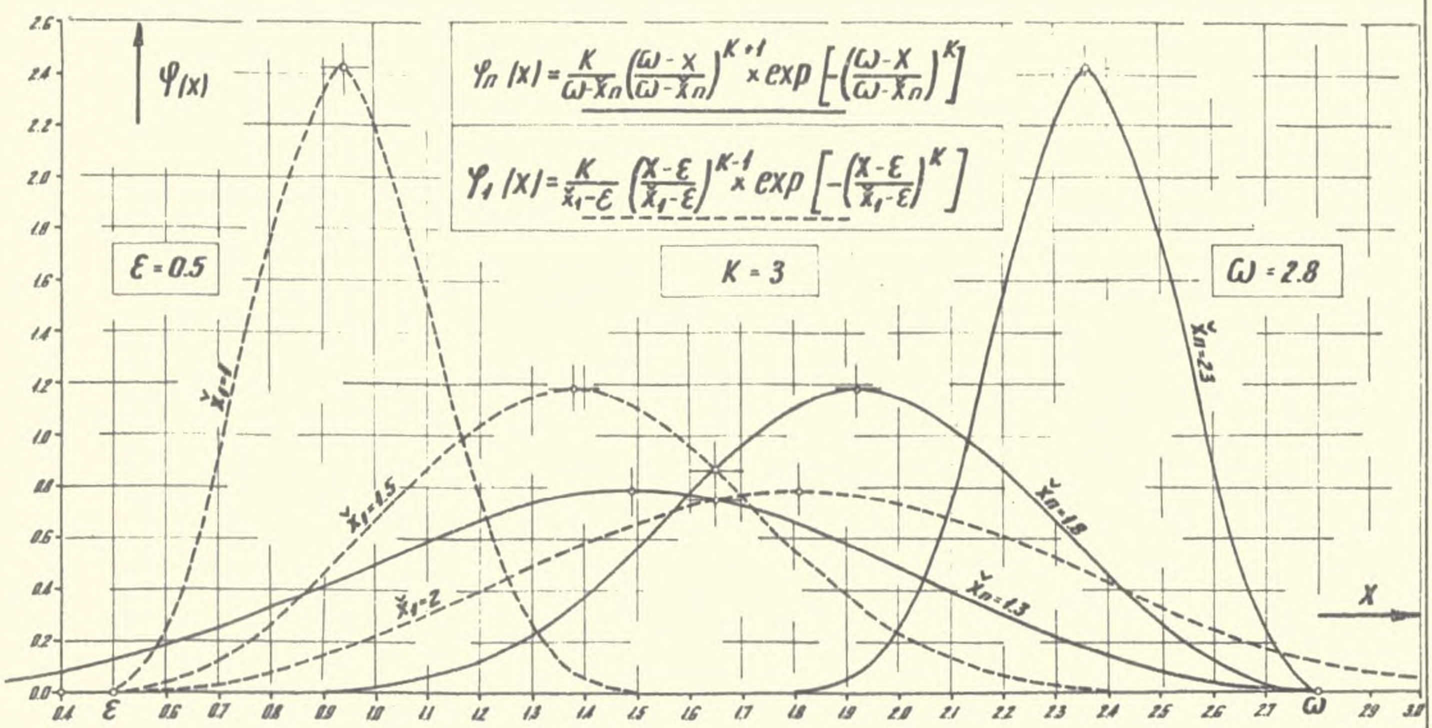
φn (x) = K/(ω-x̌n) ((ω-x)/(ω-x̌n))^(K+1) × exp [-((ω-x)/(ω-x̌n))^K]
φ1 (x) = K/(x̌1-ε) ((x-ε)/(x̌1-ε))^(K-1) × exp [-((x-ε)/(x̌1-ε))^K]
φ(x)
ε = 0.5
K = 3
ω = 2.8
x̌1=1
x̌1=1.5
x̌1=2
x̌n=1.3
x̌n=1.8
x̌n=2.3
x
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.4 ε 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 ω 2.9 3.0

Fig. Nr 7.