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Jan Milewski

# ON CERTAIN BICALORIC PROBLEMS

1. In the paper we consider the equation

(1) $$P^2u(x,t) = 0,$$

where $Pu(x,t) = D_{x^2}\left[u(x,t)\right] - D_t\left[u(x,t)\right]$.

We shall solve the limit problem for the equation (1) in the domain $R_2^+$ with initial and boundary data. The analogous problem in the three dimentional time-space was considered in the paper [2].

2. The fundamental solution of the equation

(2) $$Pu(x,t) = 0$$

is given by formula

(3) $$U(x,t;\xi,s) = \frac{1}{\sqrt{t-s}} \exp\left[-\frac{(\xi-x)^2}{4(t-s)}\right].$$

In the paper [1] the following theorems are proved:

Theorem 1. Every solution of the equation (1) is of the form

(4) $$u(x,t) = u_1(x,t) + tu_2(x,t),$$

$u_1(x,t)$ and $u_2(x,t)$ meaning the solutions of the equation (2) and inversely if $u_1(x,t)$ and $u_2(x,t)$ are the solutions of the equation (2) then the function defined by (4) is the solution of (1).

Theorem 2. The solution of the equation (1) in the form (4) is unique

3. Let us consider the domain

$$R_2^+ = \left\{(x,t) : x>0,\ t>0\right\}.$$

We shall solve the mixed problem for the equation (1). Briefly we shall denote this problem by [B-C] problem. It concerns the construction of the function $u(x,t)$ of class $C^4$ in $R_2^+$, satysfying in $R_2^+$ the equation (1), the initial data

(5) $$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u(x,t) = f_1(x_0), \qquad \lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} D_t\big[u(x,t)\big] = f_2(x_0)$$

and the boundary data

(6) $$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} u(x,t) = f_3(t_0), \qquad \lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} Pu(x,t) = f_4(t_0),$$

$f_i(s)$ $(i = 1,2,3,4)$ being the given functions defined for $s \geqslant 0$.

We shall solve this problem by using the convenient Green-function.

Let $X(x,t)$ denote the point belonging to $R_2^+$ and $X_1(-x,t)$ the symetric point to the $t$-axis.

Let $Y(\xi,s)$ denote the arbitrary point in $\bar{R}_2^+ = \{(x,t): x \geqslant 0, t \geqslant 0\}$.

The Green-function for (2) and the domain $R_2^+$ with boundary data of Dirichlet type is of the form [3]:

(7) $$G(x,t;\xi,s) = \frac{1}{\sqrt{t-s}} \exp\left[-\frac{(\xi-x)^2}{4(t-s)}\right] - \exp\left[-\frac{(\xi+x)^2}{4(t-s)}\right].$$

4. Let

(8) $$u_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f_1(\xi)\, G(x,t;\xi,0)\, d\xi,$$

(9) $$u_2(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \big[f_2(\xi) - f_1''(\xi)\big] G(x,t;\xi,0)\, d\xi,$$

(10) $$u_3(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t f_3(s)\, D_\xi\big[G(x,t;0,s)\big]\, ds,$$

(11) $$u_4(x,t) = \frac{-1}{2\sqrt{\pi}} \int_0^t f_4(s)(t-s)\, D_\xi\big[G(x,t;0,s)\big]\, ds.$$

We shall prove that the function

(12) $$u(x,t) = u_1(x,t) + tu_2(x,t) + u_3(x,t) + u_4(x,t)$$

is the solution of the problem $[B-C]$.

5. We suppose that the functions $f_i(s)$, $(i = 1,2,3,4)$ and $f_1''(s)$ are absolutely integrable and bounded for $s \geqslant 0$.

Let

$$\sup_{s\geqslant 0}|f_i(s)| \leqslant M, \qquad \sup_{s\geqslant 0}|f_1''(s)| \leqslant M.$$

Let

$$I_{\alpha\beta}(x,t) = \int_0^\infty \frac{|\xi - x|^\alpha}{t^\beta} \exp\left[-\frac{(\xi - x)^2}{4t}\right] f(\xi)\,d\xi, \tag{13}$$

$$J_{pq}(x,t) = \int_0^t \frac{x^p}{(t-s)^q} \exp\left[-\frac{x^2}{4(t-s)}\right] g(s)\,ds, \quad x > 0, \tag{14}$$

where $\alpha \geqslant 0$, $\beta$, $p \geqslant 0$, $q > 0$ are arbitrary real numbers.

We shall prove

Lemma 1. If the function $f(\xi)$ is bounded and absolutely integrable for $\xi \geqslant 0$, then the integral $I_{\alpha\beta}$ is almost uniformly convergent in the half-plane

$$W = \{(x,t):\ -\infty < x < \infty\ ,\ t > 0\}.$$

Proof. Let $a$, $b$, $c$ by arbitrary positive numbers, $b < c$ and let

$$W_1 = \{(x,t): |x| \leqslant a,\ 0 < b \leqslant t \leqslant c\}$$

denote the rectangle in the half-plane $t > 0$ and let

$$I^R_{\alpha\beta} = \int_{\xi \geqslant R} f(\xi) \frac{|\xi - x|^\alpha}{t^\beta} \exp\left[-\frac{(\xi - x)^2}{4t}\right] d\xi$$

and $(x,t) \in W_1$.

For sufficently great $R$ and for $\xi \geqslant R$

$$\tfrac{1}{4}\xi^2 \leqslant (\xi - x)^2 \leqslant 4\xi^2 \tag{15}$$

for every $x$ satysfying the condition $|x| \leqslant a$.

From (15) we get

$$|I^R_{\alpha\beta}| \leqslant M \int_{\xi \geqslant R} \frac{(2\xi)^\alpha}{b^\beta} \exp\left[-\frac{\xi^2}{16c}\right] d\xi\ ,\ M = \sup_{\xi \geqslant 0}|f(\xi)|\ . \tag{16}$$

Since the integral in (16) is convergent, then for arbitrary $\varepsilon > 0$ there exists a number $R$ such that for each $(x,t) \in W_1$

$$|I^R_{\alpha\beta}| \leqslant \varepsilon.$$

<u>Lemma 2</u>. If the function $g(s)$ is bounded and absolutely integrable for $s \geqslant 0$, then the integral $J_{pq}(x,t)$ is almost uniformly convergent in the domain $R_2^+$.

<u>Proof</u>. Let $a_1, b_1, c_1, d_1$ are arbitrary real positive numbers, $a_1 < b_1$, $c_1 < d_1$ and let

$$W_2 = \left\{(x,t):\ 0 < a_1 \leqslant x \leqslant b_1,\ 0 < c_1 \leqslant t \leqslant d_1\right\}$$

denote the rectangle in $R_2^+$.

Using the inequality

$$k^m e^{-k} \leqslant m^m e^{-m} \tag{17}$$

for $0 < c \leqslant k < \infty$, $m$ being an arbitrary non-negative number; $c$ positive number, we get

$$|J_{pq}(x,t)| \leqslant M \int_0^t \left[\frac{x^2}{4(t-s)}\right]^q \frac{4^q}{x^{2q-p}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds \leqslant \tag{18}$$

$$\leqslant C \int_0^t ds \leqslant C_1, \qquad M = \sup_{s \geqslant 0} |g(s)|,$$

$C$, $C_1$ are the convenient constants. From (18) follows that $J_{pq}(x,t)$ is almost uniformly convergent in $R_2^+$.

Now we skall prove

<u>Theorem 3. The function $u(x,t)$ defined by (12) is of the class $C^4$ and satisfies the equation (1) in the domain $R_2^+$.</u>

<u>Proof</u>. The derivatives of the function $u_1 + tu_2$ are the linear combinations with constant coefficients of integral of the type $I_{\alpha\beta}(x,t)$. According to lemma 1 these integrals are almost uniformly convergent. From the above consideration follows that the function $u_1 + tu_2$ is of class $C^4$ in $R_2^+$. Since the function $G(x,t;\xi,0)$ satisfies the equation (2). Applying theorem 1, we conclude that the function $u_1 + tu_2$ is the solution of the equation (1). From the lemma 2 and from the condition

$$\lim_{\substack{s \to t \\ x > 0}} D_\xi\left[G(x,t;\ 0,s)\right] = 0,$$

$$P_x\left\{D_\xi\left[G(x,t;\ 0,s)\right]\right\} = 0$$

follows that the function $u_3 + u_4$ is of the class $C^4$ and satisfies the equation (1) in $R_2^+$.

Lemma 3. If the function $f_1(\xi)$ is continuous at the point $x_0$, then

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u_1(x,t) = f_1(x_0).$$

Proof.

$$u_1(x,t) = I_1(x,t) - I_2(x,t),$$

where

$$I_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f_1(\xi) \frac{1}{\sqrt{t}} \exp\left[-\frac{(\xi-x)^2}{4t}\right] d\xi ,$$

$$I_2(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f_1(\xi) \frac{1}{\sqrt{t}} \left[\exp - \frac{(\xi-x)^2}{4t}\right] d\xi .$$

We have the inequality

$$\left| \frac{1}{2\sqrt{\pi}} \int_0^\infty f_1(\xi) \frac{1}{\sqrt{t}} \exp\left[-\frac{(\xi+x)^2}{4t}\right] d\xi \right| \leqslant$$

$$\leqslant C\sqrt{t} \int_0^\infty \frac{d\xi}{(\xi+x)^2} = \frac{C\sqrt{t}}{x} ,$$

C - positive constant. It follows from the above inequality that

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} I_2(x,t) = 0.$$

Let us consider the integral $I_1(x,t)$.

Let

$$\bar{f}_1(\xi) = \begin{cases} f_1(\xi) & \text{for } \xi > 0, \\ 0 & \text{for } \xi < 0. \end{cases}$$

We have

$$I_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^\infty \bar{f}_1(\xi) \frac{1}{\sqrt{t}} \exp\left[-\frac{(\xi-x)^2}{4t}\right] d\xi .$$

It follows from ([3], p.460) that

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} L_1(x,t) = f_1(x_0).$$

Lemma 4.

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} tu_2(x,t) = 0.$$

The proof is analogous to that of the lemma 3.

Lemma 5.

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u_3(x,t) = 0.$$

Proof. We have the inequality

$$\left|\int_0^t D_t\left[G(x,t:0,s)\right]f_3(s)\,ds\right| =$$

$$= 16\left|\int_0^t \frac{x^4}{\left[4(t-s)\right]^2}\exp\left[-\frac{x^2}{4(t-s)}\right]\frac{\sqrt{t-s}}{x^3}\,f_3(s)\,ds\right| \leqslant$$

$$\leqslant M\,\frac{16}{\delta^3}\int_0^t \sqrt{t-s}\,ds = M\,\frac{16}{\delta^3}\,\frac{2}{3}\,t^{\frac{3}{2}} \leqslant Ct^{\frac{3}{2}},$$

$x \geqslant \delta > 0$, C - positive constant. The thesis of lemma 5 follows from the above inequality.

Lemma 6.

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u_4(x,t) = 0.$$

The proof is analogous to that of the lemma 5. From lemmas 3-6 follows

Theorem 4. If the assumptions of lemmas 3-6 are satisfied, then the function defined by formula (12) satisfies the initial condition

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u(x,t) = f_1(x_0).$$

<u>Lemma 7</u>. If the function $f_1(\xi)$ is of the class $C^2$ in a certain neighbourhood of the point $x_0$, then

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} D_t\left[u_1(x,t)\right] = f_1''(x_0).$$

The proof of this lemma is analogous to the one is given in [3] /p.464/.

<u>Lemma 8</u>. If the function $f_1(\xi)$ is of the class $C^2$ in a certain neighbourhood of the point $x_0$ and the function $f_2(\xi)$ is continuous at the point $x_0$, then

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} D_t\left[u_1(x,t)+tu_2(x,t)\right] = f_2(x_0).$$

<u>Proof</u>. We have

$$D_t\left[u_1(x,t)+tu_2(x,t)\right] = D_t\left[u_1(x,t)\right] + u_2(x,t) + tD_t\left[u_2(x,t)\right].$$

According to lemma 7

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} D_t\left[u_1(x,t)\right] = f_1''(x_0)$$

and

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} tD_t\left[u_2(x,t)\right] = 0.$$

From lemma 3 follows that

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} u_2(x,t) = f_2(x_0) - f_1''(x_0).$$

Hence we have the thesis.

<u>Lemma 9</u>.

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x_0>0,\ t>0}} D_t\left[u_3(x,t)\right] = 0.$$

<u>Proof</u>. We have

$$D_t\left[u_3(x,t)\right] = \frac{1}{2\sqrt{\pi}} \; f_3(t) D_\xi\left[ G(x,t;\ 0,\ t)\right] +$$

$$+ \frac{1}{2\sqrt{\pi}} \int_0^t f_3(s) D_t\left\{ D_\xi\left[ G(x,t;\ 0,\ s)\right]\right\} ds.$$

It is easy to observe that

$$\lim_{\substack{s \to t \\ s < t}} D_\xi\left[ G(x,t;\ 0,\ s)\right] = 0.$$

Using similar estimates as in lemma 5 we have

$$\lim_{\substack{(x,t)\to(x_0,0) \\ x_0 > 0,\, t > 0}} \frac{1}{2\sqrt{\pi}} \int_0^t f_3(s) D_t\left\{D_\xi\left[ G(x,t;\ 0,\ s)\right]\right\} ds = 0.$$

Finally we obtain the thesis.

<u>Lemma 10</u>.

$$\lim_{\substack{(x,t)\to(x_0,0) \\ x_0 > 0,\ t > 0}} D_t\left[u_4(x,t)\right] = 0.$$

The proof is analogous to that of lemma 9.
From lemmas 7–10 follows

<u>Theorem 5</u>. <u>If the asumption of the lemmas 7–10 are satisfied, then the function $u(x,t)$ defined by formula (12) satisfies the initial condition</u>

$$\lim_{\substack{(x,t)\to(x_0,0) \\ x_0 > 0,\ t > 0}} D_t\left[u(x,t)\right] = f_2(x_0).$$

<u>Lemma 11</u>. If $t > 0$, $x > 0$, then

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^t \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds = 1.$$

<u>Proof</u>. Applying the change of the variable

$$z = \frac{x}{2\sqrt{t-s}}$$

we have

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{t} \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz = 1.$$

<u>Lemma 12</u>. If the function $f_3(s)$ is continuous at the point $t_o$, then

$$\lim_{\substack{(x,t)\to(0,t_o) \\ x>0,\ t_o>0}} u_3(x,t) = f_3(t_o).$$

<u>Proof</u>. Let

$$\bar{f}_3(s) = \begin{cases} f_3(s) & \text{for } s \geqslant 0, \\ 0 & \text{for } s < 0. \end{cases}$$

In view of lemma 11

$$u_3(x,t) = f_3(t_o) + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{t} \left[\bar{f}_3(s) - f_3(t_o)\right] \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds.$$

From the continuity of the function $f_3(s)$ at the point $t_o$ follows that for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$, such that $|\bar{f}_3(s) - f_3(t_o)| \leqslant \frac{\varepsilon}{2}$ when $|s - t_o| \leqslant \delta$. In view of this condition

$$\left| \int_{|s-t_o|\leqslant\delta} \left[\bar{f}_3(s) - f_3(t_o)\right] \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds \right| \leqslant \frac{\varepsilon}{2}.$$

Since $t \to t_o$; we can suppose that $|t - t_o| \leqslant \delta$.

Introducing a new variable in the integral

$$z = \frac{x}{2\sqrt{t-s}}$$

we get

$$\left| \int_{-\infty}^{t_o-\delta} \left[\bar{f}_3(s) - f_3(t_o)\right] \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds \right| \leqslant 8M \int_0^{\frac{x}{2\sqrt{\delta}}} e^{-z^2} dz.$$

As $x \to 0^+$ both integration limits in the last integral tend to 0; therefore there exists the positive number $\varrho(\delta)$, such that for $0 < x < \varrho$ the inequality

$$\left| \int_{-\infty}^{t_o-\delta} \left[\bar{f}_3(s) - f_3(t_o)\right] \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds \right| \leqslant \frac{\varepsilon}{2}$$

holds.

Lemma 13.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} u_4(x,t) = 0 \quad .$$

Proof. Let $\delta$ be an arbitrary positive number and $0<t-s\leqslant\delta$ . We have

$$|u_4(x,t)| \leqslant M\delta \frac{1}{2\sqrt{\pi}}\left|\int_0^t \frac{x}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{x^2}{4(t-s)}\right] ds\right|$$

From lemma 11 follows that for every $\varepsilon>0$ there exists $\delta(\varepsilon)$ such that the inequality

$$|u_4(x,t)| \leqslant \frac{\varepsilon}{2}$$

holds.

For $t-s>\delta$ there exists positive number $\varsigma(\delta)$ , such that for $0<x<\varsigma$ the inequality

$$|u_4(x,t)| \leqslant \frac{\varepsilon}{2}$$

holds. Finally we get the thesis.

Lemma 14.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} u_1(x,t) = 0.$$

Proof. In view of lemma 1 and the condition

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} G(x,t;\xi,0) = 0$$

we get the thesis.

Lemma 15.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} t\, u_2(x,t) = 0.$$

The proof is analogous to that of the lemma 14. From lemmas 11-15 follows

Theorem 6. Under the assumption of the lemmas 11-15 the function defined by formula (12) satisfies the boundary condition

$$\lim_{\substack{(x,t)\to(0,\ )\\ x>0,\ t_0>0}} u(x,t) = f_3(t_0).$$

Lemma 16. If the function $f_4(s)$ is continuous the point $t_0$, then

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} Pu_4(x,t) = f_4(t_0).$$

The proof is analous to that of the lemma 12.

Lemma 17.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} P\,u_3(x,t) = 0.$$

Proof. We have

$$Pu_3(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t f_3(s) P_X\left\{D_\xi\left[G(x,t;0,s)\right]\right\} ds -$$

$$- f_3(t) D_\xi\left[G(x,t;0,t)\right].$$

It is easy to verify that

$$\lim_{\substack{s\to t\\ x>0}} D_\xi\left[G(x,t;0,s)\right] = 0$$

and

$$P_X\left\{D_\xi\left[G(x,t;0,s)\right]\right\} = 0.$$

In view of these conditions we get the thesis.

Lemma 18.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\, t_0>0}} Pu_1(x,t) = 0.$$

Proof. In view of lemma 1

$$Pu_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f_1(\xi)\ P_X\left[G(x,t;\xi,0)\right] d\xi .$$

It is easy to observe that

$$P_X\left[G(x,t;\xi,0)\right] = 0.$$

Hence we have the thesis.

Lemma 19.

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} P\,tu_2(x,t) = 0.$$

Proof. In virtue of lemma 1

$$P_X\left[tu_2(x,t)\right] = \frac{1}{2\sqrt{\pi}}\, t \int_0^{\infty}\left[f_2(\xi) - f_1''(\xi)\right]P_X\left[G(x,t;\xi,0)\right] d\xi - u_2(x,t).$$

Since

$$P_X\left[G(x,t;\xi,0)\right] = 0$$

and

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} u_2(x,t) = 0$$

we have the thesis.

From lemmas 16–19 follows

Theorem 7. Under the assumptions of the lemmas 16–19 the function $u(x,t)$ defined by formula (12) satisfies the boundary condition

$$\lim_{\substack{(x,t)\to(0,t_0)\\ x>0,\ t_0>0}} Pu(x,t) = f_4(t_0).$$

From the theorems 1–7 follows

Theorem 8. If the functions $f_i(s)$, $i = 2,3,4$ are bounded and continuous for $s \geqslant 0$ and the function $f_1(s)$ of the class $C^2$, then the function $u(x,t)$ defined by formula (12) is the solution of the problem [B-C].

B i b l i o g r a p h y

[1] Craiu Mariana, Resculet Marcel N.: Ecuatia

$$\frac{\partial^{2m} u}{\partial x_1^{2m}} + \frac{\partial^{2m} u}{\partial x_2^{2m}} + \dots + \frac{\partial^{2m} u}{\partial x_n^{2m}} - \frac{\partial u}{\partial t} = 0 \text{ iterata, Studii}$$

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