ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE

Zeszyt 51 Prace matematyczne VII Rok 1974

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ON PEAKLESS FUNCTIONS

H. Busemann in his monography [1] p.109 introduces the peakless functions by definition: "A function $f(\tau)$ which is defined and continuous on a convex set of the real τ -axis is called "peakless" if

$$f(\tau_2) \leqslant \max\{f(\tau_1), f(\tau_3)\} \quad \text{for} \quad \tau_1 < \tau_2 < \tau_3$$

and the equality sign implies $f(\tau_1) = f(\tau_3)$."

The author states: "A function is peakless if and only if it belongs to one of the following types: $f(\tau)$

1/ is constant,

2/ is strictly increasing or decreasing,

3/ takes its minimum at one point, decreases strictly to the left of the point and increases strictly to the right,

4/ takes its minimum at all points of an interval and decreases strictly to the left and increases strictly to the right of this interval".

The purpose of our note is to find the solution of the following problem: Let $(A,\leqslant)$, $(B,\leqslant)$ with $\bar{\bar{A}} \geqslant 2$ and $B \neq \emptyset$ be two linearly ordered sets.

Let $f: A \to B$ be a function with domain A and values in B and have two properties:

$$\text{(i)} \qquad \bigwedge_{\tau_1, \tau_2, \tau_3 \in A} \left[\tau_1 < \tau_2 < \tau_3 \Rightarrow f(\tau_2) \leqslant \max\{f(\tau_1), f(\tau_3)\}\right],$$

$$\text{(ii)} \qquad \bigwedge_{\tau_1, \tau_2, \tau_3 \in A} \left[\tau_1 < \tau_2 < \tau_3 \wedge f(\tau_2) = \max\{f(\tau_1), f(\tau_3)\} \Rightarrow f(\tau_1) = f(\tau_3)\right].$$

Such functions, by analogy to H. Busemann, will be called "peakless functions". The problem is to find all the peakless functions, when the ordered sets

$(A,\leqslant)$, $(B,\leqslant)$ are given.

Definition 1. The symbols $<, >$, $\max\{a,b\}$ are used as usuall in the theory of ordered sets.

Definition 2. For the nonempty subset A^* of A and the function $f : A \to B$, f is called non-decreasing in A^* /non-increasing in A^*, strictly increasing in A^*, strictly decreasing in A^*/ if and only if for each $\tau_1, \tau_2 \in A^*$ the inequality $\tau_1 < \tau_2$ implies $f(\tau_1) \leqslant f(\tau_2)$ /$f(\tau_1) \geqslant f(\tau_2)$, $f(\tau_1) < f(\tau_2)$, $f(\tau_1) > f(\tau_2)$/ respectively.

Lemma 1.

Let $f : A \to B$ be non-decreasing in A and satisfy the conditions (i) and (ii). Then

(a) f is constant in A or

(b) f is strictly increasing in A or

(c) there exist nonempty sets A_1, A_2 with $A_1 \cup A_2 = A$ and $\tau_1 < \tau_2$ for $\tau_i \in A_i$, $i = 1,2$ such that f is constant in A_1 and strictly increasing in A_2 /naturally $f(\tau_1) \leqslant f(\tau_2)$ for $\tau_i \in A_i$/.

Proof. Suppose, that f is not constant in A. Define:

$$A_2 = \left\{ \tau \in A : \bigvee_{u \in A} u < \tau \wedge f(u) < f(\tau) \right\}.$$

By our supposition, there is $A_2 \neq \emptyset$.

If $\tau \in A_2$ and $\tau < \tau^*$, we find $u \in A$ with $u < \tau$ and $f(u) < f(\tau)$. Thus we have $u < \tau^*$ and $f(u) < f(\tau) \leqslant f(\tau^*)$, which implies $\tau^* \in A_2$. Observe, that f is strictly increasing in A_2. For, if $\tau_1, \tau_2 \in A_2$ and $\tau_1 < \tau_2$ there exists u with $u < \tau_1$ and $f(u) < f(\tau_1)$. By (i) is $f(\tau_1) \leqslant \max\{f(u), f(\tau_2)\} = f(\tau_2)$ and $f(\tau_1) = f(\tau_2)$ owing to (ii) leads to $f(\tau_1) = f(\tau_2) = f(u)$, thus $f(\tau_1) < f(\tau_2)$. In the case $A_2 = A$ f is strictly increasing in A. Consider now the case $\overline{\overline{A - A_2}} = 1$ and let $v \in A - A_2$. Obviously is $v < \tau$ for $\tau \in A_2$ and $v = \min A$. There is also $f(v) \leqslant f(\tau)$ for $\tau \in A_2$. The equality sign for any $\tau \in A_2$ implies that $f(\tau^*) = f(\tau)$ for each $\tau^* \in A$ and $\tau^* < \tau$, which is impossible, because of the definition of A_2. Thus f is strictly increasing in A.

The last case $\overline{\overline{A - A_2}} \geqslant 2$ leads to the set $A_1 \overset{df}{=} A - A_2$. By its definition we have $\tau \in A_1$ if and only if

$$\bigwedge_{u \in A} \left[u < \tau \longrightarrow f(u) \geqslant f(\tau) \right].$$

Taking $\tau, \tau' \in A_1$ $\tau < \tau'$ we obtain $f(\tau) \geqslant f(\tau')$ because of the above condition and $f(\tau) \leqslant f(\tau')$ by supposition that f is non- decreasing. Thus f is constant in A_1, and obviously $\tau_1 < \tau_2$ for $\tau_i \in A_i$. In this case f satisfies the property (c) of our lemma.

Lemma 2.

Let $f : A \to B$ be non-increasing in A and satisfy the conditions (i) and (ii). Then

(a) f is constant in A or

(d) f is strictly decreasing in A or

(e) there exist nonempty sets A_1, A_2 with $A_1 \cup A_2 = A$ and $\tau_1 < \tau_2$ for $\tau_i \in A_i$, $i = 1,2$, such that f is strictly decreasing in A_1 and constant in A_2.

The proof of lemma 2 is analogous to that of lemma 1.

Lemma 3.

Assume that $f : A \to B$ with properties (i) and (ii) is not non-decreasing and not non-increasing in A. Then either

(f) there exist nonempty sets A_1, A_2 with $A_1 \cup A_2 = A$ and $\tau_1 < \tau_2$ for τ_i A_i, $i = 1,2$, such that f is strictly decreasing in A_1 and strictly increasing in A_2 or

(g) there exist nonempty sets A_1, A_2, A_3 with $A_1 \cup A_2 \cup A_3 = A$ and $\tau_1 < \tau_2 < \tau_3$ for $\tau_i \in A_i$, $i = 1,2,3$, such that f is strictly decreasing in A_1, constant in A_2 and strictly increasing in A_3 and moreover $f(\tau_1) \geq f(\tau_2)$, $f(\tau_2) \leq f(\tau_3)$ for $\tau_i \in A_i$, $i = 1,2,3$.

Proof. Our assumptions imply the existence of u_1, u_2, u_3, u_4 with $u_1 < u_2$, $u_3 < u_4$ and $f(u_1) > f(u_2)$, $f(u_3) < f(u_4)$. Observe, that for these u_1, u_4 there must be $u_1 < u_4$. If not, then $u_4 \leq u_1$ and $u_3 < u_4 \leq \leq u_1 < u_2$. By (i) we have

$$f(u_4) \leq \max\{f(u_3), f(u_2)\} \quad \text{and}$$
$$f(u_1) \leq \max\{f(u_3), f(u_2)\}.$$

Thus when $f(u_3) \leq f(u_2)$ we obtain $f(u_1) \leq f(u_2)$, and $f(u_4) \leq f(u_3)$ in the other case. These inequalities contradict the definition of the elements u_1, u_2, u_3, u_4.

We define next

$$\tilde{A} \overset{df}{=} \{\tau \in A : \bigvee_{u \in A} \tau < u \wedge f(\tau) > f(u)\},$$
$$A^{*} \overset{df}{=} \{\tau \in A : \bigvee_{u \in A} u < \tau \wedge f(u) < f(\tau)\}.$$

Evidently $u_1 \in \tilde{A}$ and $u_4 \in A^{*}$.

If $\tau_0 \in \tilde{A} \cap A^{*}$ we find $\tilde{u}$, u^{*} with $\tilde{u} < \tau_0 < u^{*}$ and $f(\tilde{u}) < f(\tau_0)$, $f(\tau_0) > f(u^{*})$. Because of (i) we have $f(\tau_0) \leq \max\{f(\tilde{u}), f(u^{*})\}$, which contradicts the above inequalities. Thus $\tilde{A} \cap A^{*} = \emptyset$.

As in the proof of lemma 1 we state now that:
for $\tau \in \tilde{A}$ and $\tau' < \tau$ is $\tau' \in \tilde{A}$ and for $\tau \in A^*$ and $\tau < \tau'$ is $\tau' \in A^*$. In consequence if $\tilde{\tau} \in \tilde{A}$ and $\tau^* \in A^*$ then there is $\tilde{\tau} < \tau^*$. Let now $\tau_1, \tau_2 \in \tilde{A}$ and $\tau_1 < \tau_2$. Find $u > \tau_2$ with $f(\tau_2) > f(u)$. By (i) is

$$f(\tau_2) \leqslant \max\left\{f(\tau_1),\ f(u)\right\} = f(\tau_1)$$

and the equality sign is imposible because of (ii) and $f(u) < f(\tau_2)$. Hence f is strictly decreasing in $\tilde{A}$. In the same way we state, that f is strictly increasing in A^*. In the case $A = \tilde{A} \cup A^*$ defining $A_1 \overset{df}{=} \tilde{A}$, $A_2 \overset{df}{=} A^*$ in consequence we have the statement (f). When the set. $A - (\tilde{A} \cup A^*)$ has the unique element v we define

$A_1 \overset{df}{=} \tilde{A} \cup \{v\}$ and $A_2 \overset{df}{=} A^*$ and there is again the statement (f). Consider now the last case $A - (\tilde{A} \cup A^*) \geqslant 2$.

Let $A_1 \overset{df}{=} \tilde{A}$, $A_2 \overset{df}{=} A - (\tilde{A} \cup A^*)$, $A_3 \overset{df}{=} A^*$. It follows from the properties of $\tilde{A}$ and A^* that $\tau_1 < \tau_2 < \tau_3$ for $\tau_i \in A_i$. Furthermore if $\tau \in A_2$, by the definitions of A_i, $\tilde{A}$ and A^*, we have

$$\bigwedge_{u \in A} \left[\tau < u \Rightarrow f(\tau) \leqslant f(u)\right] \quad \text{and}$$

$$\bigwedge_{u \in A} \left[u < \tau \Rightarrow f(u) \geqslant f(\tau)\right].$$

In consequence $f(\tau)$ for $\tau \in A_2$ is the minimal value of function f. Thus f is constant in A_2 and $f(\tau) \leqslant f(\tau')$ for $\tau \in A_2$ and $\tau' \in A_1 \cup A_3$. The proof of lemma 3 is complete.

Since $f : A \to B$ is nondecreasing in A or is nonincreasing in A or is not nondecreasing in A and is not nonicreasing in A by lemmas 1,2,3 we have proved the following theorem:

Theorem 1.

If $f : A \to B$ is a peakless function, then f satisfies one of the conditions (a), (b), (c), (d), (e), (f),(g) from lemmas 1, 2, 3. It is very easy to verify that conversely

Theorem 2.

Function $f : A \to B$ satisfying one of the conditions (a), (b), (c),(d), (e), (f), (g) is a peakless function.

Our considerations show that H. Busemann does not give all the classes of his peakless functions. The classes (c),(e) are omitted in [1].

B i b l i o g r a p h y

[1] Herbert Busemann, The geometry of geodesics, Academic Press,Inc. New York, 1955.