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ON THE FATOÛ PROBLEM FOR HARMONIC FUNCTIONS IN THE HALF-SPACE

In this paper we shall give the proof of the theorem being a three dimentional analogoue of the Fatoû theorem for the half-plane ([1]). We shall suppose that the boundary function is of a divergent form.

1. Let E_2 denote a plane (s,t), E_3^+ denote a half-space $z > 0$. In [2] the following theorem is proved.

Theorem 1. If $\varphi(s,t)$ is a continuous and bounded function in E_2, then the function

(1) $$u(x,y,z) = \frac{z}{2\pi}\int_{E_2}\left((x-s)^2+(y-t)^2+z^2\right)^{-\frac{3}{2}}\varphi(s,t)\,dsdt$$

is harmonic in E_3^+ and

$$u(x,y,z) \longrightarrow (x_0,y_0) \quad \text{as} \quad (x,y,z)\longrightarrow(x_0,y_0,0).$$

If $\varphi(s,t)$ is bounded and mesurable in E_2, then the function $u(x,y,z)$ defined by the formula (1) is harmonic in E_3^+.

Let

$$M(x-s,y-t,z) = \frac{z}{2\pi}\left((x-s)^2+(y-t)^2+z^2\right)^{-\frac{3}{2}}.$$

We suppose that

(2) $$\varphi(s,t) = D_sf_1(s,t) + D_tf_2(s,t),$$

where f_1, f_2 are bounded and mesurable in E_2. Applying the Green formula, we get

$$u(x,y,z) = \lim_{R\to\infty}\int_{C_R}\left[f_1(s,t)\cos(n,s) - f_2(s,t)\cos(n,t)\right]M(x-s,y-t,z)\,d\sigma +$$

$$-\int_{E_2}\left[f_1(s,t)\,D_sM(x-s,y,t,z) + f_2(s,t)\,D_tM(x-s,y-t,z)\right]dsdt,$$

where C_R denotes the circumference $s^2 + t^2 = R^2$, n - outward normal. Since f_1, f_2 are bounded, we obtain

$$\lim_{R\to\infty} \int_{C_R} \left[f_1(s,t)\cos(n,s) - f_2(s,t)\cos(s,t)\right] M(x-s,y-t,z)\, d\sigma = 0 \tag{3}$$

in the neighbourhood of every point $(x,y,z) \in E_3^+$. From (3) it follows that

$$u(x,y,z) = -\int_{E_2} \left[f_1(s,t) D_s(x-s,y-t,t) + f_2(s,t) D_t(x-s,y-t,z)\right] dsdt. \tag{4}$$

2. Let $Q = (x,y,z) \in E_3^+$, $Q_0 = (x_0, y_0, 0)$ and L denote an arbitrary positive constant. Let Ω denote a set of the points Q for which

$$\frac{|x-x_0|}{z} < L \quad \text{and} \quad \frac{|y-y_0|}{z} < L. \tag{5}$$

Let

$$P_1(s,t,Q) = -s\, D_s M(x-x_0-s, y-y_0-t, z),$$

$$P_2(s,t,Q) = -t\, D_t M(x-x_0-s, y-y_0-t, z).$$

Applying the change of the variables $s = s' + x_0$, $t = t' + y_0$ to the integral (4), we get

$$u(x,y,z) = -\int_{E_2} f_1(x_0+s, y_0+t) D_s M(x-x_0-s, y-y_0-t, z)\, dsdt +$$

$$-\int_{E_2} f_2(x_0+s, y_0+t) D_t M(x-x_0-s, y-y_0-t, z)\, dsdt =$$

$$= \int_{E_2} \frac{f_1(x_0+s, y_0+t)}{s} P_1(s,t,Q) dsdt +$$

$$+ \int_{E_2} \frac{f_2(x_0+s, y_0+t)}{t} P_2(s,t,Q) dsdt.$$

We shall prove the following lemma

__Lemma__. If $Q \in \Omega$, then

$$\int_{E_2} P_1(s,t,Q)\, dsdt = 1, \tag{6}$$

(7) there exist constants K_1 such that

$$\int_{E_2} |P_1(s,t,Q)|\, dsdt < K_1,$$

(8) for arbitrary $r > 0$

$$\lim_{\substack{Q \to Q_0 \\ Q \in \Omega}} \int_D |P_i(s,t,Q)|\, ds dt = 0,$$

where $D = \{(s,t): s^2 + t^2 \geqslant r^2\}$, $i = 1,2$.

<u>Proof</u>. We shall give the proof in the case $i = 1$. For $i = 2$ the proof is analogous. Since

$$\lim_{R \to \infty} \int_{C_R} s\, M(x-x_0-s, y-y_0-t, z) \cos(n,s)\, d\sigma = 0$$

for every $Q \in E_2$, we get

$$\int_{E_2} P_1(s,t,Q)\, ds dt = -\lim_{R\to\infty} \int_{C_R} sM(x-x_0-s, y-y_0-t, z)\cos(n,s)\, d\sigma + $$
$$- \int_{E_2} M(x-x_0-s, y-y_0-t, z)\, ds dt = 1,$$

and consequently (6).

Applying the change of the variables $x-x_0-s = s'$, $y-y_0-t = t'$ to the integral (7), we get

$$\int_{E_2} |P_1(s,t,Q)|\, ds dt = \int_{E_2} |sD_s M(x-x_0-s, y-y_0-t, z)|\, ds dt =$$
$$= \int_{E_2} |(x-x_0-s) D_s M(s,t,z)|\, ds dt \leqslant |x-x_0| \int_{E_2} |D_s M(s,t,z)|\, ds dt +$$
$$+ \int_{E_2} |s\, D_s M(s,t,z)|\, ds dt.$$

On the other hand

$$\int_{E_2} |D_s M(s,t,z)|\, ds dt = \frac{3z}{2\pi} \int_{E_2} |s| (s^2 + t^2 + z^2)^{-\frac{5}{2}}\, ds dt \leqslant \frac{C_1}{z},$$

$$\int_{E_2} |sD_s M(s,t,z)|\, ds dt = \frac{3z}{2\pi} \int_{E_2} s^2 (s^2 + t^2 + z^2)^{-\frac{5}{2}}\, ds dt \leqslant C_2,$$

where C_1, C_2 are certain positive constants. Hence

$$\int_{E_2} |P_1(s,t,Q)|\, ds dt \leqslant C_1 L + C_2 = K_1,$$

and consequently (7).

Let $r > 0$ be given. Applying the change of the variables $-s'z = x-x_0-s$, $-t'z = y-y_0-t$ to the integral (8), we get

$$\int_D |P_1(s,t,Q)|\,dsdt = \frac{3z}{2\pi}\int_D |s(x-x_0-s)|\left((x-x_0-s)^2+(y-y_0-t)^2+z^2\right)^{-\frac{5}{2}} dsdt =$$

$$= \frac{3}{2\pi z}\int_{D_0} |(sz-x-x_0)s|(s^2+t^2+1)^{-\frac{5}{2}} dsdt \leqslant$$

$$\leqslant C_3\int_{D_0} s^2(s^2+t^2+1)^{-\frac{5}{2}} dsdt +$$

$$+ C_4\int_{D_0} |s|(s^2+t^2+1)^{-\frac{5}{2}} dsdt,$$

where C_3, C_4 are certain positive constants, and

$$D_0 = \left\{(s,t) : (x-x_0-sz)^2 + (y-y_0-tz)^2 \geqslant r^2\right\}.$$

Let $\varepsilon > 0$ be given; then there exists a number $r_0 > 0$ such that

$$C_3\int_{D_1} s^2(s^2+t^2+1)^{-\frac{5}{2}} dsdt + C_4\int_{D_1} |s|(s^2+t^2+1)^{-\frac{5}{2}} dsdt < \varepsilon,$$

for $R > r_0$, $D_1 = \left\{(s,t): s^2 + t^2 \geqslant R^2\right\}$.

Let D_2 denote a sphere with centre Q_0 such that

$$R^2 = \frac{r^2}{z^2} - 2L^2 \geqslant r_0^{\,2}$$

for every point $Q \in \Omega \cap D_2$, where L is the convenient constant from the inequality (5). If $Q \in D_2$ and $(s,t) \in D_0$, then $(s,t) \in D_1$. Hence

$$\int_D |P_1(s,t,Q)|dsdt \leqslant C_3\int_{D_0} s^2(s^2+t^2+1)^{-\frac{5}{2}} dsdt + C_4\int_{D_0} |s|(s^2+t^2+1)^{-\frac{5}{2}} dsdt \leqslant$$

$$\leqslant C_3\int_{D_1} s^2(s^2+t^2+1)^{-\frac{5}{2}} dsdt + C_4\int_{D_1} |s|(s^2+t^2+1)^{-\frac{5}{2}} dsdt$$

for every point $Q \in \Omega \cap D_2$.

<u>Theorem 2</u>. If φ, f_1, f_2 are bounded and mesurable functions in E_2 and satisfy the condition (2), the function $D_s f_1(x_0,t)$ is continuous

with respect to t for $t = y_0$, the function $D_t f_2(s,y_0)$ is continuous with respect to s for $s = x_0$, then the function $u(x,yz)$, defined by the formula (1) is harmonic in E_3^+ and satisfies the boundary condition

$$\lim_{\substack{Q \to Q_0 \\ Q \in \Omega}} u(Q) = \varphi(Q_0).$$

Proof. Let

$$u_1(x,y,z) = \int_{E_2} \frac{f_1(s+x_0,t+y_0)}{s} P_1(s,t,Q)\,dsdt,$$

$$u_2(x,y,z) = \int_{E_2} \frac{f_2(s+x_0,t+y_0)}{t} P_2(s,t,Q)\,dsdt.$$

We shall prove that

$$u_1(x,y,z) \longrightarrow D_s f_1(x_0,y_0) \quad \text{as} \quad Q \to Q_0 \quad \text{and} \quad Q \in \Omega.$$

Applying the Green formula and (3), we get

$$\int_{E_2} f_1(x_0,y_0+t)\, D_s M(x-x_0-s,y-y_0-t,z)\,dsdt = 0$$

and consequently

$$u_1(x,y,z) = \int_{E_2} \frac{f_1(x_0+s,y_0+t) - f_1(x_0,y_0+t)}{s} P_1(s,t,Q)\,dsdt.$$

Let $\varepsilon > 0$ be given; then there exists a number $r > 0$ such that

$$\left|\frac{f_1(x_0+s,y_0+t) - f_1(x_0,y_0+t)}{s} - D_s f_1(x_0,y_0)\right| \leqslant \frac{\varepsilon}{2K_1}$$

for $s^2 + t^2 \leqslant r^2$. From the lemma it follows that

$$|u_1(x,y,z) - D_s f_1(x_0,y_0)| \leqslant \int_{E_2}\left|\left[\frac{f_1(x_0+s,y_0+t) - f_1(x_0,y_0+t)}{s} - D_s f_1(x_0,y_0)\right] P_1(s,t,Q)\right| dsdt$$

$$= \int_{E_2 \setminus D}\left|\left[\frac{f_1(x_0+s,y_0+t) - f_1(x_0,y_0+t)}{s} - D_s f_1(x_0,y_0)\right] P_1(s,t,Q)\right| dsdt +$$

$$+ \int_{D}\left|\left[\frac{f_1(x_0+s,y_0+t) - f_1(x_0,y_0+t)}{s} - D_s f_1(x_0,y_0)\right] P_1(s,t,Q)\right| dsdt \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + 2M \int |P_1(s,t,Q)|\,dsdt \leqslant \varepsilon,$$

where

$$M = \max\left\{ \sup_{\substack{s>0 \\ t>0}} \left| \frac{f_1(x_0+s, y_0+t) - f_1(x_0, y_0+t)}{s} \right|, \left| D_s f_1(x_0, y_0) \right| \right\},$$

for every point $Q \in \Omega \cap D_2$, D_2 being the convenient neighbourhood of point Q_0. The condition (8) implies the existence of D_2.

Similary we can prove that

$$u_2(x,y,z) \longrightarrow D_t f_2(x_0,y_0) \quad \text{as} \quad Q \to Q_0 \quad \text{and} \quad Q \in \Omega.$$

From (2), we get

$$u(x,y,z) \longrightarrow \varphi(x_0,y_0) \quad \text{as} \quad Q \to Q_0 \quad \text{and} \quad Q \in \Omega.$$

Since the function $u(x,y,z)$ given by the formula (1) is harmonic in E_3^+, thus we get the thesis of theorem 2.

B i b l i o g r a p h y

[1] K. Hoffman, Banach spaces of analytic functions, Prentice-Hall, INC, 1962.

[2] M. Krzyżański, Partial differential equations of second order, vol.I, PWN, Warszawa 1972.