ROCZNIK NAUKOWO-DYDAKTYCZNY WSP W KRAKOWIE

Zeszyt 51 Prace matematyczne VII Rok 1974

Eugeniusz Wachnicki

ON THE RIQUIER PROBLEM FOR THE EQUATION $(\Delta - c^2)^2 u(x,y) = 0$ IN THE HALF-PLANE

1. In this paper we construct the solution of the equation

$$(\Delta - c^2)^2 u(x,y) = \Delta^2 u(x,y) - 2c^2 \Delta u(x,y) + c^4 u(x,y) = 0, \tag{1}$$

with c as a positive constant, in the half-plane $y > 0$ satisfying the boundary data

$$\lim_{(x,y)\to(x_0,0)} u(x,y) = f_1(x_0), \qquad \lim_{(x,y)\to(x_0,0)} \Delta u(x,y) = f_2(x_0), \tag{2}$$

f_1, f_2 being given functions.

2. Let $P(x,y)$, $Q(s,t)$, $P \neq Q$, denote the points of the plane and let

$$r^2 = (x-s)^2 + (y-t)^2.$$

Let

$$U(r) = U(P,Q) = cr\, K_1(cr), \tag{3}$$

where $K_n(cr)$ denotes the Mac Donald functions with index n ([3], p.115).

Using the formulae ([3], p.117)

$$\begin{cases} \dfrac{d}{dz}\left(z^n K_n(z)\right) = -z^n K_{n-1}(z), \\[2mm] \dfrac{d}{dz}\left(z^{-n} K_n(z)\right) = -z^{-n} K_{n+1}(z), \end{cases} \tag{4}$$

we get

$$\Delta_Q U(P,Q) = -2c^2 K_0(cr) + c^3 r K_1(cr) = -2c^2 K_0(cr) + c^2 U(r). \tag{5}$$

The conditions (5) and ([1], p.115)

$$\Delta_Q K_0(cr) - c^2 K_0(cr) = 0, \quad P \neq Q,$$

imply that the function defined by (3) satisfies the equation (1) with respect to the point $P \neq Q$.

3. Let D denote the bounded domain and S its boundary of the class C^1.

Theorem 1. If the function $u(P)$ is of class C^4 in D and of class C^3 in its closure and $u(P)$ satisfies the equation (1) in the domain D, then

(6)
$$\frac{1}{4\pi c^2}\int_S \Big\{u(Q)\Big[\frac{d\Delta_Q U(P,Q)}{dn_Q} - 2c^2\frac{dU(P,Q)}{dn_Q}\Big] + - \frac{du(Q)}{dn}\Big[\Delta_Q U(P,Q) - 2c^2 U(P,Q)\Big] + \Delta u(Q)\frac{dU(P,Q)}{dn_Q} + - \frac{d\Delta u(Q)}{dn}U(P,Q)\Big\}ds_Q = \begin{cases} -u(P) & \text{if } P\in D\setminus S,\\ 0 & \text{if } P\in(\overline{D\cup S})', \end{cases}$$

n denotes the inward normal.

Proof. From the formulae ([2], p.230 v.I and p.200 v.II) we get

(7)
$$\int_D \big[u(\Delta^2 v - 2c^2\Delta v + c^4 v) - v(\Delta^2 u - 2c^2\Delta u + c^4 u)\big]dxdy = \int_S\Big[v\frac{d\Delta u}{dn} - 2c^2 v\frac{du}{dn} + 2c^2 u\frac{dv}{dn} - \Delta u\frac{dv}{dn} + \Delta v\frac{du}{dn} - u\frac{d\Delta v}{dn}\Big]ds,$$

where n denotes the inward normal.

Let $v(Q) = U(P,Q)$. If $P\in(\overline{P\cup S})'$ then the function $U(P,Q)$ satisfies the equation (1) in D with respect to the point Q; hence (6) is satisfied. Let P be an interior point of domain D and K_R denote the circle with center P and radius R. The function $U(P,Q)$ satisfies the equation (1) with respect to the point Q in the domain $D\cap K_R$. From (7) we get

$$0 = \int_{S\cup S_R}\Big[U(P,Q)\frac{d\Delta u(Q)}{dn} - 2c^2 U(P,Q)\frac{du(Q)}{dn} + 2c^2 u(Q)\frac{dU(P,Q)}{dn_Q} + - \Delta u(Q)\frac{dU(P,Q)}{dn_Q} + \Delta_Q U(P,Q)\frac{du(Q)}{dn} - u(Q)\frac{d\Delta_Q U(P,Q)}{dn_Q}\Big]ds_Q$$

where S_R denotes the circumference of the circle K_R.

From (3), (4) and (5) follows that

$$\frac{dU(P,Q)}{dn_Q} = c^2 R\, K_0(cR),\quad \frac{d\Delta_Q U(P,Q)}{dn_Q} = -2c^2 K_1(cR) + c^4 R\, K_0(cR)$$

on S_R.

Hence we have

$$\int_S \Big[U(P,Q)\,\frac{d\Delta u(Q)}{dn} - 2c^2U(P,Q)\,\frac{du(Q)}{dn} + 2c^2u(Q)\,\frac{dU(P,Q)}{dn} +$$

$$-\Delta u(Q)\,\frac{dU(P,Q)}{dn_Q} + \Delta_Q U(P,Q)\,\frac{du(Q)}{dn} - u(Q)\,\frac{d\Delta_Q U(P,Q)}{dn_Q}\Big]\,ds_Q =$$

$$= cRK_1(cR)\int_{S_R}\frac{d\,u(Q)}{dn}ds - 2c^3RK_1(cR)\int_{S_R}\frac{du(Q)}{dn}ds + 2c^4RK_0(cR)\int_{S_R}u(Q)\,ds$$

$$- c^2RK_0(cR)\int_{S_R}u(Q)\,ds + \left[-2c^2K_0(cR) + c^3RK_1(cR)\right]\int_{S_R}\frac{du(Q)}{dn}\,ds +$$

$$-\left[-2c^2K_1(cR) + c^4RK_0(cR)\right]\int_{S_R}u(Q)\,ds.$$

From the asymptotic properties of the Mac Donald functions ([3],p. 146) follows that

$$\lim_{R\to 0^+} RK_0(cR) = 0, \qquad \lim_{R\to 0^+} cR\,K_1(cR) = 1.$$

Hence

$$\left|cR\,K_1(cR)\int_{S_R}\frac{d\Delta u(Q)}{dn}\,ds\right| \leqslant M_1\;cR\,K_1(cR)\,2\pi R \to 0, \quad \text{when} \quad R\to 0,$$

$$\left|K_0(cR)\int_{S_R}\frac{du(Q)}{dn}\,ds\right| < M_2\;K_0(cR)\,2\pi R\to 0, \quad \text{when} \quad R \longrightarrow 0;$$

where

$$M_1 = \sup_{Q\in S_R}\left|\frac{d\Delta u(Q)}{dn}\right|, \quad M_2 = \sup_{Q\in S_R}\left|\frac{du(Q)}{dn}\right|.$$

By means of a similar estimation we get

$$c^2R\,K_1(cR)\int_{S_R}\frac{du(Q)}{dn}\,ds \longrightarrow 0, \quad \text{when} \quad R\longrightarrow 0.$$

From the continuity of the function u follows that for every $\varepsilon > 0$ there exist $R_0 > 0$ such that for every $Q\in S_R$ and $0 < R < R_0$

$$|u(Q) - u(P)| < \varepsilon,$$

hence

$$\left|\int_{S_R}\left[u(Q) - u(P)\right]ds_Q\right| \leqslant 2\pi\varepsilon R$$

and

$$\lim_{R\to 0^+} 2c^2K_1(cR)\int_{S_R} u(Q)\,ds = \lim_{R\to 0^+} 4\pi c^2 RK_1(cR)u(P) = 4\pi c^2 u(P).$$

Finally, we get

$$4\pi c^2 u(P) = \int_S \Big[U(P,Q)\frac{d\Delta u(Q)}{dn} - 2c^2U(P,Q)\frac{du(Q)}{dn} + $$

$$+ 2c^2u(Q)\frac{dU(P,Q)}{dn_Q} - \Delta u(Q)\frac{dU(P,Q)}{dn_Q} +$$

$$+ \Delta_Q U(P,Q)\frac{du(Q)}{dn} - u(Q)\frac{d\Delta_Q U(P,Q)}{dn_Q}\Big]\,ds_Q.$$

From the above formula we get (6) when P is the interior point of the domain D.

4. Let E_2^+ denote the half-plane $t>0$ and $P(x,y)$, $Q(s,t)\in E_2^+$. Let

$$r_1^2 = (x-s)^2 + (y+t)^2, \quad R^2 = (x-s)^2 + y^2.$$

Theorem 2. The function

(8) $$G(P,Q) = U(r) - U(r_1)$$

is the Green function for the equation (1), for E_2^+ with a pole P with the boundary conditions

(9) $$G(P,Q)\Big|_{t=0} = 0, \qquad \Delta_Q G(P,Q)\Big|_{t=0} = 0,$$

Proof. The function $U(r_1)$ is of class C^4 for $t>0$ with respect to Q and satisfies the equation (1). For $t=0$ we have $r = r_1 = R$, $G(P,Q)\Big|_{t=0} = 0$.

If

$$\Delta_Q G(P,Q) = -2c^2\big[K_0(cr) - K_0(cr_1)\big] + c^2\big[crK_1(cr) - cr_1K_1(cr_1)\big],$$

then $\Delta_Q G(P,Q)\Big|_{t=0} = 0$.

Applying the formula (6) for $D = E_2^+$, formally, we get

$$u(P)=\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}\Big\{u(s,t)\big[2c^2D_tU(r)-D_t\Delta_QU(r)\big]+$$
$$+D_tu(s,t)\big[\Delta_QU(r)-2c^2U(r)\big]+$$
$$-\Delta u(s,t)D_tU(r)+D_t\Delta u(s,t)\;U(r)\Big\}\Big|_{t=0}\,ds,$$

$$0=\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}u(s,t)\big[2c^2D_tU(r_1)-D_t\Delta_QU(r_1)\big]+$$
$$+D_tu(s,t)\big[\Delta_QU(r_1)-2c^2U(r_1)\big]+$$
$$-\Delta u(s,t)D_tU(r_1)+U(r_1)D_t\Delta u(s,t)\Big|_{t=0}\,ds.$$

From the above formulae using (2) and (9) we obtain

$$(10)\qquad u(P)=\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_1(s)\big[2c^2D_tG(P,Q)-D_t\Delta_QG(P,Q)\big]\Big|_{t=0}\,ds+$$
$$-\frac{1}{4\pi c^2}\int_{-\infty}^{\infty}f_2(s)D_tG(P,Q)\Big|_{t=0}\,ds.$$

From (4) and (8) it follows that

$$D_tG(P,Q)\Big|_{t=0}=2c^2yK_0(cR),$$

$$D_t\Delta_QG(P,Q)\Big|_{t=0}=-4c^3R^{-1}yK_1(cR)+2c^4yK_0(cR),$$

hence

$$(11)\qquad u(P)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f_1(s)\big[2c\,R^{-1}yK_1(cR)+c^2y\,K_0(cR)\big]ds\;+$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}f_2(s)y\,K_0(cR)\,ds.$$

5. We shall prove under convenient assumptions of the functions f_1, f_2 that the function $u(x,y)$ given by the formula (11) satisfies the equation (1) in E_2^+ and the boundary data (2).

We shall give the following lemmas [4].

Lemma 1. If the function f is absolutely integrable in the interval $(-\infty,\infty)$, then the integral

$$I(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\,K_0(cR)\,ds$$

is of class C^∞ in E_2^+ and

$$D_{x^p y^q} I(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\, D_{x^p y^q} K_o(cR)\, ds.$$

Lemma 2. If the funtion f is absolutely integrable in the interval $(-\infty,\infty)$ and continuos at the point x_o, then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(s) c y R^{-1} K_1(cR) ds \longrightarrow f(x_o) \quad \text{as} \quad (x,y)\to(x_o,0^+).$$

We shall prove

Lemma 3. If the function f is absolutely integrable in the interval $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(s) y K_o(cR) ds \longrightarrow 0 \qquad \text{as} \quad (x,y)\longrightarrow(x_o,0).$$

Proof. The function $K_o(cR)$ is non-negative and decreasing for $R>0$ ([3], p.146). From the asymptotic formula ([3], p.146)

$$K_o(cy) \underset{y\to 0^+}{\approx} \ln \frac{2}{cy},$$

we get $\lim\limits_{y\to 0^+} y\, K_o(cy) = 0.$

Because $R>y>0$, then

$$\left|\int_{-\infty}^{\infty} f(s) y K_o(cR) ds\right| \leqslant y K_o(cy)\int_{-\infty}^{\infty} |f(s)| ds \to 0, \quad \text{when} \quad y\to 0^+.$$

Theorem 3. If the functions f_1, f_2 are absolutely integrable in the interval $(-\infty,\infty)$ and continuous at the point x_o, then the function $u(x,y)$ defined by the formula (11) is a solution of the equation (1) in the half-plane $y>0$ and satisfies the boundary conditions (2).

Proof. It is easy to observe that

$$2cR^{-1}K_1(cR) = -\,2D_y K_o(cR).$$

From lemma 1 follows that the function $u(x,y)$ given by the formula (10) resp. (11) is of class C^∞ in E_2^+ and

$$(\Delta - c^2)^2 u(x,y) = \frac{1}{4\pi c^2}\int_{-\infty}^{\infty} f_1(s)(\Delta_P - c^2)^2\left[2c^2 D_t G(P,Q) - D_t\Delta_Q G(P,Q)\right]\Big|_{t=0} ds +$$

$$-\frac{1}{4\pi c^2}\int_{-\infty}^{\infty} f_2(s)(\Delta_P - c^2)^2 D_t G(P,Q)\Big|_{t=0} ds \;=$$

$$= \frac{1}{4\pi c^2}\int_{-\infty}^{\infty} f_1(s)\Big[2c^2 D_t(\Delta_P - c^2)^2 G(P,Q) +$$
$$- D_t \Delta_Q(\Delta_P - c^2)^2 G(P,Q)\Big]\Big|_{t=0} ds +$$
$$- \frac{1}{4\pi c^2}\int_{-\infty}^{\infty} f_2(s) D_t(\Delta_P - c^2)^2 G(P,Q)\Big|_{t=0} ds = 0,$$

because the function $G(P,Q)$ satisfies the equation (1) for $P \neq Q$.

From lemmas 2 and 3 follows that the function $u(P)$ satisfies the boundary condition

$$\lim_{(x,y)\to(x_0,0)} u(P) = f(x_0).$$

Concerning the proof of the second boundary condition we apply the formula

$$\Delta_P\big(yK_0(cR)\big) = y\Delta_P K_0(cR) + 2D_y K_0(cR).$$

Since the function $K_0(cR)$ satisfies the equation $\Delta u - c^2 u = 0$ in the half-plane $y > 0$, we obtain

$$\Delta_P\big(yK_0(cR)\big) = c^2 y\, K_0(cR) + 2D_y K_0(cR).$$

From (4) $D_y K_0(cR) = -cR^{-1} y K_1(cR)$, hence

$$\Delta_P\big(cR^{-1} y K_1(cR)\big) = -\Delta_P D_y K_0(cR) = -D_y\big(\Delta_P K_0(cr)\big) =$$
$$= -D_y\big(c^2 K_0(cR)\big) = -c^3 y R^{-1} K_1(cR).$$

Using lemma 1, we get

$$\Delta u(P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(s) c^4 y K_0(cR) ds + \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(s)\big[2c\, R^{-1} y K_1(cR) - c^2 y K_0(cR)\big] ds.$$

Applying once more lemmas 2 and 3 we obtain

$$\lim_{(x,y)\to(x_0,0)} \Delta u(P) = f_2(x_0).$$

B i b l i o g r a p h y

[1] B.M. Budak, A.A. Samarski, A.N. Tichonow, Zadania i problemy fizyki matematycznej, PWN, Warszawa 1963.

[2] M. Krzyżański, Partial differential equations of second order, vol.I, vol.II, PWN, Warszawa 1971.

[3] N.N. Lebiediew, Funkcje specjalne i ich zastosowania, PWN, Warszawa 1957.

[4] E. Wachnicki, Rozwiązanie pewnych zagadnień brzegowych dla równania $\Delta u - c^2 u = 0$, Rocznik Naukowo-Dydaktyczny WSP w Krakowie /Prace Matematyczne/, z.41, p.165-174.