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Adam Wachułka

ON THE NEUMANN PROBLEM FOR CERTAIN ANGULAR DOMAINS

1. In this paper we shall give the solution of the Neumann problem for the equation

(1) $$\Delta u(x,y) = 0$$

in the domain

(2) $$D = \{(x,y) : 0 < y < x, 0 < x < \infty\}$$

with the boundary conditions

(3) $$D_n u(x,0) = f_1(x),$$

(4) $$D_n u(x,x) = f_2(x),$$

where n denotes an inward normal.

2. In order to solve the problem (1), (3), (4) we shall construct a convenient Green function using the method of symmetric images. Let l_1 denote the straight line $t=0$, l_2: $t=s$, l_3: $s:0$, l_4: $t=-s$. Let $X_1(x,y)$ denote the point of the plane (s,t), $X_1 \in D$, $X_2(y,x)$ the point symmetric to X_1 with respect to l_2, $X_3(-y,x)$ the point symmetric to X_2 with respect to l_3, $X_4(-x,y)$ the point symmetric to X_3 with respect to l_4, $X_5(-x,y)$ the point symmetric to X_4 with respect to l_1, $X_6(-y,-x)$ the point symmetric to X_5 with respect to l_2, $X_7(y,-x)$ the point symmetric to X_6 with respect to l_3, $X_8(x,-y)$ the point symmetric to X_7 with respect to l_4.

Let $Y(s,t)$ denote an arbitrary point belonging to the closure $\bar{D}$ of the domain D.

Let

(5) $$r_1^2 = (s-x)^2 + (t-y)^2, \quad r_2^2 = (s-y)^2 + (t-x)^2,$$

$$r_3^2 = (s+x)^2 + (t-y)^2, \quad r_4^2 = (s+y)^2 + (t-x)^2,$$

$$r_5^2 = (s+x)^2 + (t+y)^2, \quad r_6^2 = (s+y)^2 + (t+x)^2,$$

$$r_7^2 = (s-x)^2 + (t+y)^2, \quad r_8^2 = (s-y)^2 + (t+x)^2.$$

If $Y \in l_2$, then

(6) $r_1 = r_2 = R_1,\ r_3 = r_4 = R_2,\ r_5 = r_6 = R_3,\ r_7 = r_8 = R_4.$

If $Y \in l_1$, then

(7) $r_1 = r_7 = R_5,\ r_3 = r_5 = R_6,\ r_2 = r_8 = R_7,\ r_4 = r_6 = R_8.$

We shall prove

Theorem 1. The function

(8) $$G(x,y;\ s,t) = \sum_{i=1}^{8} \ln r_i$$

is the Green function for the domain D and the Neumann problem.

Proof. The function $G(X,Y)$ is a harmonic function with recpect to the point Y.

$$D_n G(X,Y)\Big|_{t=o} = D_t G\Big|_{t=o} = \frac{1}{r_1}\frac{t-y}{r_1} + \frac{1}{r_7} + \frac{t-y}{r_7} + \frac{1}{r_3}\frac{t-y}{r_3} + \frac{1}{r_5}\frac{t+y}{r_5} +$$

$$= \frac{1}{r_2}\frac{t-x}{r_2} + \frac{1}{r_8}\frac{t+x}{r_8} + \frac{1}{r_6}\frac{t+x}{r_6}\Big|_{t=o} = 0.$$

For $Y \in l_2$ we get

$$D_n G = \frac{\sqrt{2}}{2}\left(D_t G - D_s G\right)\Big|_{t=s} = 0.$$

For $\ln r_3 + \ln r_8$ we obtain

$$D_n(\ln r_3 + \ln r_8) = \frac{1}{r_3}\frac{t-y}{r_3} + \frac{1}{r_8}\frac{t+y}{r_8} - \frac{1}{r_3}\frac{s+x}{r_3} - \frac{1}{r_8}\frac{s-y}{r_8}.$$

Hence

$$D_n(\ln r_3 + \ln r_8)\Big|_{t=o} = 0.$$

Similarly

$$D_n(\ln r_i + \ln r_j)\Big|_{s=t} = 0, \quad \text{for} \quad \begin{aligned} i &= 1, j = 2,\\ i &= 4, j = 7,\\ i &= 5, j = 6. \end{aligned}$$

3. Applying the formulae (6), (7) we get

(9) $$G(x,y;s,o) = \ln R_5 + \ln R_6 + \ln R_7 + \ln R_8,$$

(10) $$G(x,y;s,s) = \ln R_1 + \ln R_2 + \ln R_3 + \ln R_4.$$

Let

(11) $$u(x,y)=\frac{1}{\pi}\int_0^\infty f_1(s)\,G(x,y;s,o)\,ds+\frac{1}{\pi}\int_0^\infty f_2(s)\,G(x,y;s,s)\,ds=\sum_{i=1}^{8} J_i(x,y),$$

where

(12) $$J_1(x,y)=\frac{1}{\pi}\int_0^\infty f_2(s)\ln R_1 ds,\quad J_2(x,y)=\frac{1}{\pi}\int_0^\infty f_2(s)\ln R_2 ds,$$

$$J_3(x,y)=\frac{1}{\pi}\int_0^\infty f_2(s)\ln R_3 ds,\quad J_4(x,y)=\frac{1}{\pi}\int_0^\infty f_2(s)\ln R_4 ds,$$

$$J_5(x,y)=\frac{1}{\pi}\int_0^\infty f_1(s)\ln R_5 ds,\quad J_6(x,y)=\frac{1}{\pi}\int_0^\infty f_1(s)\ln R_6 ds,$$

$$J_7(x,y)=\frac{1}{\pi}\int_0^\infty f_1(s)\ln R_7 ds,\quad J_8(x,y)=\frac{1}{\pi}\int_0^\infty f_1(s)\ln R_8 ds.$$

4. In order to prove that $u(x,y)$ is a solution of the problem (1), (3), (4) we shall give the convenient lemmas. Let W denote a rectangle

(13) $$W=\{(x,y);\ a<x<A,0<b<y<B\},$$

W_1 the set

(14) $$W_1=\{(x,y);\ c<x<C,0<d<y<x<D\},$$

a,A,c,C being arbitrary numbers and b,B,d,D arbitrary positive numbers.

We assume that the functions $f(s)$, $f_2(s)$ are bounded and continuous and for every $N>0$

(I) $$\int_N^\infty |f_1(s)||\ln s|\,ds<\infty$$

and

(II) $$\int_N^\infty |f_2(s)||\ln s|\,ds<\infty.$$

From (I), (II) it follows that $f_1(s)$ and $f_2(s)$ are absolutely integrable.

Let

$$M=\max(|f_1(s)|,|f_2(s)|).$$

Lemma 1. The integrals $J_k, k=1,2,3,4$ are uniformly convergent in every W.

Proof. We shall prove this lemma for the integral $J_1(x,y)$. For J_2,J_3,J_4 the proof is analogous. Let $\varepsilon>0$ be an arbitrary positive number. For $s>s_0$ we have

(15) $$\frac{1}{2}s<(s-x)^2+y^2<2s$$

for every (x,y) belonging to a bounded domain.

Hence for $s > s_0$, $N(W,\varepsilon)$ we obtain

$$\int_N^\infty |f_1(s)||\ln[(s-x)^2 + y^2]|\, ds \leqslant \int_N^\infty |f_1(s)||\ln 2s|\, ds \leqslant 2\int_N^\infty |f_1(s)||\ln s|\, ds < \varepsilon$$

for every $(x,y) \in W$. The above inequality is a sufficient condition for the uniformly convergence of the integral J_1.

Lemma 2. The integrals $J_p, p=5,6,7,8$ are uniformly convergent in every W_1.

Proof. We shall give the proof only for J_5. For J_6, J_7, J_8 the proof is analogous. Applying the inequality (15) we get

$$\int_N^\infty |f_2(s)| \ln \sqrt{(s-x)^2+(s-y)^2}\, ds \leqslant 2\int_N^\infty |f_2(s)||\ln s|\, ds < \varepsilon$$

for every point $(x,y) \in W_1$.

Let

(16) $$E_{jkl} = \int_0^\infty f_1(s) D_{x^j y^k} R_l\, ds, \quad j,k=0,1,2; \; l = 1,2,3,4,$$

(17) $$E_{jkp} = \int_0^\infty f_2(s) D_{x^j y^k} R_p\, ds, \quad j,k=0,1,2, \; p=5,6,7,8.$$

Lemma 3. The integrals (16) are uniformly convergent in every set W and the integrals (17) are uniformly convergent in every set W_1.

Proof. For $s > N$ the integral $M\int_N^\infty \frac{1}{s^2} ds$ or $\int_N^\infty |f_1(s)||\ln s|\, ds$ is a majorant for all integrals (16), (17) and this is a sufficient condition for the uniform convergence of the integrals (16), (17).

From lemma 3 follows

Lemma 4. If the assumptions of the foregoing lemma are satisfied then

(18) $$D_{x^j y^k} J_1(x,y) = \int_0^\infty f_1(s) D_{x^j y^k} R_l\, ds, \quad j, k=0,1,2, \; l=1,2,3,4,$$

(19) $$D_{x^j y^k} J_2(x,y) = \int_0^\infty f_2(s) D_{x^j y^k} R_p\, ds, \quad j, k=0,1,2, \; p=5,6,7,8.$$

Lemma 5. The function $u(x,y)$ defined by the formula (11) is of class C^2 in the domain D and satisfies the equation (1).

Proof. Applying lemmas 3,4,5 we obtain

$$\Delta u(x,y) = \frac{1}{\pi}\int_0^\infty f_1(s)\Delta_{x,y} G(x,y;s,o)\, ds + \frac{1}{\pi}\int_0^\infty f_2(s)\Delta_{x,y} G(x,y;s,s)\, ds = 0$$

since $G(x,y;s,t)$ is symmetric and

$$\Delta_{x,y} G(x,y;s,o) = 0, \quad \Delta_{x,y} G(x,y;s,s) = 0.$$

5. Now we shall prove that the function $u(x,y)$ defined by formula (11) satisfies the boundary conditions (3), (4).

Let

$$\bar{f}_i(s) = f_i(s) \quad \text{for} \quad s > 0, \quad \text{and} \quad \bar{f}_i(s) = 0 \quad \text{for} \quad s < 0, i=1,2.$$

From (18), (19) it follows that

$$D_y u(x,y) = K_1(x,y) + K_2(x,y) + K_3(x,y) + K_4(x,y) + K_5(x,y), \tag{20}$$

where

$$K_1(x,y) = \frac{1}{\pi}\int_0^\infty \bar{f}_1(s)\frac{y}{(s-x)^2+y^2}\,ds,$$

$$K_2(x,y) = \frac{1}{\pi}\int_0^\infty \bar{f}_1(s)\frac{y}{(s-x)^2+y^2}\,ds,$$

$$K_3(x,y) = \frac{1}{\pi}\int_0^\infty \bar{f}_1(s)\left[\frac{-s+y}{(s-y)^2+x^2} + \frac{s+y}{(s-y)^2+x^2}\right]ds,$$

$$K_4(x,y) = \frac{1}{\pi}\int_0^\infty \bar{f}_1(s)\left[\frac{-s+y}{(s+x)^2+(s-y)^2} + \frac{s+y}{(s-x)^2+(s+y)^2}\right]ds,$$

$$K_5(x,y) = \frac{1}{\pi}\int_0^\infty \bar{f}_1(s)\left[\frac{-s+y}{(s-x)^2+(s-y)^2} + \frac{s+y}{(s-x)^2+(s+y)^2}\right]ds.$$

$$D_n u(x,y) = \frac{\sqrt{2}}{2}\left(D_y u - D_x u\right) = K_6(x,y) + K_7(x,y) + K_8(x,y) +$$

$$= K_9(x,y) + K_{10}(x,y) + K_{11}(x,y) + K_{12}(x,y) + K_{13}(x,y),$$

where

$$K_6(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{y}{(s-x)^2+y^2} - \frac{x}{(s-y)^2+x^2}\right]ds,$$

$$K_7(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{y}{(s+x)^2+y^2} - \frac{x}{(s+y)^2+x^2}\right]ds,$$

$$K_8(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{-s+y}{(s-y)^2+x^2} + \frac{s-x}{(s-x)^2+y^2}\right]ds,$$

$$K_9(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{s+y}{(s+y)^2+x^2} - \frac{s+x}{(s+x)^2+y^2}\right]ds,$$

$$K_{10}(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{-s+y}{(s-x)^2+(s-y)^2} + \frac{s-x}{(s-x)^2+(s-y)^2}\right]ds,$$

$$K_{11}(x,y) = \frac{-1}{\pi}\int_0^\infty \bar{f}_2(s)\left[\frac{s+y}{(s-x)^2+(s+y)^2} + \frac{s+x}{(s+x)^2+(s-y)^2}\right]ds,$$

$$K_{12}(x,y) = \frac{-1}{\pi}\int_0^\infty f_2(s)\left[\frac{-s+y}{(s+x)^2+(s-y)^2} + \frac{s-x}{(s-x)^2+(s+y)^2}\right] ds,$$

$$K_{13}(x,y) = \frac{-1}{\pi}\int_0^\infty f_2(s)\left[\frac{s+y}{(s+x)^2+(s-y)^2} - \frac{s+x}{(s+x)^2+(s+y)^2}\right] ds.$$

We shall prove

Lemma 6. If the functions f_1, f_2 satisfy the assumptions of the lemmas 1-5 then

(21) $$D_y u(x,y) \longrightarrow f_1(x_0,0) \quad \text{as} \quad (x,y) \longrightarrow (x_0,0^+),\ x_0 > 0,$$

(22) $$\frac{\sqrt{2}}{2}\left(D_y u(x,y) - D_x u(x,y)\right) \longrightarrow f_2(x_0,x_0)$$

as $(x,y) \longrightarrow (x_0,x_0),\ y < x,\ x_0 > 0.$

Proof of 21 .

$$K_1(x,y) \longrightarrow f_1(x_0) \quad (\text{see } [1]),$$

$$|K_2(x,y)| \leqslant \frac{1}{\pi} M y \int_0^\infty \frac{ds}{(s+x_0)^2} < \frac{1}{\pi} M y \int_{x_0}^\infty \frac{1}{t^2}\, dt \longrightarrow 0.$$

Applying the uniform convergence of the integrals $K_j, j=3,4,5,$ we get

$$K_j(x,y) \longrightarrow K_j(x_0,0) = 0, \quad j = 3,4,5.$$

Proof of (22). Applying the uniform convergence of the convenient integrals we get

$$K_j(x,y) \text{---} K_j(x_0,x_0) = 0, \quad j = 6,7,8,9,\ 11,12,13.$$

After the change of the variable

$$s-x = t(x-y),\ s-y = (t-1)(x-y),$$

we get

(23) $$\frac{1}{\pi}\int_{-\infty}^\infty \frac{(x-y)^2}{(x-y)^2(t^2+(t-1))^2}\, dt = 1.$$

Since

$$K_{10}(x,y) = \frac{1}{\pi}\int_{-\infty}^\infty \bar{f}_2(s)\frac{x-y}{(s-x)^2+(s-y)^2}\, ds =$$

$$= f(x)\frac{1}{\pi}\int_{-\infty}^\infty \frac{x-y}{(s-x)^2+(s-y)^2}\, ds + \frac{1}{\pi}\int_{-\infty}^\infty\left[\bar{f}_2(s) - f_2(x_0)\right]\frac{(x-y)\, ds}{(s-x)^2+(s-y)^2},$$

$$K_{10}(x,y) - f_2(x_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\bar{f}_2(s) - f_2(x_0)\right]\frac{x-y}{(s-x)^2+(s-y)^2}\,ds = P_1(x,y).$$

Let ε denote an arbitrary positive number and δ such a number that

$$|f_2(s) - f_2(x_0)| < \varepsilon \ , \quad \text{if} \quad |s-x_0| < \delta \ , \ s > 0.$$

Let E_1 denote the set $E_1 = \{ s : |s-x_0| > \delta \}$, and E_2 the set $E_2 = \{ s : |s-x| > \frac{\delta}{4} \}. \ |x-x_0| < \frac{\delta}{4} \}$. Since $E_2 \supset E_1$ we get

$$\left|\frac{1}{\pi}\int_{|s-x|>\frac{\delta}{4}}\left(\bar{f}_2(s) - f_2(x_0)\right)\frac{x-y}{(s-x)^2+(s-y)^2}\,ds\right| \leqslant \frac{2M}{\pi}(x-y)\int_{-\infty}\frac{ds}{\left(\frac{\delta}{4}\right)^2 + (s-y)^2} \leqslant$$

$$\leqslant \frac{2M}{\pi}(x-y)\int_{E_2}\frac{du}{\left(\frac{\delta}{4}\right)^2 + u^2} \longrightarrow 0, \text{as } x \to x_0, y \to x_0, x > y.$$

For $P_1(x,y)$ we get the estimation

$$|P_1(x,y)| \leqslant \frac{1}{\pi}\int_{|s-x_0|<\delta}\left|\left(\bar{f}_2(s) - f_2(x_0)\right)\right| \frac{x-y}{(s-x)^2+(s-y)^2}\,ds +$$

$$+ \frac{2M}{\pi}\int_{|s-x_0|\geqslant\delta}\frac{x-y}{(s-x)^2+(s-y)^2}\,ds.$$

From the above inequality it follows that

$$K_{10}(x,y) \longrightarrow f_2(x_0), \text{ as } (x,y) \longrightarrow (x_0,x_0), \ x > y, x_0 > 0.$$

From lemmas 1,2,3,4,5,6 follows

Theorem 2. If the functions f_1, f_2 are continous, bounded and satisfy the assumptions (I), (II) then the function $u(x,y)$ defined by the formula (11) is the solution of the problem (1), (3), (4).

B i b l i o g r a p h y

[1] M. Krzyżański, Partial Differential Equations of Second Order, vol.1, Warszawa 1972.