ROCZNIK NAUKOWO—DYDAKTYCZNY WSP W KRAKOWIE

Zeszyt 61 Prace matematyczne VIII Rok 1977

Andrzej Grząślewicz

ON THE SOLUTIONS OF THE GENERALIZING EQUATION OF HOMOMORPHISM

Introduction

We shall consider the equation

$$H_1(x) \ast H_2(y) = H_3(x \cdot y), \tag{i}$$

where the functions H_1, H_2, H_3 are defined on some subsets of $A \times A \times G$, $(G, \cdot)$ is an arbitrary semigroup with a unit "e" and the values of these functions belong to a set K, in which an partially associative operation "$\ast$" is defined.

In the first part we give the fundamental definitions and theorems, which are necessary for the later parts of this paper. We define the solution of equation (i), the extension of the solution, and, generalizing the definitions given by J. Aczel in the paper [1] we define the right regular element, left regular element and strictly regular element.

In the second part we give the general solution of equation (i) /in a particular case we obtain the results of J. Aczél given in [1]/ and the theorems concerning the extensions. Moreover, we give one theorem of the extension of the solution of equation (i) in the case, where the functions H_1, H_2, H_3 are defined on some subset of the Ehresmann groupoid, and the values of this functions belong to an Ehresmann´s groupoid.

More detailed informations about earlier results regarding the sulutions of the equation (i) are given in this paper.

Chapter I

Preliminary definitions and theorems

By $[X \multimap\!\!\rightarrow Y]$ we shall denote the set of all functions /called partial functions/ the domain of which is contained in the set X, and the range of which is contained in the set Y.
By $[X \longrightarrow Y]$ we shall denote the set of all functions the domain of which is the set X, and the range of which is contained in the set Y. The do-

main of the function f will be denoted by D_f and the range will be denoted by $\mathcal{Q}_f$. If the function f belongs to the set $[X \times Y \longrightarrow Z]$ then we shall write:

$$D_f^1 = \{x \in X : \bigvee_{y \in Y} ((x,y) \in D_f)\},$$

$$D_f^2 = \{x \in Y : \bigvee_{y \in X} ((y,x) \in D_f)\}.$$

Definition 1. The pair $(A, \cdot)$ where A is an arbitrary set and "$\cdot$" is an arbitrary element of the set $[A \times A \longrightarrow A]$ will be called the binary algebraic structure /shortly - structure/.
If $B \subset A$ and $C \subset A$ then

$$B \cdot C := \{x \cdot y : (x,y) \in D_\cdot \wedge x \in B \wedge y \in C\}.$$

Definition 2. Let $(A, \cdot)$ be an arbitrary structure and let S be an arbitrary subset of the set A. The pair $(S, *)$, where "$*$" is the restriction of the function "$\cdot$" to the set $S \times S$ will be called the substructure of A, and, for simplicity, note $(S, \cdot)$.

Definition 3. We shall call the substructure $(S, *)$ of the structure $(A, \cdot)$ closed, if $\mathcal{Q}_* \subset S$.

Definition 4. We call the structure $(A, \cdot)$ associative, if for arbitrary $x,y,z \in A$ the following conditions are fulfilled:

1° $\quad [(x,y) \in D_\cdot \wedge (y,z) \in D_\cdot] \Longrightarrow (x, y \cdot z) \in D_\cdot ,$

2° $\quad [(x,y) \in D_\cdot \wedge (y,z) \in D_\cdot] \Longrightarrow (x \cdot y, z) \in D_\cdot ,$

3° $\quad [(x,y) \in D_\cdot \wedge (x \cdot y, z) \in D_\cdot] \Longrightarrow (y,z) \in D_\cdot ,$

4° $\quad [(y,z) \in D_\cdot \wedge (x, y \cdot z) \in D_\cdot] \Longrightarrow (x,y) \in D_\cdot ,$

5° $\quad [(x,y) \in D_\cdot \wedge (y,z) \in D_\cdot] \Longrightarrow [(x \cdot y) \cdot z = x \cdot (y \cdot z)] .$

After W. Waliszewski ([4], p.6) we will use the following definition of a groupoid.

Definition 5. A structure $(A, \cdot)$ is called a groupoid, if it is associative and if the following conditions are fulfilled:

1° $\quad \bigwedge_{x,y,z} \{[(x,y) \in D_\cdot \wedge (x,z) \in D_\cdot \wedge (x \cdot y = x \cdot z)] \Longrightarrow y = z\},$

$2^o \quad \bigwedge_{x,y,z} \{[(y,x) \in D_\bullet \wedge (z,x) \in D \wedge (y \cdot x = z \cdot x)] \Longrightarrow y = z\},$

$3^o \quad \bigwedge_{x \in A} \bigvee_{y \in A} \{(x,y) \in D_\bullet \wedge x \cdot y \in A_o\},$

where

$$A_o := \{e : e \in A \wedge (e,e) \in D_\bullet \wedge e.e = e\}.$$

Definition 5 is equivalent to the definition of the Ehresmann groupoid ([3], p.9), and therefore we will call the groupoid in the sense of definition 5 the Ehresmann groupoid.

<u>Definition 6</u>. The Ehresmann groupoid $(A,\cdot)$ in which the following condition is fulfilled

$$\bigwedge_{x,y} \bigvee_{z} [(x,z) \in D_\bullet \wedge (z,y) \in D_\bullet]$$

will be called the Brandt groupoid.

In the paper [4] the following theorem is proved:

<u>Theorem 1</u>. The pair $(A,\cdot)$ is the Ehresmann groupoid iff there exists a decomposition U of the set A on such disjoint sets, that

$$D_\bullet \subset \bigcup \{M \times M : M \in U\}$$

and every set $\bar{A} \in U$ with the operation "$\cdot$" restricted to the set $\bar{A} \times \bar{A}$ is the Brandt groupoid. If $(A,\cdot)$ is the Ehresmann groupoid, then such decomposition is synonymous.

<u>Definition 7</u>. We say that the structures $(A,\cdot)$ and (B,o) are isomorphic, if there exists a bijection $f : A \longrightarrow B$ such that the following conditions

a/ $\quad \bigwedge_{x,y \in A} [(x,y) \in D_\bullet \Longleftrightarrow (f(x), f(y)) \in D_o],$

b/ $\quad \bigwedge_{(x,y) \in D_\bullet} [f(x \cdot y) = f(x) \circ f(y)]$

are fulfilled.

One can prove ([3], p.111-112) that:
Every Brandt groupoid is isomorphic to some groupoid of the form $(A \times A \times G, *)$ /called the product groupoid/ where A is the set, $(G,\cdot)$ is the group and „$*$" is an operation defined as follows:

a/ $((x,y,\alpha),(z,v,\beta)) \in D_{*} \Longleftrightarrow y = z,$

b/ $((x,y,\alpha),(z,v,\beta)) \in D_{*} \Longrightarrow (x,y,\alpha) * (z,v,\beta) = (x,v,\alpha\cdot\beta)$

It is easy to verify, that every product groupoid is a Brandt groupoid. In a particular case, when $(G,\cdot)$ is one element group the product Brandt groupoid is called the pair Brandt groupoid and denoted by $(A \times A, *)$. Let $(A,\cdot)$ be an arbitrary structure.

Definition 8. The element $a \in A$ will be called right /left/ regular, if for arbitrary two elements $x \in A$, $y \in A$ such that $(x, a) \in D$, and $(y, a) \in D$, $((a, x) \in D, (a, y) \in D)$ there is

$$[x \cdot a = y \cdot a] \Longrightarrow x = y \qquad ([a \cdot x = a \cdot y] \Longrightarrow x = y).$$

The set of all right regular elements of A will be designed by A_{rr}, and the set of all left regular elements of A will be designed by A_{lr}. Using definition 8 it is easy to prove the following

Lemma 1. If a is an arbitrary element of the set A_{rr}, b an arbitrary element of the set A_{lr} and c an arbitrary element of the set A, then each of the equations

a/ $x \cdot a = c$ and b/ $b \cdot x = c$

have no more than one solution.

From lemma 1 there follows

Corollary 1. Every element $a \in A_{rr}$ has at most one left unit a_l and every element $b \in A_{lr}$ has at most one right unit b_r.

Lemma 2. If $(A,\cdot)$ is an associative structure, a is an arbitrary element of the set $A_{rr} \cap A_{lr}$ having the right unit a_r and the left unit a_l and one of the two equations

a/ $a \cdot x = a_l$ and b/ $x \cdot a = a_r$

has a solution, then the other one has a solution, too, and the solutions are equal.

Proof. Let, for example, the equation a/ possess the solution x_1, i.e. $a \cdot x_1 = a_l$.

Multiplying this equality by a from the right side we obtain

$$a \cdot x_1 \cdot a = a_l \cdot a = a \cdot a_r.$$

Since $a \in A_{lr}$ we get

$$x_1 \cdot a = a_r.$$

and then x_1 is the solution of equation b/. One can prove analogously, that the solution of equation b/ is also the solution of equation a/.

Lemma 3. If $(A,\cdot)$ is an associative structure, a is an arbitrary element of the set $A_{rr} \cap A_{lr}$ possessing the left unit a_l and right unit a_r such that $a_l \in A_{lr}$ and $a_r \in A_{rr}$ and x is the solution of the equation a/ or b/ from lemma 2 then $x \in A_{rr} \cap A_{lr}$.

Proof. Let x be, for example, the solution of equation a/ from lemma 2, let x_1, x_2, y_1, y_2 be arbitrary elements such that

$(x,x_1) \in D_\cdot$, $(x, x_2) \in D_\cdot$, $(y_1,x) \in D_\cdot$, $(y_2, x) \in D_\cdot$ and let

$$x \cdot x_1 = x \cdot x_2 \qquad \text{and} \qquad y_1 \cdot x = y_2 \cdot x .$$

Then we have

$$a \cdot x \cdot x_1 = a \cdot x \cdot x_2 \qquad \text{and} \qquad y_1 \cdot x \cdot a = y_2 \cdot x \cdot a,$$

and consequently

$$a_l \cdot x_1 = a_l \cdot x_2 \qquad \text{and} \qquad y_1 \cdot a_r = y_2 \cdot a_r .$$

Thus

$$x_1 = x_2 \qquad \text{and} \qquad y_1 = y_2 .$$

So $\qquad x \in A_{rr} \cap A_{lr}$,

which, thanks to lemma 2, completes the proof.

Lemma 4. If $(A,\cdot)$ is an associative structure, a is an arbitrary element of the set A_{rr} possessing a left unit a_l, and b is an arbitrary element of the set A_{lr} possessing a right unit b_r, then

$$a_l \in A_{rr} \qquad \text{and} \qquad b_r \in A_{lr} .$$

Proof. Let a be an arbitrary element of the set A_{rr}, a_l its left unit and let x,y be arbitrary elements of the set A such that

$$x \cdot a_l = y \cdot a_l .$$

Multiplying this equality by a from the right side we obtain

$$x \cdot a = y \cdot a.$$

$a \in A_{rr}$, so we have

$x = y,$

thus $a_1 \in A_{rr}$. One can prove analogously that $b_r \in A_{lr}$.

Lemma 5. If $(A,\cdot)$ is an associative structure, then $(A_{rr},\cdot)$ and $(A_{lr},\cdot)$ are closed substructures of $(A,\cdot)$.

Proof. Let a, b be arbitrary elements of the set A_{rr} such that $(a,b) \in D$, and let for some $x,y \in A$ the following equality hold $x\cdot(a\cdot b) = y\cdot(a\cdot b)$.

From the associativity of the structure $(A,\cdot)$ and from the regularity of a and b we obtain

$x = y,$

thus $a\cdot b \in A_{rr}$. One can prove analogously that $a\cdot b \in A_{lr}$.
Consequently $(A_{rr},\cdot)$ and $(A_{lr},\cdot)$ are closed substructures of $(A,\cdot)$.

Definition 9. We will call an element a of the set A strictly regular, if the following conditions hold:

1^o $a \in A_{rr} \cap A_{lr}$,

2^o $\bigvee_{a_r} \bigvee_{a_1} \left[(a_1,a) \in D, \cdot (a,a_r) \in D, \wedge \left(a_1 \cdot a = a \cdot a_r = a\right)\right]$,

3^o $a_r \in A_{rr}$ and $a_1 \in A_{lr}$,

4^o equation $a \cdot x = a_1$

possesses a solution.

We will denote the set of all strictly regular elements of the set A by A_r. From conditions 1^o and 2^o of definition 9 and from lemma 4 we obtain

$$a_1 \in A_{rr} \quad \text{and} \quad a_r \in A_{lr}$$

for $a \in A_r$.

Condition 3^o in definition 9 does not follow from other conditions of this definition. It is illustrated by following

Example 1. Let R be the set of real numbers and $([R \to R], o)$ be the structure with the superposition „o" defined as follows

1^o. $(f,g) \in D_o \Longleftrightarrow D_g \subset D_f,$

2^o. $(f,g) \in D_o \Longrightarrow f o g = \{(x,y) : x \in Dg \wedge y = f(g(x))\}$.

It is easy to verify that

a/ the function f is left regular iff it is an one-to-one function,
b/ the function f is right regular iff $D_f = R$.

It is easy to see that for the function

$$f : x \longrightarrow \ln x \qquad \text{for } x > 0$$

the conditions 1^0, 2^0, 4^0 of definition 9 hold and that the function

$$f_r : x \longrightarrow x \qquad \text{for } x > 0$$

being the right unit for the function f is not right regular, because $D_{f_r} \neq R$.

<u>Lemma 6</u>. If $(A,\cdot)$ is an associative structure, then $(A_r,\cdot)$ is the Ehresmann groupoid.

<u>Proof.</u> At first we shall prove that $(A_r,\cdot)$ is a closed substructure of $(A,\cdot)$. Let a, b be arbitrary elements of the set A_r such that $(a,b) \in D_\cdot$. From lemma 5 and condition 1^0 of definition 9 we obtain that $a\cdot b \in A_{rr} \cap A_{lr}$, hence condition 1^0 of definition 9 holds for the element $a\cdot b$. Moreover, we have

$$a \cdot b = (a_l \cdot a)\cdot b = a_l \cdot (a \cdot b)$$

and

$$a \cdot b = a \cdot (b \cdot b_r) = (a \cdot b) \cdot b_r .$$

From corollary 1 we obtain that a_l and b_r are the only units of the element $a \cdot b$, thus conditions 2^0 and 3^0 of definition 9 are fulfilled. Let y, z be arbitrary elements of the set A such that

$$a \cdot y = a_l \qquad \text{and} \qquad b \cdot z = b_l$$

/a_l and b_l exist, because $a \in A_r$ and $b \in A_r$/.
From lemma 2 we obtain

$$y \cdot a = a_r ,$$

and, because $(a,b) \in D_\cdot$, $a \in A_r$ and $b \in A_r$, then

$$a \cdot a_r \cdot b = a \cdot a_r \cdot a_r \cdot b = a \cdot b = a \cdot b_l \cdot b .$$

Thus $(a_r, a_r) \in D_\cdot$ and $a_r = b_l$, i.e. $(z,y) \in D_\cdot$.

We have

$$(a\cdot b)\cdot(z\cdot y) = a\cdot(b\cdot z)\cdot y = a\cdot b_1\cdot y = (a\cdot a_r)\cdot y = a\cdot y = a_1 = (a\cdot b)_1,$$

what means that the equation

$$(a\cdot b)\cdot x = (a\cdot b)_1$$

has a solution. Thus for $a\cdot b$ condition 4^o of definition 9 holds.
We have shown yet, that $(A_r,\cdot)$ is a closed substructure of $(A,\cdot)$. From the associativity of $(A,\cdot)$ we obtain the associativity of $(A_r,\cdot)$.
Let now a, b, c be arbitrary elements of A_r. Because a, b, c are elements of the set $A_{rr}\cap A_{1r}$, then for a, b, c conditions 1^o and 2^o of definition 5 are fulfilled. From condition 4^o of definition 9 we obtain that condition 3^o of definition 5 holds. It is easy to see, that the set A_o from definition 5 is the set of all units of the set A_r.
From the above considerations we obtain that for $(A_r,\cdot)$ conditions $1^o - 3^o$ of definition 5 of the Ehresmann groupoid are fulfilled, hence $(A_r,\cdot)$ is an Ehresmann groupoid, which completes the proof of the lemma.
Now we will denote the element inverse to the element a by a^{-1}.

Definitions 8 and 9 generalize the definitions given by J. Aczél in the paper [1], p.40. J. Aczél formulated these definitions for semigroups.
Let now $(A,\cdot)$ and $(B,*)$ be arbitrary structures.

<u>Definition 10</u>. The triplet of functions (H_1,H_2,H_3) from the set $[A \multimap B]^3$ will be called the solution of the equation

$$H_1(x) * H_2(y) = H_3(x\cdot y) \tag{1}$$

if for arbitrary $x,y\in A$ such that $(x,y)\in D$, the following condition holds:

$$[x\in D_{H_1} \wedge y\in D_{H_2} \wedge x\cdot y\in D_{H_3}] \Longrightarrow [(H_1(x), H_2(y))\in D_* \wedge (H_1(x) * H_2(y) = H_3(x\cdot y))].$$

<u>Definition 11</u>. We shall say that the triplet $(\bar{H}_1,\bar{H}_2,\bar{H}_3)\in[A \multimap B]^3$ is the extension of the solution (H_1,H_2,H_3) of equation (1), if the following conditions are fulfilled:

1^o. $D_{H_i} \subset D_{\bar{H}_i}$ for $i = 1,2,3,$

2^o. $\bar{H}_i|D_{H_i} = H_i$ for $i = 1,2,3$

3^o. the triplet $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ is the solution of equation (1).

We shall say that the triplet (H_1,H_2,H_3) can be extended to the triplet $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ or that the triplet (H_1,H_2,H_3) can be extended on the triplet of sets $(D_{\bar{H}_1},D_{\bar{H}_2},D_{H_3})$.

Chapter II

Solution of the equation $H_1(a,b,\alpha)\cdot H_2(b,c,\beta) = H_3(a,c,\alpha\cdot\beta)$

Let φ be a function of four variables. We shall denote:

$$D^1_\varphi := \left\{x : \bigvee_{v,y,z} \left[(x,v,y,z) \in D_\varphi\right]\right\},$$

$$D^2_\varphi := \left\{x : \bigvee_{v,y,z} \left[(v,x,y,z) \in D_\varphi\right]\right\},$$

$$D^3_\varphi := \left\{x : \bigvee_{v,y,z} \left[(v,y,x,z) \in D_\varphi\right]\right\},$$

$$D^4_\varphi := \left\{x : \bigvee_{v,y,z} \left[(v,y,z,x) \in D_\varphi\right]\right\}.$$

If φ is a function of two or three variables, we will use analogous notations.

Let A be an arbitrary set, $(G,\cdot)$ an arbitrary semigroup with the unit e and let $(K,\cdot)$ be an arbitrary associative structure. Let us consider a structure $(A \times A \times G, \ast)$, defining the operation „$\ast$" as follows:

1^o. $\quad \big((a,b,\alpha),(c,d,\beta)\big) \in D_\ast \Longleftrightarrow b = c,$

2^o. $\quad \big((a,b,\alpha),(c,d,\beta)\big) \in D_\ast \Longrightarrow \big[(a,b,\alpha) \ast (c,d,\beta) = (a,d,\alpha\cdot\beta)\big].$

Let us consider a subset Γ_1 of the set $[A \times A \times G \xrightarrow{o} K]^3$ defined as follows:

Definition 12. The triplet of functions (H_1,H_2,H_3) belongs to the set Γ_1 iff the following conditions are fulfilled:

1^o. $\quad D_{H_1} \ast D_{H_2} = D_{H_3},$

2^o. $\quad D^2_{H_1} = D^1_{H_2},$

3°. $\bigvee_{\bar{a}} \bigwedge_{b \in D^2_{H_1}} \bigwedge_{\alpha \in G} \left[(\bar{a},b,\alpha) \in D_{H_1} \wedge H_1(\bar{a},b,e) \in K_r \right],$

4°. in the set A there exists such elements $\bar{c}$ and $\bar{b}$ that the following conditions are satisfied:

a/ $(a,\bar{b},\alpha) \in D_{H_1}$ for $a \in D^1_{H_1}$,

b/ $(\bar{b},a,\alpha) \in D_{H_2}$ for $a \in D^2_{H_2}$,

c/ $H_2(\bar{b},\bar{c},e) \in K_{rr}$,

5°. $\bigwedge_{(a,b,\alpha)} \bigwedge_{\beta \in G} \left[(a,b,\alpha) \in D_{H_i} \Longrightarrow (a,b,\beta) \in D_{H_1} \right], \quad i = 1,2,3.$

<u>Theorem 2</u>. Every solution $(H_1,H_2,H_3) \in \Gamma_1$ of the equation

(2) $$H_1(a,b,\alpha) \cdot H_2(b,c,\beta) = H_3(a,c,\alpha\cdot\beta)$$

can be extended in a unique way on the triplet of sets

(3) $$\left(D^1_{H_1} \times D^2_{H_1} \times G,\ D^1_{H_2} \times D^2_{H_2} \times G,\ D_{H_3}\right),$$

and this extending belongs to the set Γ_1.

<u>Proof</u>.

Let the triplet of functions (H_1,H_2,H_3) belonging to the set Γ_1 satisfy equation (2) and let a, b, c be arbitrary elements of the set A satisfying conditions 3° and 4° of definition 12. From conditions 3° and 4° of definition 12 it follows that

$$D_{H_3} = D^1_{H_3} \times D^2_{H_3} \times G.$$

Let the triplet of the functions $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ be an extension on the triplet of sets (3) of the solution (H_1,H_2,H_3). Then, evidently

(4) $$\bar{H}_3 = H_3.$$

Let $(a,b,\alpha) \in D_{\bar{H}_2}$. Then we have

$$\bar{H}_1(\bar{a},a,e) \cdot \bar{H}_2(a,b,\alpha) = \bar{H}_3(\bar{a},b,\alpha) .$$

From (2) and conditions 2^0 and 3^0 of definition 12 it follows, that $(\bar{a},a,e) \in D_{H_1}$. That fact and (4) implies, that

$$H_1(\bar{a},a,e)\cdot\bar{H}_2(a,b,\alpha) = H_3(\bar{a},b,\alpha) .$$

Now, using the fact that $H_1(\bar{a},a,e) \in K_r$, we obtain

$$(5) \qquad \bar{H}_2(a,b,\alpha) = H_1^{-1}(\bar{a},a,e)\cdot H_3(\bar{a},b,\alpha) .$$

Let $(a,b,\alpha) \in D_{\bar{H}_1}$. We obtain

$$\bar{H}_1(a,b,\alpha)\cdot\bar{H}_2(b,\bar{c},e) = \bar{H}_3(a,\bar{c},\alpha) .$$

Regarding conditions (4) and (5) we have

$$\bar{H}_1(a,b,\alpha)\cdot H_1^{-1}(\bar{a},b,e)\cdot H_3(\bar{a},\bar{c},e) = H_3(a,\bar{c},\alpha) .$$

Hence

$$\bar{H}_1(a,b,\alpha)\cdot H_1^{-1}(\bar{a},b,e)\cdot H_1(\bar{a},b,e)\cdot H_2(\bar{b},\bar{c},e) = H_1(a,\bar{b},\alpha)\cdot H_2(\bar{b},\bar{c},e) .$$

$H_1(\bar{a},b,e) \in K_r$ and $H_1(\bar{a},\bar{b},e) \in K_r$ and $H_2(\bar{b},\bar{c},e) \in K_{rr}$, therefore

$$(6) \quad \bar{H}_1(a,b,\alpha) = H_1(a,\bar{b},\alpha)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},b,e) .$$

From (4), (5), (6) we obtain that $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ is the unique extension of the solution (H_1,H_2,H_3). Moreover,

$$\bar{H}_1(\bar{a},b,e) = H_1(\bar{a},b,e) \qquad \text{for } b \in D^2_{\bar{H}_1}$$

and

$$\bar{H}_2(\bar{b},\bar{c},e) = H_2(\bar{b},\bar{c},e),$$

whence

$\bar{H}_1(\bar{a},b,e) \in K_r$ for $b \in D^2_{\bar{H}_1}$ and $\bar{H}_2(\bar{b},\bar{c},e) \in K_{rr}$ which means that the triplet $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ is an element of the set Γ_1.

Now we shall prove that the triplet $(\bar{H}_1,\bar{H}_2,\bar{H}_3)$ where the functions $\bar{H}_1,\bar{H}_2,\bar{H}_3$ are defined by conditions (6), (5), (4), is a solution of equation (2).

Let $(a,b,\alpha) \in D_{\bar{H}_1}$, $(b,c,\beta) \in D_{\bar{H}_2}$. Than $(a,c,\alpha\cdot\beta) \in D_{\bar{H}_3}$ and we have

$$\bar{H}_1(a,b,\alpha)\cdot\bar{H}_2(b,c,\beta) = H_1(a,\bar{b},\alpha)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},b,e)\cdot H_1^{-1}(\bar{a},b,e)\cdot H_3(\bar{a},c,\beta) =$$
$$= H_1(a,\bar{b},\alpha)\, H_1^{-1}(\bar{a},\bar{b},e)\cdot H_3(\bar{a},c,\beta) =$$
$$= H_1(a,\bar{b},\alpha)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},e)\cdot H_2(\bar{b},c,\beta) =$$
$$= H_1(a,\bar{b},\alpha)\cdot H_2(\bar{b},c,\beta) = H_3(a,c,\alpha\cdot\beta) = \bar{H}_3(a,c,\alpha\cdot\beta),$$

which completes the proof.

For the triplet of functions $(H_1,H_2,H_3)\in\Gamma_1$ satisfying equation (2) condition

(7) $$D_{H_i} = D^1_{H_i} \times D^2_{H_i} \times G \quad \text{for} \quad i = 1,\,2,\,3$$

need not be satisfying. It is illustrated by the following

Example 2. Let us put $A = \{1,2,3,4\}$, $G = \{e\}$, let $(K,\cdot)$ be the multiplicative group of real numbers and let the functions H_1, H_2, H_3 be defined as follows

	(1,2,e)	(1,3,e)	(4,2,e)	(2,3,e)	(3,3,e)	(4,3,e)
H_1	2	3	5			
H_2				3	2	
H_3		6				15

It is easy to verify, that the triplet (H_1,H_2,H_3) is the solution from the set Γ_1 of equation 2 and that for the function H_1

$$D_{H_1} \neq D^1_{H_1} \times D^2_{H_1} \times G.$$

From theorem 2 we can conclude that it is sufficient to consider only those solutions (H_1,H_2,H_3) from the set Γ_1, for which condition (7) is satisfying.

Definition 13. We will denote by Γ_2 the set of all triplets $(H_1,H_2,H_3)\in[A\times A\times G \xrightarrow{\;\theta\;} K]^3$, for which conditions 1^0 and 2^0 of definition 12, condition (7) and the following conditions are satisfied:

(8) $$\bigvee_{\bar{a}\in D^1_{H_1}} \bigwedge_{b\in D^2_{H_1}} \left[H_1(\bar{a},b,e) \in K_r\right],$$

$$(9) \qquad \bigvee_{\bar{b} \in D^1_{H_2}} \bigvee_{\bar{c} \in D^2_{H_2}} \left[H_2(\bar{b},\bar{c},e) \in K_{rr} \right].$$

It is easy to see that Γ_2 is the set of all triplets $(H_1,H_2,H_3) \in \Gamma_1$, for which condition (7) is satisfying.

Lemma 7. If the triplet of functions $(H_1,H_2,H_3) \in \Gamma_2$ is the solution of equation (2), then $H_2(b,\bar{c},e) \in K_{rr}$ for every $b \in D^1_{H_2}$, where $\bar{c}$ is an arbitrary element, for which condition (9) holds for some $\bar{b}$.

Proof. Let b be an arbitrary element of the set $D^1_{H_2}$, let $\bar{b}$ and $\bar{c}$ be arbitrary elements for which condition (9) holds, and let $\bar{a}$ be an arbitrary element, for which condition (8) holds. We have

$$H_1(\bar{a},b,e) \cdot H_2(b,\bar{c},e) = H_1(\bar{a},\bar{b},e) \cdot H_2(\bar{b},\bar{c},e)$$

and, because $H_1(\bar{a},b,e) \in K_r$,

$$H_2(b,\bar{c},e) = H_1^{-1}(\bar{a},b,\ ) \cdot H_1(\bar{a},\bar{b},e) \cdot H_2(\bar{b},\bar{c},e).$$

From lemmas 6 and 5 we obtain

$$H_2(b,\bar{c},e) \in K_{rr}.$$

From lemma 7 it follows that we can replace condition (9) in definition 13 by

$$(9') \qquad \bigvee_{\bar{c} \in D^2_{H_2}} \bigwedge_{b \in D^1_{H_2}} \left[H_2(b,\bar{c},e) \in K_{rr} \right],$$

because such a substitution does not change the set of solutions from the set Γ_2 of equation 2 .

From the above considerations it follows that it is sufficient to consider equation 2 in the set Γ defined as follows

Definition 14. We will denote by Γ the set of all triplets of functions (H_1,H_2,H_3) belonging to the set $[A \times A \times G \xrightarrow{\ \theta\ } K]^3$, for which conditions 1^o and 2^o of definition 12 and conditions (7), (8), (9) hold.

Theorem 3 given below is a generalization of the theorem, formulated by J. Aczél in paper [1], p.39/40. The results of J. Aczél concern the case, when $(G,\cdot)$ is an one-element group, $(K,\cdot)$ is a semigroup and the functions H_1,H_2,H_3 are defined on the set $A \times A$.

Theorem 3. The triplet of functions $(H_1, H_2, H_3) \in [A \times A \times G \xrightarrow{e} K]^3$ is a solution from the set Γ of equation (2) iff (H_1, H_2, H_3) have the form

(10) $$H_1(a,b,\alpha) = f_1(a) \cdot g(\alpha) \cdot f_2^{-1}(b),$$

(11) $$H_2(a,b,\alpha) = f_2(a) \cdot g(\alpha) \cdot f_3(b),$$

(12) $$H_3(a,b,\alpha) = f_1(a) \cdot g(\alpha) \cdot f_3(b),$$

where f_1, f_2, f_3 are arbitrary functions from the set $[A \xrightarrow{e} K]$ such that

(13) there exists such an $\bar{a} \in D_{f_1}$, that $f_1(\bar{a}) \in K_r$,

(14) there exists such a $\bar{c} \in D_{f_3}$, that $f_3(\bar{c}) \in K_{rr}$,

(15) $f_2 \in [A \xrightarrow{e} K_r]$,

(16) g is an arbitrary homomorphism of $(G,\cdot)$ into $(K,\cdot)$, such that $g(e) \in K_r$,

(17) $(f_1(a), g(\alpha)) \in D_\cdot$, $(g(\alpha), f_2^{-1}(b)) \in D_\cdot$, $(f_2(b), g(\alpha)) \in D_\cdot$, $(g(\alpha), f_3(c)) \in D_\cdot$

for $a \in D_{f_1}$, $b \in D_{f_2}$, $c \in D_{f_3}$, $\alpha \in G$.

Proof. Let us assume that the functions H_1, H_2, H_3 have the form (10) - (12) and let $\bar{a}$, $\bar{c}$ be arbitrary elements such that conditions (13) and (14) are fulfilled.
$g(e) \in K_r$, therefore $g(e)$ is an unit in $(K,\cdot)$ and we have

$$H_1(\bar{a},b,e) = f_1(\bar{a}) \cdot f_2^{-1}(b), \quad \text{i.e. } H_1(\bar{a}, b, e) \in K_r \quad \text{for} \quad b \in D^2_{H_1},$$

and

$$H_2(b,\bar{c},e) = f_2(b) \cdot f_3(\bar{c}) \quad \text{for} \quad b \in D^1_{H_2}.$$

Thus, by lemma 5, $H_2(b,\bar{c},e) \in K_{rr}$.
It is easy to see that conditions 1^0 and 2^0 of definition 12 and condition (7) are implicated by (17). Hence the triplet (H_1, H_2, H_3) belongs to the set Γ. Let now $(a,b,\alpha) \in D_{H_1}$ and $(b,c,\beta) \in D_{H_2}$.
Then $(a,c,\alpha \cdot \beta) \in D_{H_3}$ and we have

$$H_1(a,b,\alpha) \cdot H_2(b,c,\beta) = f_1(a) \cdot g(\alpha) \cdot f_2^{-1}(b) \cdot f_2(b) \cdot g(\beta) \cdot f_3(c) = f_1(a) \cdot g(\alpha) \cdot g(\beta) \cdot f_3(c) =$$
$$= f_1(a) \cdot g(\alpha \cdot \beta) \cdot f_3(c) = H_3(a,c,\alpha \cdot \beta),$$

hence (H_1, H_2, H_3) is the solution of equation (2).
Let us suppose now that $(H_1, H_2, H_3) \in \Gamma$ satisfies (2) and let $\bar{a}$ and $\bar{c}$ be arbitrary elements, for which conditions (8) and (9) hold. Then

$$H_1(\bar{a},b,e) \cdot H_2(b,\bar{c},\alpha) = H_1(\bar{a},c,\alpha) \cdot H_2(c,\bar{c},e) \quad \text{for} \quad b,c \in D^2_{H_1},\ \alpha \in G,$$

and therefore

$$(18) \qquad \left(H_1^{-1}(\bar{a},b,e),\ H_1(\bar{a},c,\alpha)\right) \in D_{\circ}.$$

Let $\bar{b}$ be an arbitrary, fixed element from the set $D^2_{H_1}$.
We put:

$$f_1(a) := H_1(a,\bar{b},e) \qquad \text{for} \quad a \in D^1_{H_1},$$

$$f_2(a) := H_1^{-1}(\bar{a},a,e) \cdot f_1(\bar{a}) \qquad \text{for} \quad a \in D^2_{H_1},$$

$$f_3(a) := H_2(\bar{b},a,e) \qquad \text{for} \quad a \in D^2_{H_2},$$

$$g(\alpha) := H_1^{-1}(\bar{a},\bar{b},e) \cdot H_1(\bar{a},\bar{b},\alpha) \quad \text{for} \quad \alpha \in G.$$

For $a \in D_{f_1}$, $\alpha \in G$ we have

$$(19) \qquad \left(H_1(a,\bar{b},\alpha),\ H_2(\bar{b},\bar{c},e)\right) \in D_{\circ}.$$

Moreover,

$$H_1(\bar{a},\bar{b},e) \cdot H_2(\bar{b},\bar{c},e) = H_3(\bar{a},\bar{c},e)$$

and hence, since $H_1(\bar{a},\bar{b},e) \in K_r$,

$$H_2(\bar{b},\bar{c},e) = H_1^{-1}(\bar{a},\bar{b},e) \cdot H_3(\bar{a},\bar{c},e),$$

i.e., by (19)

$$(20) \qquad \left(H_1(a,\bar{b},\alpha),\ H_1^{-1}(\bar{a},\bar{b},e)\right) \in D_{\circ}.$$

Thus

$$\left(f_1(a),\ g(\alpha)\right) \in D_{\circ} \qquad \text{for} \quad a \in D_{f_2},\ \alpha \in G$$

and

$$\left(g(\alpha),\ f_2^{-1}(a)\right) \in D_{\circ} \qquad \text{for} \quad a \in D_{f_2},\ \alpha \in G.$$

The triplet (H_1, H_2, H_3) satisfies (2), therefore

$$\left(g(\alpha),\ f_3(a)\right) \in D_{\circ} \qquad \text{for} \quad a \in D_{f_3},\ \alpha \in G.$$

From the definitions of functions f_1, f_2, f_3 we obtain that conditions (13) - (15) are fulfilled. We shall show that for the function g condition (16) holds. From the definition of the function g we obtain that $g(e) \in K_r$.

Let α, β be arbitrary elements of the set G. We get

$$g(\alpha)\cdot g(\beta)\cdot H_2(\bar{b},\bar{c},e) =$$

$$= H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\alpha)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\beta)\cdot H_2(\bar{b},\bar{c},e) =$$

$$= H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\alpha)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},e)\cdot H_2(\bar{b},\bar{c},\beta) =$$

$$= H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\alpha)\cdot H_2(\bar{b},\bar{c},\beta) =$$

$$= H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\alpha\cdot\beta)\cdot H_2(\bar{b},\bar{c},e) = g(\alpha\cdot\beta)\cdot H_2(\bar{b},\bar{c},e) .$$

$H_2(\bar{b},\bar{c},e) \in K_{rr}$ therefore

$$g(\alpha)\cdot g(\beta) = g(\alpha\cdot\beta),$$

i.e. g is a homomorphism of $(G,\cdot)$ into $(K,\cdot)$.

Let (a,b,α) be an arbitrary element of the set D_{H_3}. By (20) we get

$$H_3(a,b,\alpha) = H_1(a,\bar{b},e)\cdot H_2(\bar{b},b,\alpha) =$$

$$= H_1(a,\bar{b},e)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},e)\cdot H_2(\bar{b},b,\alpha) =$$

$$= H_1(a,\bar{b},e)\cdot H_1^{-1}(\bar{a},\bar{b},e)\cdot H_1(\bar{a},\bar{b},\alpha)\cdot H_2(\bar{b},b,e) = f_1(a)\cdot g(\alpha)\cdot f_3(b),$$

thus H_3 has form (12).

Let (a,b,α) be an arbitrary element of the set D_{H_2}. We get

$$H_1(\bar{a},a,e)\cdot H_2(a,b,\alpha) = H_3(a,b,\alpha).$$

Hence, because $H_1(\bar{a},a,e) \in K_r$ and H_3 has the form (12) we have

$$H_2(a,b,\alpha) = H_1^{-1}(\bar{a},a,e)\cdot H_3(\bar{a},b,\alpha) =$$

$$= H_1^{-1}(\bar{a},a,e)\cdot f_1(\bar{a})\cdot g(\alpha)\, f_3(b) =$$

$$= f_2(a)\cdot g(\alpha)\cdot f_3(b).$$

thus H_2 has the form (11).

Let (a,b,α) be an arbitrary element of the set D_{H_1}. We get

$$H_1(a,b,\alpha)\cdot H_2(b,c,e) = H_3(\bar{a},c,\alpha),$$

whence

$$H_1(a,b,\alpha)\cdot f_2(b)\cdot f_3(\bar{c}) = f_1(a)\cdot g(\alpha)\cdot f_3(\bar{c}).$$

$f_3(\bar{c}) \in K_{rr}$ and $f_2(b) \in K_r$, therefore

$$H_1(a,b,\alpha) = f_1(a)\cdot g(\alpha)\cdot f_2^{-1}(b),$$

thus H_1 has the form (10), which completes the proof.

<u>Theorem 4</u>. If the triplet $(H_1,H_2,H_3)\in\Gamma$ satisfies (2), $f_1\in[D^1_{H_1}\longrightarrow K]$, $f_2\in[D^2_{H_1}\longrightarrow K_r]$, $f_3\in[D^2_{H_2}\longrightarrow K]$, g is an homomorphism of $(G,\cdot)$ into $(K,\cdot)$, and conditions (10) - (12) and (17) hold for the functions H_1,H_2,H_3 and f_1,f_2,f_3,g, then $f_1(\bar{a})\in K_r$, $g(e)\in K_r$, $f_3(\bar{c})\in K_{rr}$, where $\bar{a}$, $\bar{c}$ are arbitrary elements satisfying (8) and (9).

<u>Proof</u>. Let us put $a=\bar{a}$, $\alpha=e$ in (10). We obtain

$$H_1(\bar{a},b,e) = f_1(\bar{a})\cdot g(e)\cdot f_2^{-1}b) \quad \text{for} \quad b\in D^2_{H_1}.$$

Because $H_1(\bar{a},b,e)\in K_r$ and $f_2^{-1}(b)\in K_r$, we have

(21) $$f_1(\bar{a})\cdot g(e)\in K_r.$$

The function g is a homomorphism, thus

$$f_1(\bar{a})\cdot g(e) = f_1(\bar{a})\cdot g(e)\cdot g(e),$$

and therefore, by (21), $g(e)$ is an unit in $(K_r,\cdot)$. By (21) we obtain also that $f_1(\bar{a})\in K_r$.
Let us put $a=\bar{a}$, $b=\bar{c}$, $\alpha=e$ in (11). We obtain, using the above considerations and lemma 5, that

$$f_3(\bar{c})\in K_{rr},$$

which completes the proof.

Let us denote by Δ the set of all quadruples of the functions (f_1,f_2,f_3,g), for which conditions (13) - (17) are fulfilled.

<u>Theorem 5</u>. Two quadruples of the functions (f_1,f_2,f_3,g_1) and (k_1,k_2,k_3,g_2) from the set Δ dictate the same solution of equation (2) iff

1^0 $\quad D_{f_i} = D_{k_i} \quad$ for $i=1,2,3,$

2°. there exists $m \in K_r$ such that $(f_1(a), m) \in D_\circ$, $(f_2(b), m) \in D_\circ$, $(m^{-1}, g(\alpha)) \in D_\circ$, $(g(\alpha), m) \in D_\circ$ and

$k_1(a) = f_1(a) \cdot m,$

$k_2(b) = f_2(b) \cdot m,$

$k_3(c) = m^{-1} \cdot f_3(c),$

$g_2(\alpha) = m^{-1} \cdot g_1(\alpha) \cdot m$

for arbitrary $a \in D_{f_1}$, $b \in D_{f_2}$, $c \in D_{f_3}$, $\alpha \in G$.

Proof. It is easy to verify, by (10) - (12) and 1°, 2°, that both quadruples dictate the same solution of equation (2). Let now the quadruples $(f_1, f_2, f_3, g_1) \in \Delta$ and $(k_1, k_2, k_3, g_2) \in \Delta$ dictate the same solution of equation (2). We get

$$D_{f_1} = D^1_{H_1} = D_{k_1}, \quad D_{f_2} = D^1_{H_2} = D_{k_2}, \quad D_{f_3} = D^2_{H_2} = D_{k_3} \quad \text{and}$$

a/ $f_1(a) \cdot g_1(\alpha) \cdot f_2^{-1}(b) = k_1(a) \cdot g_2(\alpha) \cdot k_2^{-1}(b)$ for $a \in D_{f_1}$, $b \in D_{f_2}$, $\alpha \in G$,

b/ $f_2(a)\, g_1(\alpha) \cdot f_3(b) = k_2(a) \cdot g_2(\alpha) \cdot k_3(b)$ for $a \in D_{f_2}$, $b \in D_{f_3}$, $\alpha \in G$.

Let $\bar{a}$, $\bar{c}$ be arbitrary, fixed elements, such that for the functions f_1, f_3 conditions (13) and (14) hold.
Let us put $a = \bar{a}$, $\alpha = e$ in a). We get

$$f_1(\bar{a}) \cdot f_2^{-1}(b) = k_1(\bar{a}) \cdot k_2^{-1}(b) \quad \text{for } b \in D_{f_2},$$

whence, because $k_2(b) \in K_r$, we obtain

$$k_1(\bar{a}) = f_1(\bar{a}) \cdot f_2^{-1}(b) \cdot k_2(b),$$

hence

$$k_1(\bar{a}) \in K_r.$$

Let us put $b = \bar{c}$, $\alpha = e$ in b). We get

$$f_2(a) \cdot f_3(\bar{c}) = k_2(a) \cdot k_3(\bar{c}) \quad \text{for } a \in D_{f_2},$$

and therefore, because $k_2(a) \in K_r$,

$$k_3(\bar{c}) = k_2^{-1}(a)\cdot f_2(a)\cdot f_3(\bar{c}).$$

From lemma 5 we have

$$k_3(\bar{c}) \in K_{rr}.$$

Let us put $a = \bar{a}$, $\alpha = e$ in a/. We get

$$k_1(\bar{a})\cdot k_2^{-1}(b) = f_1(\bar{a})\cdot f_2^{-1}(b) \quad \text{for} \quad b \in D_{f_2},$$

thus

$$(22) \quad f_1^{-1}(\bar{a})\cdot k_1(\bar{a}) = f_2^{-1}(b)\cdot k_2(b) .$$

Let us denote

$$(23) \quad m := f_1^{-1}(\bar{a})\cdot k_1(\bar{a}) .$$

Of course, $m \in K_r$. By a/, putting $\alpha = e$, we obtain

$$k_1(a)\cdot k_2^{-1}(b) = f_1(a)\cdot f_2^{-1}(b) \quad \text{for} \quad a \in D_{f_1},\ b \in D_{f_2},$$

i.e.

$$k_1(a) = f_1(a)\cdot f_2^{-1}(b)\cdot k_2(b).$$

Thus, by (22) and (23)

$$k_1(a) = f_1(a)\cdot m .$$

By (22) we obtain

$$k_2(b) = f_2(b)\cdot f_1^{-1}(\bar{a})\cdot k_1(\bar{a}) \quad \text{for} \quad b \in D_{f_2},$$

i.e.

$$k_2(b) = f_2(b)\cdot m.$$

By b/, putting $\alpha = e$ we obtain

$$f_2(a)\cdot f_3(b) = k_2(a)\cdot k_3(b) \quad \text{for} \quad a \in D_{f_2},\ b \in D_{f_3},$$

hence

$$k_3(b) = k_2^{-1}(a)\cdot f_2(a)\cdot f_3(b),$$

i.e.

$$k_3(b) = \left[f_2^{-1}(a)\cdot k_2(a)\right]^{-1}\cdot f_3(b).$$

Thus, by (22) and (23)

$$k_3(b) = m^{-1}\cdot f_3(b).$$

From condition a/, putting $a = \bar{a}$, we have

$$k_1(\bar{a})\cdot g(\alpha)\cdot k_2^{-1}(b) = f_1(\bar{a})\cdot g_1(\alpha)\cdot f_2^{-1}(b) \quad \text{for} \quad b \in D_{f_2},\ \alpha \in G.$$

Hence

$$g_2(\alpha) = k_1^{-1}(\bar{a})\cdot f_1(\bar{a})\cdot g_1(\alpha)\cdot f_2^{-1}(b)\cdot k_2(b),$$

i.e.

$$g_2(\alpha) = m^{-1}\cdot g_1(\alpha)\cdot m,$$

which completes the proof.
By theorem 5 we obtain the following

<u>Corollary 2</u>. If two quadruples $(f_1,f_2,f_3,g_1) \in \Delta$ and $(k_1,k_2,k_3,g_2) \in \Delta$ dictate the same solution of equation (2) and $k_1 = f_1$ or $k_2 = f_2$ or $k_3 = f_3$, then these quadruples are identical.

<u>Theorem 6</u>. If the solution (H_1,H_2,H_3) of equation (2) may be dictated by a quadruple $(f_1,f_2,f_3,g) \in \Delta$, then it may be extended on the triplet of sets $(A \times A \times G,\ A \times A \times G,\ A \times A \times G)$ and this extending belongs to the set Γ.
To prove this it is sufficient to extend the functions f_1 and f_3 on the set A in an arbitrary manner and to extend the function f_2 on the set A so, that for any $a \in A$ there is $\bar{f}_2(a) \in K_r$.

Let us observe that the structure of the sets $D_{H_1}, D_{H_2}, D_{H_3}$ determined by the definition of the set Γ in an essential manner affects the form of the solution of equation (2). It is illustrated by the following

<u>Example 3</u>. Let us put:

$A = \{1,2,3,4\}$,

$D_{H_1} = D_{H_2} = D_{H_3} = \{(1,2,e),(1,3,e),(4,2,e),(1,1,e),(2,2,e),(3,3,e),(4,4,e)\}$,

where $(\{e\},\cdot)$ is a group.

Let H be a function defined as follows:

	(1,2,e)	(1,3,e)	(4,2,e)	(4,3,e)	(1,1,e)	(2,2,e)	(3,3,e)	(4,4,e)
H	2	4	3	5	1	1	1	1

D_H is the subset of the set $A \times A \times \{e\}$, of course. It is easy to verify that $(D_H, \cdot)$ is the closed substructure of the product Brandt groupoid $(A \times A \times \{e\}, \ast)$. Let K be the multiplicative group of real numbers. It is easy to verify that the triplet (H,H,H) satisfies equation (2). Let us suppose that H have the form

$$H(a,b,e) = f_1(a) \cdot g(e) \cdot f_2^{-1}(b).$$

Then g is the homomorphism and therefore $g(e) = 1$. We have

$$f_1(1) \cdot f_2^{-1}(2) = 2 \qquad \text{and} \qquad f_1(4) \cdot f_2^{-1}(2) = 3,$$

hence

(24) $$f_1(1) \cdot f_1^{-1}(4) = \frac{2}{3}.$$

Moreover, we have

$$f(1) \quad f_2^{-1}(3) = 4 \qquad \text{and} \qquad f_1(4) \cdot f_2^{-1}(3) = 5,$$

hence

$$f_1(1) \cdot f_1^{-1}(4) = \frac{4}{5},$$

which is contrary to (24). Thus the solution (H,H,H) of equation (2) has not the form (10), (11), (12).
It is easy to verify also that this solution (H,H,H) of equation (2) can not be extended on the triplet of sets $(A \times A \times \{e\}, A \times A \times \{e\}, A \times A \times \{e\})$.

Now we shall consider equation (2) on the Ehresmann groupoid.
Let (R, o) be an arbitrary closed substructure of the Ehresmann groupoid (E, o). We will denote by R^{-1} a subset of E defined as follows:

$$R^{-1} := \{x : x \in E \wedge x^{-1} \in R\}.$$

<u>Theorem 7</u>. If the triplet of functions $(H_1, H_2, H_3) \in [R \longrightarrow E_1]^3$, where $(E_1, \cdot)$ is an Ehresmann groupoid and $R \cup R^{-1} = E$, is the solution of the equation

(25) $$H_1(x) \cdot H_2(y) = H_3(xoy),$$

then there exists an extension of this solution on the triplet of sets (E,E,E), and it is assigned in an unique manner.

<u>Proof</u>. We shall show first that (R^{-1}, o) is a closed substructure of (E, o). Let x, y be arbitrary elements of the set R^{-1} such that $(x,y) \in D_o$. Then $y^{-1} \in R$, $x^{-1} \in R$, $(y^{-1}, x^{-1}) \in D_o$, whence $y^{-1} \circ x^{-1} \in R$. Thus $x \circ y \in R^{-1}$. It is easy to verify that every unit of the groupoid (E, o) belongs to the set $R \cap R^{-1}$. The following cases are possible:

1°. $x \in R$, $y \in R$, $x \circ y \in R$,

2°. $x \in R$, $y \notin R$, $x \circ y \in R$,

3°. $x \notin R$, $y \in R$, $x \circ y \in R$,

4°. $x \notin R$, $y \notin R$, $x \circ y \notin R$,

5°. $x \in R$, $y \notin R$, $x \circ y \notin R$,

6°. $x \notin R$, $y \in R$, $x \circ y \notin R$.

Let x be an arbitrary element of the set R. The triplet

(H_1, H_2, H_3) satisfies equation (25), therefore

$$H_1(x_1) \cdot H_2(x_1) = H_3(x_1),$$

$$H_1(x_r) \cdot H_2(x_r) = H_3(x_r),$$

$$H_1(x) \cdot H_2(x_r) = H_3(x),$$

$$H_1(x_1) \cdot H_2(x) = H_3(x).$$

Using the above equalities and the fact that in Ehresmann groupoid $x_r = (x^{-1})_1$ we get:

$$\big(H_1(x_r)\big)_r = \big(H_2(x_r)\big)_1 = \big(H_1(x)\big)_r = \big(H_1^{-1}(x)\big)_1,$$

$$\big(H_1(x_1)\big)_1 = \big(H_3(x)\big)_1 = \big(H_1(x)\big)_1 = \big(H_1^{-1}(x)\big)_r,$$

$$\big(H_2(x_r)\big)_r = \big(H_3(x)\big)_r = \big(H_2(x)\big)_r = \big(H_2^{-1}(x)\big)_1,$$

$$\big(H_2(x_1)\big)_1 = \big(H_1(x_1)\big)_r = \big(H_2(x)\big)_1 = \big(H_2^{-1}(x)\big)_r,$$

$$\big(H_3(x_r)\big)_r = \big(H_2(x_r)\big)_r = \big(H_3(x)\big)_r = \big(H_3^{-1}(x)\big)_1,$$

$$\big(H_3(x_1)\big)_1 = \big(H_1(x_1)\big)_1 = \big(H_3(x)\big)_1 = \big(H_3^{-1}(x)\big)_r,$$

whence

$$(26)\quad \begin{aligned}&\left(H_1(x_r),\ H_1^{-1}(x)\right)\in D_0,\ \left(H_1^{-1}(x),\ H_1(x_l)\right)\in D_0,\ \left(H_2(x_r),\ H_2^{-1}(x)\right)\in D_0,\\ &\left(H_2^{-1}(x),\ H_2(x_l)\right)\in D_0,\ \left(H_3(x_r),\ H_3^{-1}(x)\right)\in D_0,\ \left(H_3^{-1}(x),\ H_3(x_l)\right)\in D_0.\end{aligned}$$

Let $x\in R\cap R^{-1}$. Then $x^{-1}\in R\cap R^{-1}$ and we have

$$H_1(x_l)\cdot H_1^{-1}(x^{-1})\cdot H_1(x_r)=H_1(x_l)\cdot H_2(x)\cdot H_2^{-1}(x)\cdot H_1^{-1}(x^{-1})H_1(x_r)=$$

$$=H_1(x_l)\cdot H_2(x)\cdot\left[H_1(x^{-1})\cdot H_2(x)\right]^{-1}\cdot H_1(x_r)=$$

$$=H_1(x)\cdot H_2(x_r)\cdot\left[H_1(x_r)\cdot H_2(x_r)\right]^{-1}\cdot H_1(x_r)=$$

$$=H_1(x)\cdot H_2(x_r)\cdot H_2^{-1}(x_r)\cdot H_1^{-1}(x_r)\cdot H_1(x_r)=H_1(x).$$

We can show analogously that

$$H_2(x_l)\cdot H_2^{-1}(x^{-1})\cdot H_2(x_r)=H_2(x)$$

and

$$H_3(x_l)\cdot H_3^{-1}(x^{-1})\cdot H_3(x_r)=H_3(x)\ .$$

Let us put:

$$\overline{H}_i(x)=\begin{cases}H_i(x) & \text{for}\quad x\in R,\\ H_i(x_l)\cdot H_i^{-1}(x^{-1})\cdot H_i(x_r) & \text{for}\quad x\in R^{-1},\end{cases}$$

where $i=1,2,3$.

From the above considerations and by (26) it follows that $\overline{H}_1,\overline{H}_2,\overline{H}_3$ are functions.

We shall show now that the triplet $(\overline{H}_1,\overline{H}_2,\overline{H}_3)$ is an extension of the solution (H_1,H_2,H_3) of equation (25) on the triplet of sets (E,E,E).

Of course

$$\overline{H}_i\big|_R=H_i \qquad \text{for}\quad i=1,2,3.$$

In case 1^o the equality $\overline{H}_1(x)\cdot\overline{H}_2(y)=\overline{H}_3(xoy)$ holds, of course.

In case 2^o we get

$$\overline{H}_1(x)\ \overline{H}_2(y)=H_1(x)\cdot H_2(y_l)\cdot H_2^{-1}(y^{-1})\cdot H_2(y_r)=$$

$$=H_3(x)\cdot H_2^{-1}(y^{-1})\cdot H_1^{-1}(xoy)\cdot H_1(xoy)\cdot H_2(y_r)-$$

$$= H_3(x)\cdot\left[H_1(xoy)\cdot H_2(y^{-1})\right]^{-1}\cdot H_3(xoy) =$$

$$= H_3(x)\cdot H_3^{-1}(x)\cdot H_3(xoy) = H_3(xoy) = \overline{H}_3(xoy).$$

In case 4^o we have

$$\overline{H}_1(x)\cdot\overline{H}_2(y) = H_1(x_l)\cdot H_1^{-1}(x^{-1})\cdot H_1(x_r)\cdot H_2(y_l)\cdot H_2^{-1}(y^{-1})\cdot H_2(y_r) =$$

$$= H_1(x_l)\cdot H_2(x_l)\cdot H_2^{-1}(x_l)\cdot H_1^{-1}(x^{-1})\cdot H_1(x_r)\cdot H_2(y_l)\cdot H_2^{-1}(y^{-1})\cdot H_1^{-1}(y_r)\cdot H_1(y_r)\cdot H_2(y_r) =$$

$$= H_3(x_l)\cdot\left[H_1(x^{-1})\cdot H_2(x_l)\right]^{-1}\cdot H_3(x_r)\cdot\left[H_1(y_r)\cdot H_2(y^{-1})\right]^{-1}\cdot H_3(y_r) =$$

$$= H_3(x_l)\cdot\left[H_1(x_r)\cdot H_2(x^{-1})\right]^{-1}\cdot H_1(x_r)\cdot H_2(y_l)\cdot\left[H_1(y^{-1})\cdot H_2(y_l)\right]^{-1}\cdot H_3(y_r) =$$

$$= H_3(x_l)\cdot H_2^{-1}(x^{-1})\cdot H_1^{-1}(y^{-1})\cdot H_3(y_r) =$$

$$= H_3(x_l)\cdot\left[H_1(y^{-1})\cdot H_2(x^{-1})\right]^{-1}\cdot H_3(y_r) =$$

$$= H_3((xoy)_l)\cdot H_3^{-1}((xoy)^{-1})\cdot H_3((xoy)_r) = \overline{H}_3(xoy).$$

In cases 3^o, 5^o, 6^o the proof is analoqous. It follows from the above considerations that every solution $(H_1,H_2,H_3)\in[R\longrightarrow E_1]^3$ of equation (25) can be extended on the triplet of sets (E,E,E).
From the properties of the structure (R,o) and from the fact that in the Ehresmann groupoid the inverse elements are uniquely assigned it follows that the extension is uniquely assigned.
From corollary 7 of the paper [2] we can conclude that in the case, when $(E_1,\cdot)$ is not an Ehresmann groupoid, then the solution (H_1,H_2,H_3) from the set $[R\longrightarrow E_1]^3$ of equation (25) cannot be extended on the triplet of sets (E,E,E).

Let $(A\times A\times G,\ast)$ be an arbitrary product Brandt groupoid, let $(R,\ast)$ be its arbitrary closed substructure, such that $R\cup R^{-1} = A\times A\times G$ and let $(B,\cdot)$ be an arbitrary Brandt groupoid. Moreover, let (H_1,H_2,H_3) be an arbitrary triplet of functions from the set $[R\longrightarrow B]^3$. From theorems 3 and 7 we obtain

<u>Theorem 8</u>. The triplet of functions (H_1,H_2,H_3) satisfies equation (2), iff the functions H_1,H_2,H_3 have the form

$$H_1(x,y,\alpha) = f_1(x)\cdot g(\alpha)\cdot f_2^{-1}(y) \qquad \text{for} \quad (x,y,\alpha)\in R,$$

$$H_2(x,y,\alpha) = f_2(x)\cdot g(\alpha)\cdot f_3(y) \qquad \text{for} \quad (x,y,\alpha)\in R,$$

$$H_3(x,y,\alpha) = f_1(x)\cdot g(\alpha)\cdot f_3(y) \qquad \text{for} \quad (x,y,\alpha)\in R,$$

where f_1, f_2, f_3 are arbitrary functions from the set $[A \longrightarrow B]$, g is a homomorphism of $(G,\cdot)$ into $(B,\cdot)$ and for the functions f_1, f_2, f_3, g condition (17) holds.

Now we shall show that there exist such essential closed substructures (R,o), that the following conditions hold:

a/ $R \cup R^{-1} = E$

and

b/ there exists in R such an element x, that $x^{-1} \in R$ and x is not a unit of $(E,\cdot)$

It is illustrated by the following

Example 4. Let us consider the pair Brandt groupoid $(\{1,2,3\} \times \{1,2,3\}, *)$ and the substructure $(R,*)$ of this groupoid such that

$$R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3), (3,2)\}.$$

It is easy to verify that $(R,*)$ is the closed substructure of our Brandt groupoid, that $R \cup R^{-1} = \{1,2,3\} \times \{1,2,3\}$ and that $x = (2,3)$ is such an element of the set R that $x^{-1} \in R$.

Example 3 shows that the assumption $R \cup R^{-1} = E$ is essential in theorem 7. The following example shows that the assumption that (R,o) is an closed substructure, is essential, too.

Example 5. Let us consider the product Brandt groupoid $(A \times A \times \{e\}, *)$ where $A = \{1,2,3\}$. Let R be the following subset of the set $A \times A \times \{e\}$:

$$R = \{(1,1,e), (1,2,e), (2,2,e), (2,3,e), (3,1,e), (3,3,e)\}.$$

Let H be a function defined as follows:

	(1,1,e)	(1,2,e)	(2,2,e)	(2,3,e)	(3,1,e)	(3,3,e)
H	1	9	1	5	7	1

It is easy to see that $(R,*)$ is not the closed substructure of $(A \times A \times \{e\}, *)$, that $R \cup R^{-1} = A \times A \times \{e\}$. Let $(E_1,\cdot)$ be the multiplicative group of real numbers. It is easy to verify that the triplet of functions (H,H,H) satisfies equation (25). We shall show that there does not exist the extension of the solution (H,H,H) on the triplet of sets $(A \times A \times \{e\}, A \times A \times \{e\}, A \times A \times \{e\})$.

Let us suppose that the triplet (H_1,H_2,H_3) is the extension.

Then we have

$$\bar{H}_3(1,3,e) = \bar{H}_1(1,2,e) \cdot \bar{H}_2(2,3,e) = H(1,2,e) \cdot H(2,3,e) = 45,$$

hence

$$\overline{H}_3(1,3,e) = \overline{H}_1(1,3,e)\cdot\overline{H}_2(3,3,e) = \overline{H}(1,3,e)\cdot H_2(3,3,e) = 45,$$

and therefore

$$\overline{H}(1,3,e) = 45.$$

Thus

$$H(1,1,e) = \overline{H}_3(1,1,e) = \overline{H}_1(1,3,e)\cdot\overline{H}_2(3,1,e) = \overline{H}_1(1,3,e)\cdot H(3,1,e) = 45\cdot 7,$$

which is contrary to the fact that $H(1,1,e) = 1$.

Now, using the above considerations and theorem 1 we shall formulate a theorem about the solutions of equation (25) on the Ehresmann groupoid. Let $(E,\cdot)$ be an arbitrary Ehresmann groupoid, let $(E_v,\cdot)_{v\in T}$ be the decomposition of this groupoid on Brandt groupoids and let $(M,\cdot)$ be an arbitrary structure. It is easy to prove the following

Theorem 9. The triplet of functions $(H_1,H_2,H_3)\in[E\longrightarrow M]^3$ satisfies equation (25), iff for any $v\in T$ the triplet of functions (H_1^v,H_2^v,H_3^v) satisfies equation (25), where H_i^v is the restriction of the function H_i, $i = 1,2,3$, to the set E_v.

From the above theorem, theorem 8, and since every Brandt groupoid is isomorphic to a product Brandt groupoid,we can easily get the form of the solution (H_1,H_2,H_3) of equation (25) in the case, where the functions H_1,H_2,H_3 belong to the set $[E\longrightarrow B]$. Now we shall consider in detail the solutions from the set Γ of equation (2), which have the form (H,H,H). Above all we shall prove the following

Lemma 8. If g is a homomorphism of the group $(G,\cdot)$ into the structure $(K,\cdot)$ and $g(e)\in K_r$, then

$$g(\alpha)\in K_r \qquad \text{for} \quad \alpha\in G.$$

Proof. Let α be an arbitrary element from the set G, let g be an arbitrary homomorphism of $(G,\cdot)$ into $(K,\cdot)$ such that $g(e)\in K_r$, let $x,y\in K$ be such elements that $(x,g(\alpha))\in D_\cdot$, $(y,g(\alpha))\in D_\cdot$ and

$$x\cdot g(\alpha) = y\cdot g(\alpha).$$

Multiplying this equality by $g(\alpha^{-1})$ from the right side, we obtain

$$x\cdot g(e) = y\cdot g(e).$$

$g(e)\in K_r$, therefore $x = y$, and thus $g(\alpha)\in K_{rr}$. Analogously we can prove that $g(\alpha)\in K_{lr}$. Hence, for $g(\alpha)$ condition 1^o of definition 9 holds.

Since

$$(g(\alpha))_r = (g(\alpha))_l = g(e),$$

for $g(\alpha)$ conditions 2^o and 3^o of definition 9 hold. $g(\alpha^{-1})$ is the solution of the equation

$$g(\alpha)\cdot x = g(e),$$

therefore for $g(\alpha)$ condition 4^o of definition 9 is fulfilled. Thus $g(\alpha) \in K_r$ which completes the proof.

From lemma 8 it follows that in the case, where $(A \times A \times G, *)$ is a Brandt groupoid, condition (16) in theorem 3 is equivalent to the following condition

(27) g is a homomorphism of the group $(G,\cdot)$ into $(K_r,\cdot)$.

It is easy to see, verifying the definition of the set Γ , that the following theorem is true:

Theorem 10. The triplet of functions $(H,H,H) \in [A \times A \times G \longrightarrow K]^3$ belongs to the set Γ , iff the following conditions hold:

(28) $\bigvee_{M \subset A} \left(D_H = M \times M \times G \right),$

(29) $\bigvee_{\bar{a} \in M} \bigwedge_{b \in M} \left(H(\bar{a},b,e) \in K_r \right),$

where M is a set satisfying (28).

Theorem 11. If the function $H \in [A \times A \times G \multimap K]$ has the property (28), M is a set satisfying (28) and the triplet (H,H,H) satisfies equation (2) then the following conditions

(30) $\bigvee_{\bar{a} \in M} \bigwedge_{b \in M} \left[H(\bar{a},b,e) \in K_r \right],$

(31) $\bigwedge_{a \in M} \left[H(a,a,e) \in K_r \right],$

(32) $\bigwedge_{a,b \in M} \left[H(a,b,e) \in K_r \right]$

are equivalent.

Proof. Let $H \in [A \times A \times G \multimap K]$ be an arbitrary function satisfying the assumptions of the theorem and let M be a set satisfying (30). First, we shall show that (30) implies (31). Let $\bar{a}$ be an arbitrary element such that (30) holds. Then we have

$$H(\bar{a},a,e)\cdot H(a,a,e) = H(\bar{a},\bar{a},e) \qquad \text{for } a \in M.$$

$H(\bar{a},a,e) \in K_r$ and $H(\bar{a},\bar{a},e) \in K_r$ therefore

(33) $\quad H(a,\bar{a},e) \in K_r.$

Moreover,

$$H(a,\bar{a},e)\cdot H(\bar{a},a,e) = H(a,a,e),$$

hence

$$H(a,a,e) \in K_r.$$

Thus condition (30) implies (31).
Let now a,b be arbitrary elements of the set M, and let $x,y \in K$ be arbitrary elements such that $(H(a,b,e), x) \in D_{\cdot}$, $(H(a,b,e), y) \in D_{\cdot}$ and

$$H(a,b,e)\cdot x = H(a,b,e)\cdot y .$$

Multiplying the above equality by $H(b,a,e)$ from the left side, we get

$$H(b,b,e)\cdot x = H(b,b,e)\cdot y.$$

$H(b,b,e) \in K_r$, thus

$$x = y,$$

hence $H(a,b,e) \in K_{1r}$. Analogously we can show that $H(a,b,e) \in K_{rr}$, thus for $H(a,b,e)$ condition 1^0 of definition 9 holds.
Moreover $(H(a,b,e))_r = H(b,b,e)$, $(H(a,b,e))_1 = H(a,a,e)$ and $H(a,a,e)$, $H(b,b,e)$ belong to the set K_r, therefore conditions 2^0 and 3^0 of definition 9 hold. It is easy to verify that $H(b,a,e)$ is the solution of the equation

$$H(a,b,e)\cdot x = H(a,a,e),$$

hence condition 4^0 of definition 9 holds. From the above considerations we can conclude that $H(a,b,e) \in K_r$. Thus condition (31) implies condition (32).
Of course, condition (32) implies condition (30), which completes the proof.
In example 9 we shall show that in condition (31) of theorem 11 we cannot replace the general quantifier by the existence quantifier. By theorems 10 and 11 we obtain the following

<u>Theorem 11'</u>. The triplet of functions $(H,H,H) \in [A \times A \times A \xrightarrow{e} K]^3$ satisfying equation (2), belongs to the set Γ, iff for the function H condition (28) and one of conditions (30), (31), (32) hold.

<u>Theorem 12</u>. Let us assume that for the function H the assumptions of theorem 11 are satisfied. Moreover, if one of conditions (30), (31),

(32), holds and $(G,\cdot)$ is a group, then

$$H(a,b,\alpha) \in K_r \qquad \text{for} \quad (a,b,\alpha) \in D_H.$$

Proof. Let M be a set satisfying (28), and let a,b be arbitrary elements belonging to the set M. Then the function g defined as follows

$$g(\alpha) = H(b,b,\alpha) \qquad \text{for } \alpha \in G,\ b \in M$$

is a homomorphism of $(G,\cdot)$ into $(K,\cdot)$, such that $g(e) \in K_r$. Therefore by lemma 8,

$$H(b,b,\alpha) \in K_r \qquad \text{for} \quad \alpha \in G,\quad b \in M.$$

Hence, from theorem 11 and from the equation (2) we obtain

$$H(a,b,\alpha) = H(a,b,e)\cdot H(b,b,\alpha),$$

thus $H(a,b,\alpha) \in K_r$, which completes the proof.

From theorems 3, 10, 11 we obtain the following corollary:

Corollary 3. The triplet of functions $(H,H,H) \in [A \times A \times G \xrightarrow{e} K]^3$ is a solution from the set Γ of equation (2), iff the function H has the form

$$H(a,b,\alpha) = f(a)\cdot g(\alpha)\cdot f^{-1}(b),$$

where

(34) f is an arbitrary element from the set $[A \xrightarrow{e} K_r]$,

35 g is an arbitrary homomorphism of the semigroup $(G,\cdot)$ into $(K,\cdot)$ such that $g(e) \in K_r$,

36 $(f(a), g(\alpha)) \in D$, and $(g(\alpha), f^{-1}(b)) \in D$. for $a,b \in D_H^1$, $\alpha \in G$.

Proof. It is sufficient to prove that for functions f_1, f_2, f_3, g for which conditions (13) - (17) hold, and which dictate, by (10) - (12), the solution (H,H,H) of equation (2), the following conditions hold:

a/ $D_{f_1} = D_{f_2} = D_{f_3}$,

b/ $\bigwedge_{a \in D_{f_1}} \left[f_1(a) = f_2(a) = f_3^{-1}(a) \right]$.

By theorem 10 we can conclude that the condition a/ is fulfilled.

Let a be an arbitrary element belonging to D_{f_1}. By (11), (12), (13) we get

$$H(a,a,e) = f_1(a)\cdot g(e)\cdot f_2^{-1}(a) = f_2(a)\cdot g(e)\cdot f_3(a) = f_1(a)\cdot g(e)\cdot f_3(a).$$

$g(e)$ is an unit in $(K_r,\cdot)$, therefore $g(e)$ is an unit in $(K,\cdot)$ and we have

(37) $$H(a,a,e) = f_1(a)\cdot f_2^{-1}(a) = f_2(a)\ f_3(a) = f_1(a)\cdot f_3(a).$$

From theorem 11 it follows that $H(a,a,e) \in K_r$. Moreover, $f_2(a) \in K_r$, thus $f_1(a) \in K_r$ and $f_3(a) \in K_r$.

Therefore, in virtue of (37) we get

$$f_2^{-1}(a) = f_3(a) \qquad \text{and} \qquad f_1(a) = f_2(a).$$

Thus condition b/ is fulfilled, and this completes the proof.

From the above corollary and lemma 8 we obtain the following corollary.

<u>Corollary 4</u>. The function H, such that $(H,H,H)\in\Gamma$ is a homomorphism of a product Brandt groupoid into $(K,\cdot)$ if H has the form

$$H(a,b,\alpha) = f(a)\cdot g(\alpha)\cdot f_2^{-1}(b),$$

where

f is an arbitrary element from the set $[A \longrightarrow K_r]$

g is an arbitrary homomorphism of group $(G,\cdot)$ into $(K_r,\cdot)$,

and for f and g condition (36) is fulfilled.

B i b l i o g r a p h y

[1] J. A c z é l, Miszelen über Funktionalgleichungen. II Mathematische Nachrichten 23, Band, Heft 1, 1961. Akademie-Verlag, Berlin 39-50.

[2] A. G r z ą ś l e w i c z, O pewnych homomorfizmach i homomorfizmach ciągłych grupoidu Brandta, Rocznik Naukowo-Dydaktyczny WSP w Krakowie. Prace Matematyczne VI, zeszyt 41, 1970, 15-30.

[3] J. T a b o r, Struktura ogólnego rozwiązania równania translacji na grupoidzie Ehresmanna oraz rozkłady niezmiennicze tego grupoidu. Rocznik Naukowo-Dydaktyczny WSP w Krakowie, Prace Matematyczne VI, zeszyt 41, 1970, 107-153.

[4] W. W a l i s z e w s k i, Categories, groupoides, pseudogroups and analitical structures, Rozprawy Matematyczne, XLV, 1965.