ROCZNIK NAUKOWO–DYDAKTYCZNY WSP W KRAKOWIE

Zeszyt 61 Prace matematyczne VIII Rok 1977

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ON THE SOLUTION OF THE EQUATION $F_1(F_2(x,\beta),\alpha) = F_3(x,\alpha\cdot\beta)$

Introduction

This note is the continuation of the note [1], and therefore we shall use here the symbols, definitions and theorems given in [1]. The numeration here will be the continuation of the numeration in [1]. In this paper we shall consider the equation

$$F_1(F_2(x,b,c,\beta),a,b,\alpha) = F_3(x,a,c,\alpha\cdot\beta), \tag{ii}$$

where the functions F_1, F_2, F_3 are partially defined in the set $X \times A \times A \times G$, and the values of these functions belong to the set X. We shall give some theorems, enabling us to find the solution of the equation (ii) from the solutions of the equation

$$H_1(a,b,\alpha) * H_2(b,c,\beta) = H_3(a,c,\alpha\cdot\beta),$$

which has been solved in [1]. In a particular case we obtain the solutions of the equation

$$F(F(x,b,c,\beta),a,b,\alpha) = F(x,a,c,\alpha\cdot\beta),$$

and the solutions of this equation with the identity condition. Dealing with the solutions of the equation (ii) we use the solution of the translation equation on the group given by Z. Moszner in [2]. More detailed informations of the earlier results regarding the solutions of the equation (ii) will be given in this paper.

Let X be an arbitrary set, $(A,\cdot)$ an arbitrary structure, „o" the superposition of functions in the set $[X \xrightarrow{\circ} X]$ defined as follows:

$$f \circ g := \{(x,y) : x \in D_g \wedge g(x) \in D_f \wedge y = f(g(x))\}.$$

It is easy to verify that $([X \xrightarrow{\circ} X], \circ)$ is a semigroup with a unit and that the identity function on X /designed by J/ is that unit.

Definition 15. The triplet of functions $(F_1,F_2,F_3) \in [X \times A \twoheadrightarrow X]^3$ will be called the solution of the equation

(38) $$F_1(F_2(x,\beta),\alpha) = F_3(x,\alpha\cdot\beta),$$

if for arbitrary $\alpha,\beta \in A$ such that $(\alpha,\beta) \in D_{\bullet}$ and for an arbitrary $x \in X$ the following condition is satisfied:

(39) $$[(x,\beta)\in D_{F_2} \wedge (F_2(x,\beta),\alpha)\in D_{F_1} \wedge (x,\alpha\cdot\beta)\in D_{F_3}] \Longrightarrow [F_1(F_2(x,\beta),\alpha) = F_3(x,\alpha\cdot\beta)].$$

Definition 16. We shall denote by $\bar{\Omega}$ the set of all triplets $(F_1,F_2,F_3) \in [X \times A \twoheadrightarrow X]^3$ such that for arbitrary $x \in X$ and arbitrary pair $(\alpha,\beta) \in D_{\bullet}$ the following condition is satisfied

(40) $$[\alpha\in D^2_{F_1} \wedge \beta\in D^2_{F_2} \wedge \alpha\cdot\beta\in D^2_{F_3}] \Longrightarrow \{[(x,\beta)\in D_{F_2} \wedge (F_2(x,\beta),\alpha)\in D_{F_1}] \Longleftrightarrow [(x,\alpha\cdot\beta)\in D_{F_3}]\}$$

Theorem 13. The triplet of functions (F_1,F_2,F_3) $[X \times A \quad X]^3$ is a solution from the set $\bar{\Omega}$ of equation (38) iff there exists in the set $[A \twoheadrightarrow [X \twoheadrightarrow X]]^3$ the triplet of functions (H_1,H_2,H_3) sat_s,ying the equation

(41) $$H_1(\alpha) \circ H_2(\beta) = H_3(\alpha\cdot\beta)$$

such that the following conditions are satisfied:

(42) $$(x,\alpha) \in D_{F_i} \Longleftrightarrow (\alpha\in D_{H_i} \wedge x\in D_{H_i(\alpha)}) \quad \text{for } x\in X,\ \alpha\in A,\ i=1,2,3,$$

(43) $$(x,\alpha) \in D_{F_i} \Longrightarrow F_i(x,\alpha) = [H_i(\alpha)](x) \quad \text{for } x\in X,\ \alpha\in A,\ i=1,2,3.$$

Proof. Let the triplet of functions $(F_1,F_2,F_3)\in\bar{\Omega}$ satisfy equation (38).
Let us put :

$$H_i(\alpha) := \{(x,y) : (x,\alpha)\in D_{F_i} \wedge y = F_i(x,\alpha)\} \quad \text{for } \alpha\in D^2_{F_i},\ i=1,2,3.$$

Let $(\alpha,\beta)\in D_{\bullet}$, $\alpha\in D_{H_1}$, $\beta\in D_{H_2}$, $\alpha\cdot\beta\in D_{H_3}$. Then

$$\alpha\in D^2_{F_1},\ \beta\in D^2_{F_2},\ \alpha\cdot\beta\in D^2_{F_3}.$$

From the above, using $(F_1,F_2,F_3)\in\bar{\Omega}$ we get

$$[(x,\beta)\in D_{F_2} \wedge (F_2(x,\beta),\alpha)\in D_{F_1}] \Longleftrightarrow [(x,\alpha\cdot\beta)\in D_{F_3}] \quad \text{for } x\in X,$$

whence

$$D_{H_1(\alpha)\circ H_2(\beta)} = D_{H_3(\alpha\cdot\beta)}.$$

For $x \in D_{H_3(\alpha\cdot\beta)}$ we obtain

$$(H_1(\alpha)\circ H_2(\beta))(x) = [H_1(\alpha)]\left([H_2(\beta)](x)\right) = F_1(F_2(x,\beta),\alpha) = F_3(x,\alpha\cdot\beta) =$$

$$= [H_3(\alpha\cdot\beta)](x),$$

thus the triplet (H_1,H_2,H_3) satisfies equation (41). From the definition of functions (H_1,H_2,H_3) it follows that conditions (42) and (43) hold.

Let us assume now that the triplet $(H_1,H_2,H_3) \in [A \twoheadrightarrow [X \leftrightarrow X]]^3$ satisfies equation (41) and let $(F_1,F_2,F_3) \in [X \times A \twoheadrightarrow X]^3$ be a triplet such that conditions (42) and (43) hold. Let $(\alpha,\beta)\in D_0$, $\alpha \in D^2_{F_1}$, $\beta \in D^2_{F_2}$, $\alpha\cdot\beta \in D^2_{F_3}$. Then

$$\alpha \in D_{H_1},\ \beta \in D_{H_2},\ \alpha\cdot\beta \in D_{H_3}$$

and we obtain

$$H_1(\alpha) \circ H_2(\beta) = H_3(\alpha\cdot\beta) \ ,$$

and hence

$$D_{H_1(\alpha)\circ H_2(\beta)} = D_{H_3(\alpha\cdot\beta)}$$

whence

$$[(x,\beta)\in D_{F_2} \wedge (F_2(x,\beta),\alpha) \in D_{F_1}] \Longleftrightarrow [(x,\alpha\cdot\beta) \in D_{F_3}] \quad \text{for } x \in X,$$

thus the triplet $(F_1,F_2,F_3) \in \bar{\Omega}$. Let us assume that $(x,\alpha\cdot\beta) \in D_{F_3}$. Then

$$(x,\beta) \in D_{F_2} \quad \text{and} \quad (F_2(x,\beta),\alpha) \in D_{F_1}$$

and we get

$$F_1(F_2(x,\beta),\alpha) = [H_1(\alpha)]\left([H_2(\beta)](x)\right) = [H_1(\alpha) \circ H_2(\beta)](x) =$$

$$= [H_3(\alpha\cdot\beta)](x) = F_3(x,\alpha\cdot\beta).$$

Therefore the triplet of functions (F_1,F_2,F_3) satisfies equation (38), which completes the proof.

Corollary 5. The triplet of functions $(F,F,F) \in \bar{\Omega}$ satisfies equation (38) iff there exists a function H in the set $[A \multimap [X \multimap X]]$ such that the triplet (H,H,H) satisfies equation (41), and the following conditions are fulfilled:

$$(44) \qquad (x, \alpha) \in D_F \Longleftrightarrow (\alpha \in D_H \wedge x \in D_{H(\alpha)}) \qquad \text{for } x \in X, \ \alpha \in A,$$

$$(45) \qquad (x, \alpha) \in D_F \Longrightarrow F(x,\alpha) = [H(\alpha)](x) \qquad \text{for } x \in X, \ \alpha \in A.$$

We shall now give some lemmas about the regularity of elements in the semigroup $([X \multimap X], o)$.

Lemma 9. Right regular elements of the set $[X \multimap X]$ are the functions the range of which is the set X, i.e.

$$f \in [X \text{—} X]_{rr} \Longleftrightarrow \mathcal{Q}_f = X.$$

Proof. Let f be an arbitrary element of the set $[X \multimap X]_{rr}$. Then for arbitrary $g_1, g_2 \in [X \multimap X]$ we have

$$(46) \qquad g_1 \circ f = g_2 \circ f \Longrightarrow g_1 = g_2.$$

Of course, $\mathcal{Q}_f \subset X$. Let us suppose that $X \setminus \mathcal{Q}_f \neq \emptyset$ and let $x_0 \in X \setminus \mathcal{Q}_f$. Let x_1, x_2 be arbitrary elements of the set X such that $x_1 \neq x_2$.
Denoting

$$g_i(x) := \begin{cases} x & \text{for } x \neq x_0, \ x \in X, \\ x_i & \text{for } x = x_0, \end{cases} \qquad \text{for } i = 1,2,$$

we obtain for $x \in X$

$$(g_1 \circ f)(x) = g_1(f(x)) = f(x) = g_2(f(x)) = (g_2 \circ f)(x),$$

i.e.

$$g_1 \circ f = g_2 \circ f,$$

whence, by (46)

$$g_1 = g_2,$$

which is contrary to the definition of g_1 and g_2. Therefore the condition $f \in [X \multimap X]_{rr}$ implies $\mathcal{Q}_f = X$, which closes the first part of the proof.

Let now f be a function such that $\mathcal{Q}_f = X$ and let g_1 and g_2 be arbitrary functions such that

(47) $\quad g_1 \circ f = g_2 \circ f.$

Let x_0 be an arbitrary element of the set D_{g_1}. There exists x in the set D_f such that $f(x) = x_0$. Thus x_0 $D_{g_1 \circ f}$. Hence and by (47)

$$x \in D_{g_2 \circ f},$$

whence $x_0 \in D_{g_2}$. Therefore $D_{g_1} \subset D_{g_2}$. One can show analogously that $D_{g_2} \subset D_{g_1}$. Thus $D_{g_1} = D_{g_2}$.

Moreover, we have

$$g_1(x_0) = g_1(f(x)) = (g_1 \circ f)(x) = (g_2 \circ f)(x) = g_2(f(x)) = g_2(x_0),$$

whence the functions g_1 and g_2 are equal. We have thus proved that the condition $\mathrm{Q}_f = X$ implies the condition $f \in [X \longrightarrow X]_{rr}$, which completes the proof.

<u>Lemma 10</u>. Left regular elements of the set $[X \xrightarrow{\circ} X]$ are the one-to-one functions on X, i.e.

$$f \in [X \longrightarrow X]_{lr} \iff [D_f = X \text{ and } f \text{ is one-to-one function}].$$

<u>Proof</u>. Let $f \in [X \xrightarrow{\circ} X]_{lr}$. Then, for arbitrary functions $g_1, g_2 \in [X \xrightarrow{\circ} X]$ we have

(48) $\quad f \circ g_1 = f \circ g_2 \Longrightarrow g_1 = g_2.$

Let us suppose that $X \setminus D_f \neq \emptyset$. Putting

$$g_1(x) := x \quad \text{for } x \in D_f,$$

$$g_2(x) := x \quad \text{for } x \in X,$$

we obtain that

(49) $\quad f \circ g_1 = f \circ g_2,$

whence, by definitions of functions g_1 and g_2, $g_1 = g_2$, which is contrary to the definitions of functions g_1 and g_2. Thus $D_f = X$.

Let us now suppose that f is not a one-to-one function, i.e. there exists $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let x_0 be an arbitrary, fixed point of the set X.

Let us put

$$g_i(x) = \begin{cases} x & \text{for } x \neq x_0, \\ x_i & \text{for } x = x_0,\ i = 1,2. \end{cases}$$

For $x \neq x_0$ we have

$$(f \circ g_1)(x) = f(g_1(x)) = f(g_2(x)) = (f \circ g_2)(x),$$

and for $x = x_0$ we get

$$(f \circ g_1)(x_0) = f(g_1(x_0)) = f(x_1) = f(x_2) = f(g_2(x_0)) = (f \circ g_2)(x_0),$$

thus equality (49) holds. Therefore $g_1 = g_2$, which is contrary to the definitions of functions g_1 and g_2. Thus f is a one-to-one function.
Let now $f \in [X \text{—}e\text{→} X]$ be an arbitrary one-to-one function with the domain X, and let g_1 and g_2 be arbitrary functions from the set $[X \text{—}e\text{→} X]$ such that

(50) $\quad f \circ g_1 = f \circ g_2.$

Let x be an arbitrary element of the set D_{g_1}. Then, $g_1(x) \in D_f$, whence $x \in D_{f \circ g_1}$, and consequently, by (50), $x \in D_{f \circ g_2}$. Thus $x \in D_{g_2}$. One can conclude from this that $D_{g_1} \subset D_{g_2}$, and show analogously that $D_{g_1} \supset D_{g_2}$.
Moreover, for $x \in D_{g_1}$ we have

$$(f \circ g_1)(x) = (f \circ g_2)(x),$$

hence

$$f(g_1(x)) = f(g_2(x)).$$

f is an one-to-one function, thus

$$g_1(x) = g_2(x),$$

i.e. $g_1 = g_2$.

Hence f belongs to the set $[X \text{——→} X]_{1r}$, which completes the proof.
Since $([X \text{—}e\text{→} X], \circ)$ is a semigroup with the unit we obtain, by lemmas (9) and (10), the following lemma

<u>Lemma 11</u>. f is a strictly regular element of the set $[X \text{—}e\text{→} X]$ iff f is an one-to-one function such that $D_f = X = \mathrm{C}_f$, i.e. f is a bijection on the set X.
Let us now put $(K, \cdot) = ([X \text{—}e\text{→} X], \circ)$ in definition 13. We obtain

<u>Theorem 13</u>. If in definition 12 we put $(K, \cdot) = ([X \text{—}e\text{→} X], \circ)$, then Γ is the set of all triplets $(H_1, H_2, H_3) \in [A \times A \times G \text{—}e\text{→} [X \text{—}e\text{→} X]]^3$ such that the following conditions are fulfilled:

(51) $D_{H_1} \times D_{H_2} = D_{H_3}$,

(52) $D^2_{H_1} = D^1_{H_2}$,

(53) $D_{H_i} = D^1_{H_i} \times D^2_{H_i} \times G$ for $i = 1,2,3$,

(54) there exists an element $\bar{a} \in D^1_{H_1}$ such that

$H_1(\bar{a},b,e)$ is a bijection on the set X for $b \in D^2_{H_1}$,

(55) there exists an element $\bar{c} \in D^2_{H_2}$ such that $H_2(b,\bar{c},e)$ is a function with the range X for $b \in D^1_{H_2}$.

Let $(F_1,F_2,F_3) \in [X \times A \times A \times G \longrightarrow X]^3$. Let us denote

$$\bar{D}_i := \left\{(a,b,\alpha) : \bigvee_x [(x,a,b,\alpha) \in D_{F_i}]\right\}, \quad i = 1,2,3.$$

Let $\bar{\Omega}$ be the set of all triplets of functions

$(F_1,F_2,F_3) \in [X \times A \times A \times G \dashrightarrow X]^3$ satisfying condition (40).

<u>Definition 17</u>. We shall denote by Ω the set of all triplets $(F_1,F_2,F_3) \in \bar{\Omega}$ satisfying the following conditions

1°. $\bar{D}_1 \times \bar{D}_2 = \bar{D}_3$,

2°. $D^3_{F_1} = D^2_{F_2}$,

3°. $\bar{D}_i = D^2_{F_i} \times D^3_{F_i} \times G$ for $i = 1,2,3$,

4°. there exists an element $\bar{a} \in D^2_{F_1}$ such that the function

$x \longrightarrow F_1(x,\bar{a},b,e)$ for $x \in \{x : (x,\bar{a},b,e) \in D_{F_1}\}$

is a bijection on the set X for $b \in D^3_{F_1}$,

5°. there exists an element $\bar{c} \in D^3_{F_2}$ such that the function

$x \longrightarrow F_2(x,b,\bar{c},e)$ for $x \in \{x : (x,b,\bar{c},e) \in D_{F_2}\}$

has the range equal to X for $b \in D^2_{F_2}$.

It is easy to see that the replacement of condition 5^0 in definition 17 by the following condition

$5'$ there exists $\bar{b} \in D^2_{F_2}$ and $\bar{c} \in D^3_{F_2}$ such that the function

$$x \longrightarrow F_2(\bar{b},\bar{c},e) \qquad \text{for } x \in \{x : (x,\bar{b},\bar{c},e) \in D_{F_2}\}$$

has the range X,

does not change the set of solutions from the set Ω of equation (38). We shall show now that for the triplet $(F_1,F_2,F_3) \in \Omega$ satisfying equation (56) the following condition

$$D_{F_i} = X \times A \times A \times G \qquad \text{for } i = 1,2,3$$

need not be satisfying. It is illustrated by the following example:

Example 6. Let X be the set of real numbers, $(G,\cdot)$ the multiplicative semigroup modulo 4, A a two-element set $\{a,b\}$. Let us define the function F as follows:

$$F(x,a,b,\alpha) = x \qquad \text{for } x \in R,\ a,b \in A,\ 0 \neq \alpha \neq 2,$$

$$F(x,a,b,\alpha) = x \qquad \text{for } x \in (0,+\infty),\ a,b \in A,\ \alpha \in \{0,2\}.$$

It is easy to verify that the triplet (F,F,F) is a solution from the set Ω of equation (56), and that the set $X \times A \times A \times G$ is not the domain of the function F.

Theorem 15. The triplet of functions $(F_1,F_2,F_3) \in [X \times A \times A \times G \leftrightarrow X]^3$ is a solution from the set Ω of the equation

$$(56) \qquad F_1(F_2(x,b,c,\beta),a,b,\alpha) = F_3(x,a,c,\alpha\cdot\beta)$$

iff there exists in the set Γ a triplet (H_1,H_2,H_3) satisfying the equation

$$(57) \qquad H_1(a,b,\alpha) \circ H_2(b,c,\beta) = H_3(a,c,\alpha\cdot\beta)$$

such that the following conditions are fulfilled:

$$(58) \qquad (x,a,b,\alpha) \in D_{F_i} \Longleftrightarrow [(a,b,\alpha) \in D_{H_i} \wedge x \in D_{H_i(a,b,\alpha)}] \quad \text{for } i = 1,2,3,$$

$$(59) \qquad F_i(x,a,b,\alpha) = [H_i(a,b,\alpha)](x) \qquad \text{for } (x,a,b,\alpha) \in D_{F_i},\ i = 1,2,3.$$

Proof. Let the triplet $(F_1,F_2,F_3) \in \Omega$ satisfy equation (56). Let us put

$$H_i(a,b,\alpha) := \{(x,y) : (x,a,b,\alpha) \in D_{F_i} \wedge y = F_i(x,a,b,\alpha)\}$$

for $(a,b,\alpha) \in \bar{D}_i$, $i = 1,2,3$.

Since $(F_1,F_2,F_3) \in \Omega$, and by lemmas 11 and 9 we obtain that

$$(H_1,H_2,H_3) \in \Gamma.$$

Let now $(a,b,\alpha) \in D_{H_1}$, $(b,c,\beta) \in D_{H_2}$, $(a,c,\alpha\cdot\beta) \in D_{H_3}$. It follows from this that $(a,b,\alpha) \in \bar{D}_1$, $(b,c,\beta) \in \bar{D}_2$, $(a,c,\alpha\cdot\beta) \in \bar{D}_3$.

Hence, since $\Omega \subset \bar{\Omega}$ we get

$$D_{H_1(a,b,\alpha) \circ H_2(b,c,\beta)} = D_{H_3(a,c,\alpha\cdot\beta)}.$$

One can show analogously as in theorem 13 that (H_1,H_2,H_3) satisfies equation (57).

Let us now assume that the triplet $(H_1,H_2,H_3) \in \Gamma$ satisfies equation (57), and let $(F_1,F_2,F_3) \in [X \times A \times A \times G \multimap X]^3$ be a triplet of functions such that conditions (58) and (57) are fulfilled. Let $(a,b,\alpha) \in \bar{D}_1$, $(b,c,\beta) \in \bar{D}_2$, $(a,c,\alpha\cdot\beta) \in \bar{D}_3$. Since (H_1,H_2,H_3) satisfies equation (57) we have

$$[(x,b,c,\beta) \in D_{F_2} \wedge (F_2(x,b,c,\beta),a,b,\alpha) \in D_{F_1}] \Longleftrightarrow [(x,a,c,\alpha\cdot\beta) \in D_{F_3}] \quad \text{for } x \in X,$$

thus $(F_1,F_2,F_3) \in \bar{\Omega}$. From theorem 14 and lemmas 9 and 11 it follows that for the triplet (F_1,F_2,F_3) the other conditions of definition 17 are fulfilled, whence $(F_1,F_2,F_3) \in \Omega$. Moreover, for $(x,a,c,\alpha\cdot\beta) \in D_{F_3}$ we get

$$F_1(F_2(x,b,c,\beta),a,b,\alpha) = [H_1(a,b,\alpha)]([H_2(b,c,\beta)](x)) =$$

$$= [H_1(a,b,\alpha) \circ H_2(b,c,\beta)](x) = [H_3(a,c,\alpha\cdot\beta)](x) = F_3(x,a,c,\alpha\cdot\beta).$$

Such the triplet (F_1,F_2,F_3) is therefore the solution of equation (56), which completes the proof.

In virtue of theorems 15 and 3, and lemmas 9 and 11 we can obtain the following theorem:

<u>Theorem 16</u>. The triplet $(F_1,F_2,F_3) \in [X \times A \times A \times G \multimap X]^3$ is a solution from the set Ω of equation (56) iff the functions F_1,F_2,F_3 have the form

(60) $\quad F_1(x,a,b,\alpha) = [f_1(a) \circ g(\alpha) \circ f_2^{-1}(b)](x)$,

(61) $\quad F_2(x,a,b,\alpha) = [f_2(a) \circ g(\alpha) \circ f_3(b)](x)$,

(62) $\quad F_3(x,a,b,\alpha) = [f_1(a) \circ g(\alpha) \circ f_3(b)](x)$,

where

(63) $f_1 \in [A \dashrightarrow [X \dashrightarrow X]]$ is a function such that there exists and element $\bar{a}$ D_{f_1} such that $f_1(\bar{a})$ is a bijection on the set X,

(64) $f_2 \in [A \dashrightarrow [X \dashrightarrow X]_r]$, i.e. $f_2(a)$ is a bijection on the set X for every $a \in D_{f_2}$,

(65) $f_3 \in [A \dashrightarrow [X \dashrightarrow X]]$ is a function such that there exists an element $\bar{c} \in D_{f_3}$ such that the range of $f_3(\bar{c})$ is the set X,

(66) g is an arbitrary homomorphism of $(G,\cdot)$ into $([X \dashrightarrow X], o)$ such that $g(e) = I$.

In particular, if $(A \times A \times G, \ast)$ is the product Brandt groupoid we can replace condition (66), in virtue of lemma 8, by the following condition

(66') g is an arbitrary homomorphism of group $(G,\cdot)$ into $([X \dashrightarrow X]_r, o)$.

Theorem 17. Let us assume that $(H_1, H_2, H_3) \in \Gamma$ satisfies equation (57), and let $(f_1, f_2, f_3, g) \in \Delta$ be the quadruple dictating this solution. Then there exists a solution $(\bar{H}_1, \bar{H}_2, \bar{H}_3) \in \Gamma$ of equation (57) such that the conditions

(67) $D_{\bar{H}_i(a,b,\alpha)} = X$ for $(a,b,\alpha) \in D_{H_i}$, $i = 1,2,3$,

(68) $\bar{H}_i(a,b,\alpha) \mid_{D_{H_i(a,b,\alpha)}} = H_i(a,b,\alpha)$ for $(a,b,\alpha) \in D_{H_i}$, $i = 1,2,3$

are fulfilled iff there exists a homomorphism $\bar{g}$ of the semigroup $(G,\cdot)$ into $([X \dashrightarrow X], o)$ satisfying the following conditions

(69) $D_{\bar{g}(\alpha)} = X$ for $\alpha \in G$,

(70) $\bar{g}(\alpha) \mid_{D_{g(\alpha)}} = g(\alpha)$ for $\alpha \in G$.

Proof. Let $(\bar{H}_1, \bar{H}_2, \bar{H}_3) \in \Gamma$ be a solution of equation (57) satisfying conditions (67) and (68). Then there exists in the set Δ the quadruple of functions $(k_1, k_2, k_3, \bar{g})$ dictating this solution. By (68) we obtain

(71) $[f_1(\bar{a}) \circ g(\alpha) \circ f_2^{-1}(b)](x) = [k_1(\bar{a}) \circ \bar{g}(\alpha) \circ k_2^{-1}(b)](x)$ for $x \in D_{H_1(\bar{a},b,\alpha)}$,

where $\bar{a}$ is an arbitrary element satisfying (8), b is an arbitrary element of the set $D^2_{H_1}$, α is an arbitrary element of the set G.

In virtue of theorem 5 we obtain that there exists a function $m \in [X \overset{\circ}{\longleftrightarrow} X]_r$ such that

(72) $k_1(\bar{a}) = f_1(\bar{a}) \circ m$ and $k_2(b) = f_2(b) \circ m$.

By (71) we get

(73) $[g(\alpha)](x) = [f_1^{-1}(a) \circ k_1(\bar{a}) \circ \breve{g}(\alpha) \circ k_2^{-1}(b) \circ f_2(b)](x)$ for $x \in D_{g(\alpha)}$.

Let us put

(74) $[\bar{g}(\alpha)](x) := [f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\alpha) \circ k_2^{-1}(b) \circ f_2(b)](x)$ for $x \in X$.

Functions occuring in the above formula are strictly regular elements of the semigroup $([X \overset{\circ}{\longrightarrow} X], \circ)$, therefore for the function $\bar{g}$ condition (69) holds. Moreover, by (73) and (74) we can conclude that for the function $\bar{g}$ condition (70) holds. From (72) we get:

$$\bar{g}(\alpha) \circ \bar{g}(\beta) = f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\alpha) \circ k_2^{-1}(b) \circ f_2(b) \circ f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\beta) \circ k_2^{-1}(b) \circ f_2(b) =$$

$$= f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\alpha) \circ m^{-1} \circ f_2^{-1}(b) \circ f_2(b) \circ f_1^{-1}(\bar{a}) \circ f_1(\bar{a}) \circ m \circ \breve{g}(\beta) \circ k_2^{-1}(b) \circ f_2(b) =$$

$$= f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\alpha) \circ \breve{g}(\beta) \circ k_2^{-1}(b) \circ f_2(b) =$$

$$= f_1^{-1}(\bar{a}) \circ k_1(\bar{a}) \circ \breve{g}(\alpha \cdot \beta) \circ k_2^{-1}(b) \circ f_2(b) = \bar{g}(\alpha \cdot \beta)$$

whence g is the homomorphism of $(G, \cdot)$ into $([X \overset{\circ}{\longrightarrow} X], \circ)$, which completes the first part of the proof.

Let now $\bar{g}$ be a homomorphism of $(G, \cdot)$ into $([X \overset{\circ}{\longrightarrow} X], \circ)$ satisfying conditions (69) and (70). Let us put for $a \in D_{f_1}$, $b \in D_{f_3}$:

$$[k_1(a)](x) := \begin{cases} [f_1(a)](x) & \text{for } x \in D_{f_1(a)}, \\ x & \text{for } x \in X \setminus D_{f_1(a)}, \end{cases}$$

$$k_2 := f_2,$$

$$[k_3(b)](x) := \begin{cases} [f_3(b)](x) & \text{for } x \in D_{f_3(b)}, \\ x & \text{for } x \in X \setminus D_{f_3(b)}. \end{cases}$$

It is easy to see that the triplet $(\bar{H}_1, \bar{H}_2, \bar{H}_3)$, dictated by the quadruple $(k_1, k_2, k_3, \bar{g})$ is a solution from the set Γ of equation (57), satisfying conditions (67) and (68), which completes the proof.

We shall show now that not for every homomorphism of semigroup $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$ there exist a homomorphism $\bar{g}$ satisfying conditions (69) and (70). It is illustrated by the following example:

Example 7. Let us put

$$X = \{1,2\}, \quad G = \{\alpha, \beta, \gamma\}.$$

Let us define operation „$\cdot$" in the set G as follows:

$\cdot$	α	β	γ
α	α	β	γ
β	β	α	γ
γ	γ	γ	γ

It is easy to verify that $(G,\cdot)$ is a semigroup with the unit α and the zero γ.
Let $g(\gamma)$ be an empty function, and let

$$[g(\alpha)](x) = x \qquad \text{for } x = 1,2,$$
$$[g(\beta)](1) = 2,$$
$$[g(\beta)](2) = 1.$$

It is easy to verify that g is a homomorphism of $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$.
Let us suppose that $\bar{g}$ is a homomorphism of $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$ satisfying conditions (69) and (70). Then we would have

$$[\bar{g}(\gamma)](x) = [\bar{g}(\alpha) \circ \bar{g}(\gamma)](x) \qquad \text{for } x = 1,2,$$

which, in comparison with the definition of the function g is impossible. Thus we have shown that there does not exist the homomorphism $\bar{g}$ fulfilling conditions (69) and (70).

Lemma 12. If g is a homomorphism of the semigroup $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$ satisfying the condition

$$(75) \qquad \bigwedge_{\alpha,\beta \in G} \left[D_{g(\alpha)} \supset D_{g(\beta)} \right]$$

then there exists a homomorphism $\bar{g}$ of the semigroup $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$ satisfying conditions (69) and (70).

Proof. Let g be an arbitrary homomorphism of the semigroup $(G,\cdot)$ into $([X \overset{e}{\longrightarrow} X], \circ)$ satisfying condition (75). If for some $\alpha_0 \in G, g(\alpha_0)$

is an empty function, then by (75) we obtain that for every $\alpha \in G, g(\alpha)$ is an empty function. In this case it is sufficient to put, for example, that $[\bar{g}(\alpha)](x) = x$ for $\alpha \in G$, $x \in X$.
Let us now assume that $g(\alpha)$ is a non-empty function for every $\alpha \in G$. Let us chose from each set $D_{g(\alpha)}$ exactly one element x_α in such manner that $x_\alpha = x_\beta$ if $D_{g(\alpha)} = D_{g(\beta)}$. Let us define the function $\bar{g}$ for $\alpha \in G$ as follows:

$$[\bar{g}(\alpha)](x) = \begin{cases} [g(\alpha)](x) & \text{for } x \in D_{g(\alpha)}, \\ [g(\alpha)](x_\alpha) & \text{for } x \in X \setminus D_{g(\alpha)}. \end{cases}$$

Let α, β be arbitrary elements belonging to the set G. Of course, for $x \in D_{g(\alpha \cdot \beta)}$ the equality

$$[\bar{g}(\alpha) \circ \bar{g}(\beta)](x) = [\bar{g}(\alpha \cdot \beta)](x)$$

holds.
Suppose that $x \in X \setminus D_{g(\alpha \cdot \beta)}$. Since g is a homomorphism satisfying condition (75) therefore

$$D_{g(\beta)} = D_{g(\alpha \cdot \beta)} = D_{g(\alpha)\, g(\beta)}$$

and hence

$$\begin{aligned}
[\bar{g}(\alpha) \circ \bar{g}(\beta)](x) &= [\bar{g}(\alpha)]([\bar{g}(\beta)](x)) = [\bar{g}(\alpha)]([g(\beta)](x_\beta)) = \\
&= [g(\alpha)]([g(\beta)](x_\beta)) = [g(\alpha) \circ g(\beta)](x_\beta) = \\
&= [g(\alpha \cdot \beta)](x_\beta) = [\bar{g}(\alpha \cdot \beta)](x),
\end{aligned}$$

which completes the proof.
It is easy to see that if the homomorphism g of the semigroup $(G, \cdot)$ into $([X \overset{\circ}{\longrightarrow} X], \circ)$ satisfies the condition

$$(76) \qquad \bigwedge_{\alpha, \beta \in G} [D_{g(\alpha)} = D_{g(\beta)}],$$

then g satisfies condition (75). Hence and by lemma 12 it follows

Lemma 13. If g is the homomorphism of the semigroup $(G, \cdot)$ into $([X \overset{\circ}{\longrightarrow} X], \circ)$ satisfying condition (76), then there exists a homomorphism $\bar{g}$ of the semigroup $(G, \cdot)$ into $([X \overset{\circ}{\longrightarrow} X], \circ)$ such that conditions (69) and (70) hold.
The following example shows that if the homomorphism g of the semigroup

$(G,\cdot)$ into $([X\longleftrightarrow X],\circ)$ satisfies condition (75) then condition (76) need not be satisfied.

Example 8. Let us put

$$X := \{1,2,3\}, \qquad G = \{\alpha,\beta,\gamma\}.$$

Let us define the operation „o" in G as follows

$\cdot$	α	β	γ
α	α	β	γ
β	α	β	γ
γ	α	β	γ

It is easy to see that $(G,\cdot)$ is a semigroup. Let us put

$$[g(\alpha)](x) = 1 \qquad \text{for } x = 1,2,3,$$

$$[g(\beta)](x) = 1 \qquad \text{for } x = 1,2,$$

$$[g(\gamma)](x) = 1 \qquad \text{for } x = 1.$$

It is easy to verify that g is the homomorphism of $(G,\cdot)$ into $([X\longleftrightarrow X],\circ)$, and that it satisfies condition (75) and does not satisfy condition (76). Moreover, it is evident that the condition

$$\bigwedge_{\alpha,\beta\in G}\left[D_{g(\alpha)} = D_{g(\beta)}\right]$$

does not imply for the homomorphism g condition (75).

Lemma 14. If g is a homomorphism of the semigroup $(G,\cdot)$ into $([X\longleftrightarrow X],\circ)$ and the following condition

$$(77) \qquad \bigwedge_{\alpha,\beta\in G}\ \bigvee_{\gamma\in G}(\gamma\cdot\alpha = \beta)$$

holds, then the homomorphism g satisfies condition (76).

Proof. Let g be an arbitrary homomorphism of $(G,\cdot)$ into $([X\longleftrightarrow X],\circ)$, let condition (77) hold and let $\alpha,\beta\in G$. Then there exists in G elements γ_1 and γ_2 such that

$$\gamma_1\cdot\alpha = \beta \qquad \text{and} \qquad \gamma_2\cdot\beta = \alpha.$$

Hence, and from the fact that g is a homomorphism, it follows that $g(\gamma_1)\circ g(\alpha) = g(\beta)$ and $g(\gamma_2)\circ g(\beta) = g(\alpha)$.

Comparing the above equalities we arrive at the conclusion that condition (76) holds.
From lemmas 14 and 13 it follows

Corollary 6. For every homomorphism g of the group $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ there exists a homomorphism $\bar{g}$ of the group $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ such that conditions (69) and (70) hold.

Lemma 15. If g is a homomorphism of the semigroup $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ and

$$(78) \qquad \bigvee_{a \in X} \ \bigwedge_{\alpha \in G} [a \notin D_{g(\alpha)}],$$

then there exists a homomorphism $\bar{g}$ of the semigroup $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ satisfying conditions (69) and (70).
To prove this it is sufficient to put

$$[\bar{g}(\alpha)](x) := \begin{cases} [g(\alpha)](x) & \text{for } x \in D_{g(\alpha)} \\ a & \text{for } x \in X \setminus D_{g(\alpha)}. \end{cases}$$

Z. Moszner in paper [2] solved the equation

$$(79) \qquad F(F(x,\beta),\alpha) = F(x,\alpha\cdot\beta)$$

in the class of functions $[X \times G \longrightarrow X]$ in the case when $(G,\cdot)$ is a group. It is easy to see that this solution is equivalent to giving all the homomorphisms of the group $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$.
From corollary 6 it follows that we can obtain all homomorphisms of the group $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ by suitable restrictions of the homomorphisms of $(G,\cdot)$ into $([X \xrightarrow{\cdot} X], o)$. Therefore the problem of finding all the homomorphisms of the group $(G,\cdot)$ into $([X \xrightarrow{\ominus} X], o)$ one can regard as solved and afterwards we shall make use of this fact.
From corollaries 3 and 5 it follows

Theorem 18. The function F defined on the set $X \times A \times A \times G$ and such that $(F,F,F) \in \Omega$ satisfies the equation

$$(80) \qquad F(F(x,b,c,\beta),a,b,\alpha) = F(x,a,c,\alpha\cdot\beta)$$

iff F has the form

$$(81) \qquad F(x,a,b,\alpha) = [f(a) \circ g(\alpha) \circ f^{-1}(b)](x) \quad \text{for } (x,a,b,\alpha) \in X \times A \times G,$$

where

(82) f is an arbitrary element of the set $[A \longrightarrow [X \xrightarrow{\circ} X]_r]$,

(83) g is an arbitrary homomorphism of the semigroup $(G,\cdot)$ into $([X \longrightarrow X], o)$ such that $g(e) = I$.

Corollary 7. In the particular case, when $(A \times A \times G, *)$ is a Brandt groupoid one can replace condition (83) in theorem 18, in virtue of lemmas 8 and 12, by condition (66).

Theorem 19. The triplet of functions $(H,H,H) \in [A \times A \times G \longrightarrow [X \longrightarrow X]]^3$ satisfying equation (57) belongs to the set Γ iff

(84) $H(a,a,e) = I$

for each a belonging to the set A.

Proof. From corollary 3 it follows immediately that if the domain of the function H is the set $A \times A \times G$, and if the triplet $(H,H,H) \in \Gamma$ satisfies equation (57), then $H(a,a,e) = I$ for every $a \in A$.
Let now H be a function satisfying equation (57) such that $D_H = A \times A \times G$ and let for every $a \in A$ condition (84) hold. It follows from this that

$$H(a,b,e) \circ H(b,a,e) = H(a,a,e) \qquad \text{for } a,b \in A,$$

whence

$$D_{H(a,b,e)} = Q_{H(a,b,e)} = X.$$

Let x_1, x_2 be arbitrary, fixed elements of the set X such that $[H(a,b,e)](x_1) = [H(a,b,e)](x_2)$.
Then

$$[H(b,a,e) \circ H(a,b,e)](x_1) = [H(b,a,e) \circ H(a,b,e)](x_2),$$

whence

$$[H(b,b,e)](x_1) = [H(b,b,e)](x_2).$$

Hence and by (84) we obtain

$$x_1 = x_2.$$

Thus we have proved that if $a,b \in A$ then $H(a,b,e) \in [X \xrightarrow{\circ} X]_r$. Hence it follows immediately that $(H,H,H) \in \Gamma$, which completes the proof.
In virtue of theorem 19 it follows immediately

Theorem 20. The triplet of functions $(F,F,F) \in [X \times A \times A \times G \longrightarrow X]^3$ satisfying (56) belongs to the set Ω, iff

(85) $\quad F(x,a,a,e) = x$

for every $a \in A$ and $x \in X$.
Moreover, from theorem 19 it follows that in theorem 18 the assumption $(F,F,F) \in \Omega$ can be replaced by condition (85).
Now we shall give a solution of equation (56) which does not belong to the set Ω.

Example 9. Let X be the set of real numbers R, A the set of positive real numbers, $(G,\cdot)$ an one-element group.
Let us define the function H as follows:

$$H(a,b) = \begin{cases} x \longrightarrow x & \text{for } x \in R, \quad a = b = 1, \\ x \longrightarrow \log_b(x) & \text{for } x > 0, \quad a = 1,\ b \neq 1, \\ x \longrightarrow a^x & \text{for } x \in R, \quad a \neq 1,\ b = 1, \\ x \longrightarrow a^{\log_b(x)} & \text{for } x > 0, \quad a \neq 1,\ b \neq 1. \end{cases}$$

It is easy to verify that the function H defined above is the homomorphism of the Brandt groupoid $(A \times A, \ast)$ into $([R \xrightarrow{e} R], o)$. Therefore, in virtue of corollary 5, the function F defined as follows

$$F(x,a,b) := [H(a,b)](x) \quad \text{for} \quad (a,b) \in D_H, \quad x \in D_{H(a,b)}$$

is the solution of the equation

$$F(F(x,b,c),a,b) = F(x,a,c).$$

$(F,F,F) \notin \Omega$ because condition 4^o definition 17 does not hold.
From theorems 15, 17 and corollary 6 it follows that if $(A \times A \times G, \ast)$ is the product Brandt groupoid, then every solution $(F_1,F_2,F_3) \in \Omega$ of equation (56) (consequently every solution $(F,F,F) \in \Omega$ of equation 80) can be extended to the solution $(\bar{F}_1,\bar{F}_2,\bar{F}_3)$ of equation (56) such that

$$D_{\bar{F}_i} = X \times A \times A \times G \quad \text{for} \quad i = 1,2,3.$$

J. Tabor in paper [3] solved equation (80) in the case when $(A \times A \times G, \ast)$ is the product Brandt groupoid and $D_F = X \times A \times A \times G$. In a particular case J. Tabor received solutions of equation (80) satisfying the identity condition

$$F(x,a,a,e) = x \quad \text{for} \quad x \in X,\ a \in A.$$

It is easy to see that we obtained these solutions also in this paper as particular cases of solutions of equation (56) in the class Ω. To use the form of solution given by J. Tabor in paper [3] it is sufficient to apply in theorem 20 the following notation

$$[f(a)](x) := f_a(x),$$

$$[g(\alpha)](x) := g(x,\alpha) .$$

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