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On the homogeneous functional inequality

1. In this paper we shall deal with the homogeneous functional inequality

$$\psi[f(x)] \leqslant g(x)\psi(x) \tag{1}$$

related to the homogeneous functional equation

$$\varphi[f(x)] = g(x)\varphi(x), \tag{2}$$

where f and g are given functions and ψ and φ are unknown functions. The inequality (1) has been studied in [1] and [2]. The results concerning continuous solutions of inequality (1) in the case, where continuous solutions of equation (2) depend on an arbitrary function, one can find in the paper [1]. But the results given in that paper are incomplete. In this paper we are going to give some additional theorems concerning the inequality (1) in the considered case. These theorems, together with those given in [1], will establish more complete theory of continuous solutions of inequality (1).

2. In the sequel we shall assume the following hypothesis H:

(i) The function f is defined, strictly increasing and continuous in an interval $I = [0, \alpha)$. Moreover, $0 < f(x) < x$ for $x \in I_0 = (0, \alpha)$.

(ii) The function g is defined and continuous in the interval I and $g(x) > 0$ for $x \in I_0$.

(iii) There exists a point $x_0 \in I_0$ such that the sequence

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] \quad \text{for} \quad x \in I, \tag{3}$$

where f^i is i-th iterate of function f, i.e., $f^0(x) = x$, $f^{n+1}(x) = f[f^n(x)]$ for $x \in I$, $n = 0, 1, \ldots,$ converges to zero uniformly in the interval $[f(x_0), x_0]$.

If the hypothesis H is fulfilled, then equation (2) has continuous solutions in I depending on an arbitrary function and every continuous solution φ of equation (2) in I satisfies the condition

$$\varphi(0) = 0 \tag{4}$$

(see [4], p. 48). Since it is difficult to obtain any results for continuous solutions of inequality (1) taking at zero any value different than that of solution of equation (2), we are going to consider in this paper only continuous solutions of inequality (1) in I satisfying the initial condition

$$\psi(0) = 0\,. \tag{5}$$

However, there may exist continuous solutions of (1) which do not satisfy condition (5) (see [1]).

In the sequel we shall need some results which have been proved in [1]. We are going to quote these results here as the following

LEMMA 1. *Let the hypothesis H be fulfilled and let ψ be a continuous solution of inequality* (1) *in I satisfying condition* (5) *and the condition*

$$\psi(x) \geqslant 0 \quad \textit{for} \quad x \in I\,. \tag{6}$$

Then there exists the limit

$$\lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} \quad \text{for} \quad x \in I_0\,, \tag{7}$$

where G_n is defined by formula (3), *and the function*

$$\varphi_0(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{\psi[f^n(x)]}{G_n(x)} & \textit{for} \quad x \in I_0 \\ 0 \quad \textit{for} \quad x = 0 \end{cases} \tag{8}$$

is a solution of equation (2) *in I, continuous from above in I and continuous at zero.*

As a matter of fact, this lemma has been proved in [1] (lemma 3.7) under the additional assumption that

$$g(0) > 0\,. \tag{9}$$

But if we drop this additional assumption, the proof of the lemma does not differ in any detail from that of lemma 3.7 in [1].

The function φ_0 need not be continuous in the whole interval I, even under the assumption of lemma 1, assumption (9) included. An example of such a discontinuous φ_0 has been given in [3].

The relation between ψ and φ_0 is given by the following

LEMMA 2. *Let the hypothesis H be fulfilled and let ψ be a continuous solution of inequality* (1) *in I satisfying condition* (6). *If φ_0 is given by formula* (8), *then*

$$\psi(x) \geqslant \varphi_0(x) \quad \textit{for} \quad x \in I\,, \tag{10}$$

and φ_0 is the greatest solution of (2) *satisfying* (10) *and there exists exactly one function η, $\{f\}$-decreasing in I* [1]), *such that*

$$\psi(x) = \eta(x)\varphi_0(x) \quad \textit{for} \quad x \in I\,.$$

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[1]) A function η is called $\{f\}$ — decreasing in I if $\eta[f(x)] \leqslant \eta(x)$ for $x \in I$.

This lemma has also been proved in [1] (see lemma 3.7) under the additional assumption (9). It is easy to see, similarly like in case of lemma 1, that this assumption is unessential.

Let us denote by Ψ the family of continuous solutions of inequality (1) in I satisfying conditions (5) and (6). Further, let us denote by Ψ_0 the family of functions $\psi \in \Psi$, satisfying the condition

$$\psi(x) > 0 \quad \text{for} \quad x \in I_0 \tag{11}$$

and such that there exists the limit

$$b = \lim_{x \to 0} \frac{\psi[f(x)]}{\psi(x)} . \tag{12}$$

It is easy to see that, if $\psi \in \Psi_0$, then

$$0 \leqslant b \leqslant g(0) . \tag{13}$$

The properties of limit (12) give us some informations about the function φ_0 defined by (8).

THEOREM 1. *Let the hypothesis H and the condition* (9) *be fulfilled.*

1^0 *If there exists a* $u \in I_0$ *such that*

$$\varphi_0(u) \neq 0 , \tag{14}$$

where φ_0 *is given by formula* (8), *then*

$$\limsup_{x \to 0} \frac{\psi[f(x)]}{\psi(x)} = g(0) . \tag{15}$$

Proof. Let us assume that there exists a $u \in I_0$ such that (14) holds. Since, in view of lemma 1, φ_0 satisfies equation (2) in I, then inequality (14) implies that

$$\varphi_0[f^n(u)] \neq 0 \quad \text{for} \quad n = 0, 1, \ldots \tag{16}$$

Hence

$$\lim_{n \to \infty} \frac{\psi[f^n(u)]}{\varphi_0[f^n(u)]} = \lim_{n \to \infty} \frac{\psi[f^n(u)]}{G_n(u)\varphi_0(u)} = 1 ,$$

by virtue of (2) and (8). The last equality, (11), (16) and (2) imply that

$$\lim_{n \to \infty} \frac{\psi[f^n(u)]}{\psi[f^{n-1}(u)]} = \lim_{n \to \infty} \frac{\psi[f^n(u)]}{\varphi_0[f^n(u)]} \lim_{n \to \infty} \frac{\varphi_0[f^{n-1}(u)]}{\psi[f^{n-1}(u)]} \lim_{n \to \infty} g[f^{n-1}(u)] = g(0) ,$$

thus (15) holds, by virtue of (13).

3. Let us denote by Φ the family of continuous solutions of equation (2) in I satisfying the condition $\varphi(x) > 0$ for $x \in I_0$. Further, let us denote by Ψ_1 the family of continuous solutions ψ of inequality (1) in I satisfying conditions (5) and (6) and the condition: there exists such a function $\varphi \in \Phi$ that the limit

$$a = \lim_{x \to 0} \frac{\psi(x)}{\varphi(x)} \tag{17}$$

exists.

Let us notice that, in general, $\Psi \neq \Psi_1$ (see the example in [3]). The relation between the families Φ and Ψ_1 is given by the following

LEMMA 3. *Let the hypothesis H be fulfilled. If $\psi \in \Psi$, then $\psi \in \Psi_1$ if and only if $\varphi_0 \in \Phi$, where φ_0 is given by formula* (8).

Similarly, the relation between the families Φ and Ψ_0 is given by the following

LEMMA 4. *Let the hypothesis H and condition* (9) *be fulfilled and let $\psi \in \Psi_0$. If $\varphi_0 \in \Phi$, then limit* (12) *exists and*

$$b = g(0)\,, \tag{18}$$

where φ_0 is given by formula (8).

These lemmas, like the lemmas 1 and 2, have been proved in [1] (see corollaries 3.6—3.9), although for lemma 3 we assumed there condition (9), which, however, turns out to be unessential.

Let us notice that if, conversely, limit (12) exists and (18) holds, φ_0 need not belong to Φ. It is easy to observe that for any solution φ of equation (2) in I satisfying (4) and positive in I_0 limit (12) exists and (18) holds, even in the case of discontinuous φ. Since every solution of (2) is a solution of (1), one can easily observe that, in such a case, φ_0 equals φ. Thus condition (18) does not imply the continuity of φ_0.

4. Now we are going to give some results concerning the case where the limit a, defined by formula (17), is equal to zero.

THEOREM 2. *Let the hypothesis H be fulfilled and let $\psi \in \Psi_1$. If the limit a, defined by formula* (17), *is equal to zero, then there exists one and only one continuous function η, $\{f\}$-decreasing in I, such that*

$$\psi(x) = \eta(x)\varphi(x) \quad \textit{for} \quad x \in I \tag{19}$$

and

$$\eta(0) = 0\,, \tag{20}$$

where $\varphi \in \Phi$ is a function occurring in condition (17).

Proof. Let us assume that limit (17) exists and $a = 0$. Put

$$\eta(x) = \begin{cases} \dfrac{\psi(x)}{\varphi(x)} & \text{for} \quad x \in I_0 \\ 0 & \text{for} \quad x = 0\,. \end{cases} \tag{21}$$

Inequality (1) and equation (2) imply that $\eta[f(x)]\leqslant\eta(x)$ for $x\in I$, therefore η is an $\{f\}$-decreasing function in I. The continuity of η in I follows from the continuity of ψ and φ in I and from conditions (21) and (17). Conditions (19) and (20) are simple consequences of (21).

COROLLARY. *Let the hypothesis H be fulfilled, let $\psi\in\Psi_1$ and let φ_0 be given by formula* (8). *If the limit a, defined by formula* (17), *exists, then $a=0$ if and only if the inequality*

$$b<g(0) \tag{22}$$

holds.

Proof. Let $a=0$. Then, in view of theorem 2, we have

$$\varphi_0(x)=\lim_{n\to\infty}\frac{\psi[f^n(x)]}{G_n(x)}=\lim_{n\to\infty}\frac{\eta[f^n(x)]\varphi[f^n(x)]}{G_n(x)}=\lim_{n\to\infty}\eta[f^n(x)]\varphi(x)=0,$$

by virtue of (8), (19), (2) and (20).

Conversely, let inequality (22) hold and let us assume that $a\neq 0$. Thus $\varphi_0\in\Phi$, in view of lemma 3, what contradicts (22).

Remark. The results presented in this paper can easily be obtained for the inequality

$$\psi[f(x)]\geqslant g(x)\psi(x)\quad\text{for}\quad x\in I,$$

by the suitable changes very simple to make.

References

[1] D. Brydak: *On functional inequalities in a single variable, Dissertationes Math.* (to appear).

[2] D. Brydak and B. Choczewski: *On non-homogeneous linear functional inequality.* Zeszyty Nauk. U.J., Prace Mat. 18 (1976), pp. 89–92.

[3] D. Brydak and B. Choczewski: *Classification of continuous solutions of a functional inequality.* Zeszyty Nauk. U.J., Prace Mat. 17 (1975), pp. 33–40.

[4] M. Kuczma: *Functional equations in a single variable.* Monografie Mat. 46, Warszawa 1968.