BOGDAN CHOCZEWSKI, ERWIN TURDZA, ROMAN WĘGRZYK

On the stability of a linear functional equation

1. In this paper we shall deal with the problem of stability of the linear functional equation

$$\varphi(f(x)) = g(x)\varphi(x)+F(x), \tag{1}$$

where f, g, F are given functions defined on the interval $I = [\xi, b)$, where $f(\xi) = \xi$, and φ is an unknown function.

The problem of stability was first formulated by D. H. Hyers [2] for the Cauchy's equation. The notions of stability of the equation (1) were proposed by D. Brydak [1]. We start with rewriting his definitions.

Definition 1. *The equation* (1) *is called stable in the class* $C[I]$ *consisting of the all functions continuous in the interval I, if there exists a positive constant K such that for each number* $\varepsilon>0$ *and each solution* $\psi \in C[I]$ *of the inequality*

$$|\psi(f(x))-g(x)\psi(x)-F(x)|\leqslant\varepsilon \quad \textit{for} \quad x\in I \tag{2}$$

there exists a solution $\varphi\in C[I]$ *of the equation* (1) *fulfilling the inequality*

$$|\psi(x)-\varphi(x)|\leqslant K\varepsilon \quad \textit{for} \quad x\in I. \tag{3}$$

Let f^n denote the n-th iterate of the function f and put

$$G_n(x) = \prod_{i=0}^{n-1} g(f^i(x)) \quad \text{for} \quad x\in I,\ n = 1, 2, \ldots \tag{4}$$

Definition 2. *The equation* (1) *is called iteratively stable in the class* $C[I]$, *if there exists a positive constant K, such that for each number* $\varepsilon>0$ *and each solution* $\psi\in C[I]$ *of the system of inequalities*

$$\left|\psi(f^n(x))-G_n(x)\psi(x)-G_n(x)\sum_{i=0}^{n-1}\frac{F(f^i(x))}{G_{i+1}(x)}\right|\leqslant\varepsilon \tag{5}$$

for

$$x\in I, \quad n = 1, 2, \ldots$$

there exists a solution $\varphi\in C[I]$ *of the equation* (1), *such that inequality* (3) *holds.*

2. Putting in definitions 1 or 2 $F \equiv 0$ we obtain the definitions of stability or iterative stability of the linear homogeneous equation

$$\varphi(f(x)) = g(x)\varphi(x). \tag{6}$$

We have the following

THEOREM 1. *If the equation* (6) *is iteratively stable* (*resp. stable*) *in the class* $C[I]$ *and there exists a solution* $\bar{\varphi} \in C[I]$ *of the equation* (1), *then the equation* (1) *is also iteratively stable* (*resp. stable*) *in the class* $C[I]$.

Proof. We present the proof for the case of the iterative stability.

If $\bar{\varphi} \in C[I]$ fulfils (1), then the general continuous solution of the equation (1) has the form

$$\varphi(x) = \bar{\varphi}(x) + \phi(x) \quad \text{for} \quad x \in I,$$

where ϕ is the general continuous solution of the equation (6). As it is easily verified, the difference of any two solutions $\psi_1, \psi_2 \in C[I]$ of the system of inequalities (5) is a continuous solution of the system of inequalities

$$|\hat{\psi}(f^n(x)) - G_n(x)\hat{\psi}(x)| \leqslant 2\varepsilon \quad \text{for } x \in I,\ n = 1, 2, \ldots \tag{7}$$

Hence, any solution $\psi \in C[I]$ of the system of inequalities (5) is of the form

$$\psi(x) = \bar{\varphi}(x) + \hat{\psi}(x) \quad \text{for} \quad x \in I, \tag{8}$$

where $\hat{\psi} \in C[I]$ is a solution of the system of inequalities (7). This is so because $\bar{\varphi}$ (a solution of (1)) fulfils also the system of inequalities (5).

Since equation (6) is iteratively stable in the class $C[I]$, then there is a $K>0$ and (for given $\varepsilon>0$) a solution ϕ of the equation (6), chosen to the $\hat{\psi}$, such that

$$|\hat{\psi}(x) - \phi(x)| \leqslant 2K\varepsilon \quad \text{for} \quad x \in I.$$

Whence and by (8) we have

$$|\psi(x) - \phi(x) - \bar{\varphi}(x)| \leqslant 2K\varepsilon \quad \text{for} \quad x \in I.$$

This gives (3) with $\varphi = \bar{\varphi} + \phi$, and the proof of iterative stability of (1) is completed.

In the case where (6) is stable, the proof is similar.

3. According to the result of the preceding section we pass to deal with stability of equation (6). In this section the following hypotheses will be assumed

(H_1) $f\colon I \to I$ (where $I = [\xi, b)$) is a continuous and strictly increasing function in I,

and $$f(x) < x \quad \text{for} \quad x \in I^* := (\xi, b), \quad f(\xi) = \xi.$$

(H_2) The function $g \in C[I]$ assumes real or complex values, and

$$g(x) \neq 0 \quad \text{for} \quad x \in I.$$

The quantity of continuous solutions of the equation (1) is dependent on the behaviour of the sequence G_n given by (4). We shall deal with the case

(*) There exists an open, non-empty interval $J \subset I$, such that $G_n \to 0$ almost uniformly on J.

Let U will be the greatest open set, such that $G_n \to 0$ almost uniformly on U. As it follows from [4] (lemma 2) such a set U actually exists.

In the case (*) the continuous solution of equation (6) depends on an arbitrary function. A corresponding theorem has been proved in [4]. We quote it here as a lemma.

LEMMA 1. *Assume* (H_1) *and* (H_2). *Let* $x_0 \in (\xi, b)$. *If the functions* G_n *are commonly bounded on* $[f(x_0), x_0] \cap U$, *then for any continuous function* φ_0 *defined on* $[f(x_0), x_0]$, *and fulfilling the conditions*

$$\varphi_0(f(x_0)) = g(x_0)\varphi_0(x_0) \tag{9}$$

and

$$\varphi_0(x) = 0 \quad \text{for} \quad x \in [f(x_0), x_0] \setminus U\,, \tag{10}$$

there exists exactly one solution $\varphi \in C[I]$ *of the equation* (6), *such that*

$$\varphi(x) = \varphi_0(x) \quad \text{for} \quad x \in [f(x_0), x_0]\,. \tag{11}$$

The solution φ *has the value* 0 *at* ξ.

Stability of equation (6) has been dealt with in [1] and [3] in the case where for the interval J from (*) the inclusion $J \supset [f(x_0), x_0]$ holds true for an $x_0 \in (\xi, b)$. The aim of this paper is to examine the iterative stability of equation (6) in some more general case, where this inclusion need not occur.

4. To obtain the main result of the paper we need some lemmas.

LEMMA 2. *If the hypothesis* (H_1) *is fulfilled and the sequence* $G_n(\bar{x})$ *converges to zero at a point* $\bar{x} \in (\xi, b)$, *then for any solution* $\psi \in C[I]$ *of the system of inequalities* (5) *with* $F \equiv 0$, *we have* $|\psi(\xi)| \leqslant \varepsilon$.

Proof. Hypothesis (H_1) implies that $f^n(x) \to \xi$ as $n \to \infty$ for all $x \in I$. Since ψ is continuous in I, then setting $\bar{x}$ instead of x in (5) and passing with n to infinity, we get $|\psi(\xi) - 0| \leqslant \varepsilon$, which was to be proved.

LEMMA 3. *Let hypothesis* (H_1) *and condition* (*) *be fulfilled and let* $U \subset J$ *be the maximal open set, such that* $G_n \to 0$ *almost uniformly on* U. *If there exists a number* $A > 0$, *such that*

$$A \leqslant |G_n(x)| \quad \text{for} \quad n = 1, 2, \ldots,\ x \in I^* \setminus U\,, \tag{12}$$

then every solution $\psi \in C[I]$ *of the system of inequalities* (5) *with* $F \equiv 0$ *fulfils the condition*

$$|\psi(x)| \leqslant \frac{2}{A}\varepsilon \quad \text{for} \quad x \in I^* \setminus U\,.$$

Proof. Let $x \in I^* \setminus U$. If $\psi \in C[I]$ fulfils (5) with $F \equiv 0$, then

$$\big||\psi(f^n(x))| - |G_n(x)||\psi(x)|\big| \leqslant |\psi(f^n(x)) - G_n(x)\psi(x)| \leqslant \varepsilon,$$

whence

$$|G_n(x)|\,|\psi(x)| \leqslant \varepsilon + |\psi(f^n(x))|,$$

and

$$|\psi(x)| \leqslant \frac{\varepsilon}{|G_n(x)|} + \frac{|\psi(f^n(x))|}{|G_n(x)|} \leqslant \frac{1}{A}\big(\varepsilon + |\psi(f^n(x))|\big),$$

by (12). The last inequality yields as $n \to \infty$ (cf. lemma 2) the thesis of the lemma.

In the sequel I_0 will denote the interval $[f(x_0), x_0]$ for some $x_0 \in (\xi, b)$.

THEOREM 2. *Let hypotheses* (H_1) *and* (H_2) *and condition* $(*)$ *be fulfilled. If there exist positive numbers* A, B *and an interval* $I_0 \subset J$, *such that*

$$(13) \qquad A \leqslant |G_n(x)| \quad \text{for} \quad n = 1, 2, \ldots, \; x \in I^* \setminus U,$$

$$(14) \qquad |G_n(x)| \leqslant B \quad \text{for} \quad n = 1, 2, \ldots, \; x \in I_0 \cap U,$$

where U *is the greatest open set on which* $G_n \to 0$ *almost uniformly, and if there exists a point* $x_1 \geqslant x_0$ *and a constant* $C > 0$, *such that*

$$(15) \qquad \prod_{i=0}^{n} |g(f^{-i}(x))| \geqslant C \quad \text{for} \quad n = 1, 2, \ldots, \; b > x \geqslant x_1,$$

then equation (6) *is iteratively stable in the class* $C[I]$.

Proof. Let $\psi \in C[I]$ be a solution of the inequality (5) (with $F \equiv 0$). Put:

$$K' = \max\left\{1, \frac{2}{A}\right\}.$$

It follows from the lemma 3 that there exists a continuous function φ_0 defined on the interval I_0 and fulfilling the conditions of the lemma 1 and such that

$$\varphi_0(x_0) = \psi(x_0),$$

$$\varphi_0(f(x_0)) = g(x_0)\varphi_0(x_0),$$

$$\varphi_0(x) = 0 \quad \text{for} \quad x \in I_0 \setminus U,$$

$$(16) \qquad |\varphi_0(x) - \psi(x)| \leqslant K'\varepsilon \quad \text{for} \quad x \in I_0.$$

Assumption (13) and the lemma 1 then imply the existence of exactly one solution $\varphi \in C[I]$ of the equation (1) fulfilling the condition

$$(17) \qquad \varphi(x) = \varphi_0(x) \quad \text{for} \quad x \in I_0.$$

We are going to show that φ fulfils inequality (3) for some $K > 0$ (independent on ψ and ε).

For, if $\xi < x \leqslant f(x_0)$, then there exist $t \in I_0$, and $k \in \mathrm{N}$ such that $x = f^k(t)$. If the $t \in I_0 \cap U$, then from (13) we get $|\psi(x) - \varphi(x)| = |\psi(f^k(t)) - \varphi(f^k(t))| = |\psi(f^k(t)) - G_k(t)\varphi(t)| \leqslant |\psi(f^k(t)) - G_k(t)\psi(t)| + |G_k(t)|\,|\psi(t) - \varphi(t)| \leqslant \varepsilon + BK'\varepsilon$ and

$$(18) \qquad |\psi(x) - \varphi(x)| \leqslant (1 + BK')\varepsilon \quad \text{for} \quad x \in (\xi, f(x_0)] .$$

In the case $t \notin I_0 \cap U$ we see that $x = f^k(t) \in I \setminus U$, and lemma 3 applies yielding

$$|\psi(x) - \varphi(x)| \leqslant \frac{2}{A}\varepsilon .$$

If $x \geqslant x_0$, then there exists an $n = n(x) \in N$ such that $f^n(x) \in I_0$. Hence, and from (17) and (16) we have

$$|\psi(f^n(x)) - \varphi(f^n(x))| = |\psi(f^n(x)) - \varphi_0(f^n(x))| \leqslant K'\varepsilon$$

and therefore

$$\big||G_n(x)|\,|\psi(x) - \varphi(x)| - |\psi(f^n(x)) - G_n(x)\psi(x)|\big| \leqslant$$
$$\leqslant |\psi(f^n(x)) - G_n(x)\psi(x) + G_n(x)\psi(x) - G_n(x)\varphi(x)| \leqslant K'\varepsilon .$$

Consequently

$$(19) \qquad |\psi(x) - \varphi(x)| \leqslant \frac{K'\varepsilon}{|G_n(x)|} + \frac{|\psi(f^n(x)) - G_n(x)\psi(x)|}{|G_n(x)|} .$$

To estimate the $|G_n(x)|$ we shall use (15). First we determine the positive integers: $N = N(x)$ so that $f^N(x) \in [f(x_1), x_1)$ and s so that $f^{s+1}(x_1) \in [f(x_0), x_0)$. Then

$$f(x_0) \leqslant f^{s+1}(x_1) < x_0 \leqslant f^s(x_1)$$

and

$$f^{s+1}(x_1) \leqslant f^{N+s}(x) \leqslant f^s(x_1)$$

i.e. either $f^{N+s+1}(x)$ or $f^{N+s}(x)$ belongs to $[f(x_0), x_0)$. Since so does also $f^n(x)$, then by hypothesis (H_1) one of the two possibilities has to occur

(i) $n = N+s+1$,

(ii) $n = N+s$,

where the n and N depend on x but the s does not. Moreover, $f^i(x) \geqslant x_1$ for $i = 0, 1, \ldots, N-1$. Putting $u = f^{N-1}(x)$,
we get

$$(20) \qquad f^{-j}(u) \geqslant x_1 \quad \text{for} \quad j = 0, 1, \ldots, N-1 .$$

From the above considerations the following inequalities follow directly (cf. (4), (20), (15) and (H_2)) in the case (i):

$$|G_n(x)| = \Big|\prod_{i=0}^{N-1} g(f^i(x))\Big|\,\Big|\prod_{k=N}^{N+s} g(f^k(x))\Big| = \Big|\prod_{j=0}^{N-1} g(f^{-j}(u))\Big|\,\Big|\prod_{i=0}^{s} g(f^i(f^N(x)))\Big| \geqslant$$
$$\geqslant C \prod_{i=0}^{s} \inf_{t \in [f(x_1), x_1]} |g(f^i(t))| = K_1 > 0 .$$

Similarly we can estimate the $|G_n(x)|$ in the case (ii) by a positive constant, K_2 say, which yields the inequality

$$|G_n(x)| \geqslant K_0 \quad \text{for} \quad x \in [x_0, b) \text{ and } n = 1, 2, \ldots$$

with $K_0 = \min\{K_1, K_2\} > 0$.
Returning to (19) we hence get by (5)

$$(21) \qquad |\psi(x) - \varphi(x)| \leqslant \frac{K'\varepsilon}{K_0} + \frac{\varepsilon}{K_0}, \quad \text{for} \quad x \in [x_0, b).$$

Recall (18), (16) with (17), (21) and the inequality $|\psi(\xi) - \varphi(\xi)| = |\psi(\xi)| \leqslant \varepsilon$ resulting from the lemmas 1 and 2. We see that inequality (3) will hold for any $x \in I$, if we put

$$K = \max\left\{1 + BK', K', \frac{K'+1}{K_0}\right\}.$$

This proves the iterative stability of equation (6).

5. We terminate the paper with some remarks.

Remark 1. Condition (15) of the theorem 2 is obviously fulfilled if there is an x_1 to fulfil

$$(22) \qquad |g(x)| \geqslant 1 \quad \text{for} \quad x \in [x_1, b).$$

However, condition (15) is more general than (22) as it can be seen on the following example. Put $I = [0, \infty)$,

$$f(x) = \frac{x}{2} \quad \text{for} \quad x \in I,$$

$$g(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{for} \quad x \in [0, 1), \\ 1 + \frac{1}{x}\sin \pi x & \text{for} \quad x \in [1, \infty). \end{cases}$$

Inequality (22) is not fulfilled in any interval $[x_1, \infty)$. But theorem 2 works since hypotheses (H_1) and (H_2) are fulfilled, and for $x \geqslant 1$ we have

$$\left|\prod_{i=0}^{n} g(f^{-i}(x))\right| = \left|\prod_{i=0}^{n}\left(\frac{1}{2^i x}\sin \pi 2^i x + 1\right)\right|.$$

The last product is convergent for $x \geqslant 1$, because the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n x} \sin \pi 2^n x$$

is absolutely convergent for $x \geqslant 1$.

Moreover, for $x \geqslant 2$ we have

$$\prod_{i=0}^{n} \left(\frac{1}{2^n x} \sin \pi 2^i x + 1 \right) \geqslant \prod_{i=0}^{n} \left(\frac{-1}{2^{i+1}} + 1 \right)$$

and for the same reasons as above the last product is convergent to a positive number, so that condition (15) is fulfilled. Remaining assumptions of the theorem 2 are fulfilled in obvious way.

Remark 2. As it has been showed in [3], under some conditions, both definitions 1 and 2 are equivalent. We do not know if the same is true under hypotheses of the theorem 2.

Remark 3. The theorem 1 allows us to obtain for equation (1) an analogue of the theorem 2. To this end it is enough to add to the hypotheses of the theorem 2 the following one

(H_3) $$F \in C[I].$$

THEOREM 3. *Let the hypotheses of the theorem* 2 *be fulfilled. If the hypothesis* (H_3) *is fulfilled and there exists a continuous solution of the equation* (1), *then the equation* (1) *is iteratively stable in the class* $C[I]$.

References

[1] D. Brydak: *On the stability of the functional equation* $\varphi(f(x)) = g(x)\varphi(x) + F(x)$. Proc. Amer. Math. Soc. 26 (1970), 455–460.

[2] D. H. Hyers: *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. USA, 27 (1941), 222–224.

[3] E. Turdza: *Pewne uwagi o stabilności liniowego równania funkcyjnego* $\varphi(f(x)) = g(x)\varphi(x) + F(x)$. Rocznik Nauk.-Dydak. WSP w Krakowie, Prace Mat. VI, 41 (1970), 155–164.

[4] R. Węgrzyk: *Continuous solution of a linear homogeneous functional equation.* Ann. Polon. Math. 35(1977), p. 15–20.