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On a certain mixed boundary problem for iterated Helmholtz equation in the half-space

In the paper we shall give the solution of the equation

(1) $(\Delta - C^2)^2 u(X) = \Delta^2 u(X) - 2C^2 \Delta u(X) + C^4 u(X) = 0, \quad X = (x_1, x_2, x_3)$

C being a positive constant in the half-space

$$E_3^+ = \{(x_1, x_2, x_3)\colon |x_i| < \infty, (i = 1, 2), x_3 > 0\}$$

satysfying the mixed boundary conditions

(2) $$hu(x_1, x_2, 0) + D_{x_3} u(x_1, x_2, 0) = f_1(x_1, x_2)$$

and

(3) $$h\Delta u(x_1, x_2, 0) + D_{x_3} \Delta u(x_1, x_2, 0) = f_2(x_1, x_2),$$

where f_i $(i = 1, 2)$ are given functions defined in 2-dimensional Euclidean space E_2, h is a negative constant.
We briefly call the problem (1), (2), (3) (M)-problem.

1. Green function for the (M)-problem.

Let $X = X_1 = (x_1, x_2, x_3)$ denote an arbitrary point belonging to E_3^+ and let $X_2 = (x_1, x_2, -x_3)$ and $X_3 = (x_1, x_2, -x_3 - v)$, where $v \geqslant 0$. Next let $Y = (y_1, y_2, y_3)$ be an arbitrary point in 3-dimensional Euclidean space E_3 and $Y \neq X$ and let $r_j = |X_j Y|$, $j = 1, 2, 3$.
Let us consider the following integrals

$$I(X, Y) = \int_0^\infty e^{hv} e^{-Cr_3} dv, \quad I_{pqr}(X, Y) = \int_0^\infty e^{hv} D^{pqr}_{x_1 x_2 x_3}(e^{-Cr_3}) dv,$$

where $p, q, r = 0, 1, 2, 3, 4$ and $0 < p + q + r \leqslant 4$.
Let

$$W = \{(X, Y)\colon |x_i| \leqslant a (i = 1, 2), 0 < b_1 \leqslant x_3 \leqslant b_2, |y_i| \leqslant a, 0 < b_1 \leqslant y_3 \leqslant b_2\}$$

a, b_1, b_2 being arbitrary positive numbers.

Now we shall prove

LEMMA 1. *The integrals* $I(X, Y)$, $I_{pqr}(X, Y)$ *are uniformly convergent in the set* W.

Proof. The integral $\int_0^\infty e^{hv}dv$ is the majorant for the integral $I(X, Y)$ and therefore $I(X, Y)$ is uniformly convergent in the set W. We have

$$D^{pqr}_{x_1x_2x_3}(e^{-Cr_3}) = e^{-Cr_3}P(y_1-x_1, y_2-x_2, y_3+x_3+v, r_3),$$

where P is polynomial with the following constituents

$$(r_3)^{-\gamma}(y_1-x_1)^{\delta_1}(y_2-x_2)^{\delta_2}(y_3+x_3+v)^{\delta_3},$$

δ_j, γ $(j = 1, 2, 3)$ being positive integers.
In the sequel we shall use the inequality

(4) $$e^{-\varphi} \leqslant (\varphi)^{-\alpha} \quad \text{for} \quad \alpha \in (0, e), \varphi > 0.$$

By (4) we get

$$|I_{pqr}(X, Y)| \leqslant K(C)^{-\alpha}\int_0^\infty e^{hv}(b_1+v)^{-\alpha-\beta}dv \leqslant K(C)^{-\alpha}(b_1)^{\alpha-\beta}\int_0^\infty e^{hv}dv,$$

where K denotes the number of the terms of polynomial P, $\beta = \gamma+\delta_1+\delta_2+\delta_3$. Since the integral at the right-hand side in the above inequality is convegergent thus the integrals $I_{pqr}(X, Y)$ are uniformly convergent in every set W.

By lemma 1 we get

LEMMA 2. *The integrals* $I_{pqr}(X, Y)$ *and* $I(X, Y)$ *exist in* W *and the function* $I(X, Y)$ *is of class* $C^4(W)$ *and* $I_{pqr}(X, Y) = D^{pqr}_{x_1x_2x_3}I(X, Y)$.

The fundamental solution of the equation (1) *in* E_3 *is the function*

(5) $$V(r_j) = e^{-Cr_j} \quad (j = 1, 2, 3).$$

Indeed

(6) $$\Delta_Y V(r_j) = V(r_j)[C^2-2C(r_j)^{-1}] \; (j = 1, 2, 3)$$

and

$$\Delta_Y^2 V(r_j) = V(r_j)(C^4-4C^3r_j^{-1}) \; (j = 1, 2, 3).$$

Hence

(7) $$(\Delta_Y - C^2)^2 V(r_j) = 0 \; (j = 1, 2, 3).$$

By symetry of the points X_1, X_2 with respect to the plane E_2 we get

(8) $$r_1 = r_2 = [(y_1-x_1)^2+(y_2-x_2)^2+x_3^2]^{\frac{1}{2}} = R \quad \text{for} \quad y_3 = 0.$$

By (5) and (8) we abtain

(9) $$D_{y_3}V(r_1)|_{y_3=0} = -D_{y_3}V(r_2)|_{y_3=0} = Cx_3V(R)R^{-1}.$$

LEMMA 3. *Let the functions* $V(r_j)$ $(j = 1, 2)$ *be defined by the formula* (5). *Then*

$$D_{y_3}[\Delta_Y V(r_1) + \Delta_Y V(r_2)]|_{y_3=0} = 0 .$$

Proof. By (6) we get

$$D_{y_3}[\Delta_Y V(r_1) + \Delta_Y V(r_2)] = C D_{y_3}[V(r_1)(C - 2r_1^{-1}) + V(r_2)(C - 2r_2^{-1})] =$$
$$= C^2 D_{y_3}[V(r_1) + V(r_2)] - 2C\{D_{y_3}[V(r_1)r_1^{-1}] + D_{y_3}[V(r_2)r_2^{-1}] +$$
$$+ V(r_1)r_1^{-2} D_{y_3} r_1 + V(r_2) r_2^{-1} D_{y_3} r_2\} .$$

Since

$$D_{y_3} r_2|_{y_3=0} = -D_{y_3} r_1|_{y_3=0} = x_3 R^{-1}$$

and

$$D_{y_3}[V(r_2)r_2^{-1}]|_{y_3=0} = -D_{y_3}[V(r_1)r_1^{-1}]|_{y_3=0} = C x_3 V(R) R^{-2} + x_3 V(R) R^{-3}$$

by (9) we get the thesis of lemma 3.

Let

(10) $$G(X, Y) = h^{-1}[V(r_1) + V(r_2)] + 2J(X, Y) ,$$

where

$$J(X, Y) = \int_0^\infty e^{hv} V(r_3)\, dv = \int_{x_3+y_3}^\infty e^{h(t-x_3-y_3)} V(r_4)\, dt$$

and

$$r_4^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + t^2 .$$

Using lemmas 2 and 3 we shall prove

THEOREM 1. *The function G given by formula* 10 *is the Green function with the pole at point X, for the problem* (*M*).

Proof. We shall verify that the function G as function of the point Y $(Y \neq X)$ satisfies the equation (1) and homogeneous boundary conditions

(11) $$[hG(X, Y) + D_{y_3} G(X, Y)]|_{y_3=0} = 0$$

and

(12) $$[h\Delta_Y G(X, Y) + D_{y_3} \Delta_Y G(X, Y)]|_{y_3=0} = 0 .$$

Moreover for every fixed X we have

(13) $$\lim \Delta_Y G(X, Y) = 0 \quad \text{when} \quad |0Y| \to \infty .$$

Since

(14) $$D_{y_3} J(X, Y)|_{y_3=0} = -h \int_{x_3}^\infty e^{h(t-x_3)} V(r_4)\, dt - V(R) ,$$

thus by (8), (9) and (14) we get

$$[hG(X, Y) + D_{y_3} G(X, Y)]|_{y_3=0} = 0$$

By (6) and lemma 3 we obtain

$$(15)\quad \begin{cases} h\Delta_Y G(X, Y)|_{y_3=0} = 2CV(R)(C-2R^{-1})+ \\ \qquad\qquad +2h\int\limits_{x_3}^{\infty} CV(r_4)(C-2r_4^{-1})e^{-h(x_3-t)}dt\,, \\ D_{y_3}\Delta_Y G(X, Y)|_{y_3=0} = -2h\int\limits_{x_3}^{\infty} CV(r_4)(C-2r_4^{-1})e^{h(t-x_3)}dt+ \\ \qquad\qquad -2CV(R)(C-2R^{-1})\,. \end{cases}$$

From (15) follows the boundary condition (12).

Now we shall prove the condition (13). By (10), (6) and lemma 1 we have

$$\lim\Delta_Y G(X, Y) = h^{-1}[\lim CV(r_1)(C-2r_1^{-1})+\lim CV(r_2)(C-2r_2^{-1})]+$$

$$-2C\int\limits_0^{\infty} \lim V(r_3)(C-2r_3^{-1})e^{hv}dv = 0 \quad \text{as} \quad |0Y|\to\infty\,.$$

By lemma 2 and formulas (8) and (7) we get

$$(\Delta_Y-C^2)^2G(X, Y) = h^{-1}[(\Delta_Y-C^2)^2V(r_1)+(\Delta_Y-C^2)^2V(r_2)]+$$

$$+2\int\limits_0^{\infty} e^{hv}(\Delta_Y-C^2)^2V(r_3)dv = 0\,.$$

2. The formulae for the solution of the problem (M).

Assuming that the functions f_i $(i = 1, 2)$ are bounded and measurable in E_2 and continuous at the point $X_3^0 = (x_1^0, x_2^0)$ we shall prove that the function

$$(16)\qquad u(X) = u_1(X)+u_2(X)\,,$$

where

$$u_1(X) = A\iint\limits_{E_2} f_1(Y_3)[\Delta_Y G(X, Y)-2C^2G(X, Y)]|_{y_3=0}dY_3$$

and

$$u_2(X) = -A\iint\limits_{E_2} f_2(Y_3)G(X, Y)|_{y_3=0}dY_3$$

and

$$A = h(8\pi C)^{-1},\ Y_3 = (y_1, y_2),\ dY_3 = dy_1dy_2$$

is the solution of the problem (M).

By (15) we have

$$(16a)\quad \begin{cases} u_1(X) = -2AC\iint\limits_{E_2} f_1(Y_3)[h^{-1}V(R)(C-2R^{-1})+ \\ \qquad\qquad + \int\limits_{x_3}^{\infty} e^{h(t-x_3)}V(r_4)(C-2r_4^{-1})dt]dY_3 \\ u_2(X) = 2AC\iint\limits_{E_2} f_2(Y_3)[h^{-1}V(R)+\int\limits_{x_3}^{\infty} e^{h(t-x_3)}V(r_4)dt]dY_3\,. \end{cases}$$

3. The theorem on the change of derivation with integration for the integrals $u_i(X)$.

Let us consider the integrals

$$K_i(X) = \iint_{E_2} f_i(Y_3) V(R) R^{-1} dY_3 \quad (i = 1, 2)$$

and

$$K^i_{pqr}(X) = \iint_{E_2} f_i(Y_3) D^{pqr}_{x_1 x_2 x_3}[V(R) R^{-1}] dY_3 \quad (i = 1, 2),$$

where

$$p, q, r = 0, 1, 2, 3, 4 \text{ and } 0 < p+q+r \leqslant 4 .$$

Let $W_1 = \{(x_1, x_2, x_3)\colon |x_i| \leqslant a (i = 1, 2), 0 < b_1 \leqslant x_3 \leqslant b_2\}$, a, b_1, b_2 being arbitrary positive numbers.

LEMMA 4. *If the functions f_i $(i = 1, 2)$ are bounded and measurable in E_2, then the integrals $K_i(X)$ and $K^i_{pqr}(X)$ $(i = 1, 2)$ are uniformly convergent in the set W_1.*

Proof. We shall give the proof only for the integrals $K_1(X)$ and $K^1_{pqr}(X)$. The proof for the integrals $K_2(X)$ and $K^2_{pqr}(X)$ is similar. Applying the triangle inequality we get

$$\bigvee_{R_0 > 0} \bigwedge_{X \in W_1} \bigwedge_{Y_3} |0Y_3| > R_0 \Rightarrow \tfrac{1}{4}|0Y_3|^2 \leqslant R_0^2 \leqslant 4|0Y_3|^2 .$$

Let

$$K_{R_0} = \{Y_3\colon |0Y_3| \leqslant R_0\}, H^1(X, Y_3) = V(R) R^{-1}, M_1 = \sup_{E_2} |f_1(Y_3)| .$$

We have the inequality

$$|K_1(X)| \leqslant \iint_{E_2} |f_1(Y_3)| H^1(X, Y_3) dY_3 = K_{11}(X) + K_{12}(X) ,$$

where

$$K_{11}(X) = \iint_{K_{R_0}} |f_1(Y_3) H^1(X, Y_3) dY_3 \quad \text{and}$$

$$K_{12}(X) = \iint_{E_2 \setminus K_{R_0}} |f_1(Y_3)| H^1(X, Y_3) dY_3 .$$

Since $H^1(X, Y_3)$ is analytic function of point X for $Y_3 \in K_{R_0}$ thus $K_{11}(X)$ is also the analytic function in W_1. By (4) for $\alpha = 2$ we get

$$K_{12}(X) \leqslant M_1 C^{-2} \iint_{E_2 \setminus K_{R_0}} R^{-3} dY_3 \leqslant 8 M_1 C^{-2} \iint_{E_2 \setminus K_{R_0}} |0Y_3|^{-3} dY_3 .$$

Let ε be an arbitrary positive number. Applying the polar coordinates to the last integral we obtain

$$K_{12}(X) \leqslant 16\pi M_1 C^{-2} \int_{R_0}^{\infty} \varrho^{-2} d\varrho < \varepsilon \quad \text{for}$$

$$R_0 > 16\pi M_1 (\varepsilon C^2)^{-1} \quad \text{and every} \quad X \in W_1 .$$

Let

$$D^{pqr}_{x_1x_2x_3}H^1(X, Y_3) = H^1_{pqr}(X, Y_3).$$

We have

$$H^1_{pqr}(X, Y_3) = V(R)P(x_3, R, y_1-x_1, y_2-x_2)$$

where P is a polynomial being a finite sum of the terms

$$J(X, Y_3) = x_3^{\beta}R^{-\gamma}(y_1-x_1)^{\delta_1}(y_2-x_2)^{\delta_2}$$

where β, δ_1, δ_2 being positive integers and $\gamma \geqslant \beta+\delta_1+\delta_2+1$.
It is enough to prove that the integral $\int\int_{E_2} f_1(Y_3)J(X, Y_3)dY_3$ is uniformly convergent in the set W_1.

Applying in the last integral formula (4) and the change of variables

(17) $$y_1-x_1 = \varrho\cos\varphi, y_2-x_2 = \varrho\sin\varphi (0\leqslant\varrho<\infty, 0\leqslant\varphi\leqslant 2\pi)$$

we get

$$\Big|\int\int_{E_2} f_1(Y_3)J(X, Y_3)dY_3\Big| \leqslant 2\pi M_1 C^{-\alpha} b_2^{\beta}\int_0^{\infty}((b_1^2+\varrho^2)^{-\frac{1}{2}(\alpha+\gamma)}\varrho\, d\varrho$$

The integral on the right-hand side of the last inequality is convergent for $X\in W_1$ and consequently the integral $K^i_{pqr}(X)$ is uniformly convergent in every set W_1.

From lemma 4 follows

LEMMA 5. *If the functions f_i $(i = 1, 2)$ satisfy the assumptions of the lemma 4, then the integrals $K_i(X)$ and $K^i_{pqr}(X)$ $(i = 1, 2)$ exist in W_1 and the functions $K_i(X)$ are of class C^4 in the domain W_1 and*

$$D^{pqr}_{x_1x_2x_3}K_i(X) = K^i_{pqr}(X).$$

Let

$$L^i(X) = \int\int_{E_2} f_i(Y_3)\Big[\int_0^{\infty} e^{hv}V(\bar r_3)(\bar r_3)^{-1}dv\Big]dY_3 \quad (i = 1, 2)$$

and

$$L^i_{pqr}(X) = \int\int_{E_2} f_i(Y_3)\, D^{pqr}_{x_1x_2x_3}\Big[\int_0^{\infty} e^{hv}V(\bar r_3)(\bar r_3)^{-1}dv\Big]dY_3$$

where

$$\bar r_3 = [(y_1-x_1)^2+(y_2-x_2)^2+(x_3+v)^2]^{\frac{1}{2}}.$$

Now we shall prove the following

LEMMA 6. *If the functions f_i $(i = 1, 2)$ satisfy the assumptions of the Lemma 4, then the integrals $L^i(X)$ and $L^i_{pqr}(X)$ $(i = 1, 2)$ are uniformly convergent in every set W_1.*

Proof. Applying in the integrals $L^i(X)$ the change of variables (17) and formula (4) we get

$$|L^i(X)| \leqslant C_i\int_0^{\infty} e^{hv}\Big[\int\int_{E_2}(\bar r_3)^{-1-\alpha}dY_3\Big]dv \leqslant 2\pi C_i\int_0^{\infty} e^{hv}\Big[\int_0^{\infty}(b_1^2+\varrho^2)^{\frac{1}{2}(-1-\alpha)}\varrho\, d\varrho\Big]dv$$

where

$$C_i = M_i(C)^{-\alpha}$$

and

$$M_i = \sup_{E_2}|f_i(Y_3)| \quad (i = 1, 2)\,.$$

The integral on the right-hand side of the last inequality is convergent for $\alpha > 1$, and consequently the integrals $L^i(X)$ are uniformly convergent in every set W_1. We have

$$D^{pqr}_{x_1x_2x_3}[V(\bar{r}_3)(\bar{r}_3)^{-1}] = V(\bar{r}_3)P(y_1-x_1, y_2-x_2, x_3+v, \bar{r}_3) = H_{pqr}(X, Y_3)$$

where P is a polynomial being a finite sum the terms

$$\bar{r}_3^{-\gamma}(y_1-x_1)^{\delta_1}(y_2-x_2)^{\delta_2}(x_3+v)^{\delta_3}$$

where γ, $\delta_i (i = 1, 2)$ being positive integers and $\gamma \geqslant 1+\delta_1+\delta_2+\delta_3$. Let K denote the number of constituents of the polynomial P. Applying inequality (4) and the change of variables (17) we obtain

$$|H_{pqr}(X, Y_3)| \leqslant KC^{-\alpha}\bar{r}_3^{\frac{1}{2}(-\gamma-\alpha)}$$

and

$$|L^i_{pqr}(X)| \leqslant 2\pi KC_i \int_0^\infty e^{hv}\Big[\int_0^\infty (b_1^2+\varrho^2)^{\frac{1}{2}(-\gamma-\alpha)}\varrho\, d\varrho\Big]dv\,.$$

Hence the integrals $L^i_{pqr}(X)$ $(i = 1, 2)$ are uniformly convergent in every set W_1.

By lemma 6 we get

LEMMA 7. *If the functions* f_i $(i = 1, 2)$ *satisfy the assumptions of the Lemma* 4, *then the integrals* $L^i(X)$, $L^i_{pqr}(X)$ $(i = 1, 2)$ *exist in* W_1 *and the functions* $L^i(X)$ *are of class* C^4 *in the domain* W_1 *and*

$$L^i_{pqr}(X) = D^{pqr}_{x_1x_2x_3}L^i(X)\,.$$

From Lemmas 5 and 7 follows

THEOREM 2. *If the functions* f_i $(i = 1, 2)$ *are bounded and measurable in* E_2, *then the integrals* $u_i(X)$ *and* $D^{pqr}_{x_1x_2x_3}u_i(X)$ $(i = 1, 2)$ *exist in* W_1 *and the functions* $u_i(X)$ $(i = 1, 2)$ *are of class* C^4 *in* E_3^+ *and*

$$D^{pqr}_{x_1x_2x_3}u_1(X) = A\iint_{E_2} f_1(Y_3)\,D^{pqr}_{x_1x_2x_3}[\Delta_Y G(X, Y) - 2C^2G(X, Y)]|_{y_3=0}\,dY_3$$

and

$$D^{pqr}_{x_1x_2x_3}u_2(X) = -A\iint_{E_2} f_2(Y_3)\,D^{pqr}_{x_1x_2x_3}G(X, Y)|_{y_3=0}\,dY_3\,.$$

4. Synthesis of the problem (M).

Now we shall prove

LEMMA 8. *If the functions* f_i $(i = 1, 2)$ *are bounded and measureable in* E_2, *then the functions* u_i *satisfy the equation* (1) *in* E_3^+.

Proof. By theorem 1 and 2 we get

$$(\Delta - C^2)^2 u_1(X) =$$
$$= A \iint_{E_2} f_1(Y_3)[\Delta_Y(\Delta_X - C^2)^2 G(X, Y) - 2C^2(\Delta_X - C^2)^2 G(X, Y)]_{y_3=0} dY_3$$

and

$$(\Delta - C^2)^2 u_2(X) = -A \iint_{E_2} f_2(Y_3)(\Delta_X - C^2)^2 G(X, Y)|_{y_3=0} dY_3 .$$

We shall verify the boundary condition (2). By formula (16a) and theorem 2 we obtain

$$(18)\quad \begin{cases} D_{x_3}u_1(X) = 2AC \iint_{E_2} f_1(Y_3)[2h^{-1}x_3 V(R) R^{-3} + \\ + Ch^{-1}x_3 V(R)(R^{-1}C + 2R^{-2}) + h \int_{x_3}^{\infty} e^{h(t-x_3)} V(r_4)(C + 2r_4^{-1}) dt + \\ \qquad + V(R)(C + 2R^{-1})] dY_3 \\ D_{x_3}u_2(X) = -2A \iint_{E_2} f_2(Y_3)[Ch^{-1}x_3 V(R) R^{-1} + \\ \qquad + h \int_{x_3}^{\infty} e^{h(t-x_3)} V(r_4) dt + V(R)] dY_3 . \end{cases}$$

Let

$$F(X) = \iint_{E_2} x_3 V(R) R^{-3} dY_3, \qquad X \in E_3^+ .$$

Lemma 9.

$$F(X) \to 2\pi \quad \text{as} \quad X \to (x_1^0, x_2^0, 0^+) .$$

Proof. We get by (8)

$$F(X) = \iint_{E_2} x_3[(y_1 - x_1)^2 + (y_2 - x_2)^2 + x_3^2]^{-3/2}$$
$$\exp\{-C[(y_1 - x_1)^2 + (y_2 - x_2^2) + x_3^2]^{1/2}\} dY_3 .$$

Applying in the last integral the change of variables

(19a) $\quad y_1 - x_1 = x_3 \varrho \cos\varphi, y_2 - x_2 = x_3 \varrho \sin\varphi (0 \leqslant \varrho < \infty, 0 \leqslant \varphi \leqslant 2\pi)$

and

(19b) $\quad \varrho^2 + 1 = z^2 (1 \leqslant z < \infty)$

we get

$$2 \int_0^{\infty} (1 + \varrho^2)^{-3/2} \exp[-Cx_3(1 + \varrho^2)^{1/2}] \varrho\, d\varrho = 2\pi \int_1^{\infty} e^{-Cx_3 z} z^{-2} dz$$

We shall prove that the integral $\int_1^{\infty} e^{-Cx_3 z} z^{-2} dz$ *is* uniformly convergent for $x_3 \in \langle 0, a \rangle$.

We have

$$\bigwedge_{z \in \langle 1, \infty)} \bigwedge_{x_3 \in \langle 0, a \rangle} |z^{-2} e^{-Cx_3 z}| \leqslant z^{-2} \quad \text{and} \quad \int_1^{\infty} z^{-2} dz = 1$$

and

$$\lim 2\pi \int_1^{\infty} e^{-Cx_3 z} z^{-2} dz = 2\pi \int_1^{\infty} (\lim e^{-Cx_3 z}) z^{-2} dz = 2\pi \quad \text{as} \quad x_3 \to 0^+ .$$

Let

$$M_i(X) = 4ACh^{-1} \iint_{E_2} f_i(Y_3) V(R) R^{-3} dY_3 \ (i = 1, 2), \ X \in E_3^+ .$$

Now we shall prove

LEMMA 10. *If the functions f_i $(i = 1, 2)$ are bounded, measurable in E_2 and continuous at the point $X_3^0 = (x_1^0, x_2^0)$, then*

$$M_i(X) \to f_i(x_1^0, x_2^0) \quad \text{when} \quad X \to (x_1^0, x_2^0, 0^+) .$$

We shall prove the lemma 10 only for $M_1(X)$. The proof for the integral $M_2(X)$ is analogous.

Let

$$d(X_3^0, Y_3) = f_1(Y_3) - f_1(X_3^0)$$

and

$$M_3(X) = A_1 \iint_{E_2} d(X_3^0, Y_3) x_3 V(R) R^{-3} dY_3$$

where

$$A_1 = 4ACh^{-1} = (2\pi)^{-1} .$$

Now the integral $M_1(X)$ may be written in the form:

$$M_1(X) = A_1 f_1(X_3^0) F(X) + M_3(X) .$$

By lemma 9 we get

$$A_1 f_1(X_3^0) F(X) \to f_1(X_3^0) \quad \text{as} \quad X \to (X_3^0, 0^+) .$$

Let $K(X_3^0, \delta)$ and $K(X_3, \frac{1}{2}\delta)$ denote the circles with radii δ, $\frac{1}{2}\delta$ and centres at the points X_3^0 and X_3 respectively. From the continuity of the function f_1 we obtain

$$\bigwedge_{\varepsilon > 0} \ \bigvee_{K(X_3^0, \delta)} [Y_3 \in K(X_3^0, \delta) \Rightarrow |d(X_3^0, Y_3)| < \tfrac{1}{2}\varepsilon] .$$

Let

$$M_4(X) = (2\pi)^{-1} \iint_{K(X_3^0, \delta)} d(X_3^0, Y_3) x_3 V(R) R^{-3} dY_3$$

and

$$M_5(X) = (2\pi)^{-1} \iint_{E_2 \setminus K(X_3^0, \delta)} d(X_3^0, Y_3) x_3 V(R) R^{-3} dY_3 .$$

Then $M_3(X) = M_4(X) + M_5(X)$. For the integral $M_4(X)$ we get the estimation

$$|M_4(X)| \leq \tfrac{1}{2}\varepsilon (2\pi)^{-1} \iint_{E_2} x_3 V(R) R^{-3} dY_3 < \tfrac{1}{2}\varepsilon \quad \text{for} \quad 0 < x_3 < \delta(\varepsilon) .$$

Let

$$|X_3^0 X_3| < \tfrac{1}{2}\delta \quad \text{and} \quad D_1 = E_2 \setminus K(X_3^0, \delta),\ D_2 = E_2 \setminus K(X_3, \tfrac{1}{2}\delta) .$$

For $M_5(X)$ holds

$$|M_5(X)| \leqslant (2\pi)^{-1} \iint_{D_1} [|f_1(Y_3)| + + |f_1(X_3^0|] x_3 V(R) R^{-3} dY_3 \leqslant M_1 \pi^{-1} \iint_{D_2} x_3 V(R) R^{-3} dY_3 .$$

Applying in the integral $\iint_{D_2} x_3 V(R) R^{-3} dY_3$ the transformation (19a) we get

$$|M_5(X)| \leqslant 2M_1 \int_{s_1}^{\infty} \exp[-Cx_3(\varrho^2+1)^{1/2}](1+\varrho^2)^{-3/2} \varrho \, d\varrho \leqslant \frac{\varepsilon}{2}$$

for

$$|X_3^0 X_3| < \frac{\delta}{2} \quad \text{and} \quad 0 < x_3 < \delta(\varepsilon), \quad \text{where} \quad s_1 = (2x_3)^{-1}\delta$$

and finally

$$|M_3(X)| < \varepsilon \quad \text{for} \quad |X_3^0 X_3| < \min\left[\frac{\delta}{2}, \delta(\varepsilon)\right].$$

Let

$$N_i(X) = \iint_{E_2} f_i(Y_3) x_3 V(R) R^{-n} dY_3 (n, i = 1, 2), \; X \in E_3^+ .$$

LEMMA 11. *If the functions f_i $(i = 1, 2)$ satisfy the assumptions of the lemma 4, then $N_i(X) \to 0$ as $X \to (X_3^0, 0^+)$ $(i = 1, 2)$.*

Proof. We shall prove lemma 11 for the integral $N_1(X)$. The proof for the integral $N_2(X)$ is similar. We have

$$|N_1(X)| \leqslant M_1 \iint_{E_2} x_3 V(R) R^{-n} dY_3 .$$

Applying the transformations (19a), (19b) and the formula (4) we get

$$|N_1(X)| \leqslant 2\pi C_1 x_3^{3-n-\alpha} \int_1^{\infty} z^{1-n-\alpha} dz < \varepsilon \quad \text{for} \quad x_3 < \delta(\varepsilon) \quad \text{and} \quad 2-n < \alpha < 3-n ,$$

where

$C_1 = M_1 C^{-\alpha}$, If $n = 2$, then $\alpha \in (0, 1)$ and if $n = 1$, then $\alpha \in (1, 2)$.

Now we shall prove

THEOREM 3. *If the functions f_i $(i = 1, 2)$ are bounded, measurable in E_2 and the function f_1 is continuous at the point X_3^0, then*

$$[hu(X) + D_{x_3} u(X)] \to f_1(X_3^0) \quad \textit{as} \quad X \to (X_3^0, 0^+) .$$

Proof. By (16a) and (18) we obtain

$$hu(X) + D_{x_3} u(X) =$$
$$= 2A_1 C \iint_{E_2} f_1(Y_3) h^{-1} x_3 \{2V(R) R^{-3} + CV(R)[R^{-1} + 2R^{-2}]\} dY_3 +$$
$$+ 2A_1 C \iint_{E_2} f_2(Y_3) h^{-1} x_3 V(R) R^{-1} dY_3 .$$

In virtue of lemma 10 we have

$$4A_1 Ch^{-1} x_3 \iint_{E_2} f_1(Y_3) V(R) R^{-3} dY_3 \to f_1(X_3^0) \quad \text{when} \quad X \to (X_3^0, 0^+) .$$

Moreover from lemma 11 follows

$$\iint_{E_2} f_i(Y_3) x_3 V(R) R^{-n} dY_3 \to 0 \quad \text{as} \quad X \to (X_3^0, 0^+)(n, i = 1, 2) .$$

Now we shall prove the boundary conditions (2).

THEOREM 4. *If the functions f_i $(i = 1, 2)$ are bounded, measurable in E_2 and the function f_2 is continuous at the point X_3^0, then*

$$[h\Delta u(X) + D_{x_3}\Delta u(X)] \to f_2(X_3^0) \quad \textit{as} \quad X \to (X_3^0, 0^+) .$$

Proof. By theorem 2 and formula (16a) we get

$$\begin{aligned} h\Delta u_1(X) + D_{x_3}\Delta u_1(X) &= \\ &= A \iint_{E_2} f_1(Y_3) h[\Delta_Y^2 G(X, Y) - 2C^2 \Delta_Y G(X, Y)] + \\ &\quad + D_{x_3}[\Delta_Y^2 G(X, Y) - 2C^2 \Delta_Y G(X, Y)]|_{y_3=0} dY_3 = \\ &= -2AC^5 \iint_{E_2} f_1(Y_3) x_3 V(R) R^{-1} dY_3 \end{aligned}$$

and

$$\begin{aligned} h\Delta u_2(X) + D_{x_3}\Delta u_2(X) &= \\ &= 2A_1 Ch^{-1} \iint_{E_2} f_2(Y_3) x_3 V(R) [2R^{-3} + 2CR^{-2} - C^2 R^{-1}] dY_3 . \end{aligned}$$

By lemmas 9, and 10 we get

$$h\Delta u_1(X) + D_{x_3}\Delta u_1(X) \to 0 \quad \text{as} \quad X \to (X_3^0, 0^+)$$

and

$$h\Delta u_2(X) + D_{x_3}\Delta u_2(X) \to f_2(X_3^0) \quad \text{as} \quad X \to (X_3^0, 0^+) .$$

From the theorems 3, 4 and lemma 8 we have the fundamental

THEOREM 5. *If the functions f_i $(i = 1, 2)$ are bounded, measurable in E_2 and continuous at the point X_3^0, then the function u defined by formulae* (16) *or* (16a) *is the solution of the equation* (1) *in the domain E_3^+ and satisfies the conditions*:

$$\lim [hu(X) + D_{x_3} u(X)] = f_1(X_3^0) \quad \text{when} \quad X \to (X_3^0, 0^+)$$

and

$$\lim [h\Delta u(X) + D_{x_3}\Delta u(X)] = f_2(X_3^0) \quad \text{when} \quad X \to (X_3^0, 0^+) .$$