JAN GÓROWSKI

On a certain boundary problem for the equation $(\Delta-c^2)^2u=0$ in the half-plane

1. In the present paper we shall give the Green function and the solution of the equation

$$(\Delta-c^2)^2u(x_1,x_2)=0 \tag{1}$$

where c is a positive constant in the domain
$E_2^+=\{(x_1,x_2)\colon -\infty<x_1<\infty \wedge x_2>0\}$ satisfying the boundary conditions

$$\begin{aligned} D_{x_2}u(x_1,0)+hu(x_1,0)&=f_1(x_1)\\ D_{x_2}\Delta u(x_1,0)+h\Delta u(x_1,0)&=f_2(x_1)\,,\end{aligned} \tag{2}$$

h being a negative constant.

2. Let

$$\begin{aligned} r_1^2&=(x_1-y_1)^2+(x_2-y_2)^2\,, & r_2^2&=(x_1-y_1)^2+(x_2+y_2)^2\,,\\ r_3^2&=(x_1-y_1)^2+(x_2+y_2+v)^2\,, & \varrho^2&=(x_1-y_1)^2+x_2^2\,,\\ R^2&=(x_1-y_1)^2+(x_2+v)^2\,, & X&=(x_1,x_2)\,,\ Y=(y_1,y_2)\,,\end{aligned}$$

$X, Y\in E_2^+$, $X\neq Y$ and let $K_n(z)$ denote the Mac-Donald function of order n ([2] p. 116).
We shall prove

THEOREM 1. *The function*

$$G(X,Y)=\frac{1}{h}\big(cr_1K_1(cr_1)+cr_2K_1(cr_2)\big)+I(X,Y)\,, \tag{3}$$

where

$$I(X,Y)=c\int_0^\infty e^{hv}r_3K_1(cr_3)\,dv$$

is the Green function for the problem (1), (2).

Proof. From ([2], p. 132) the integral $I(X,Y)$ is quasi-uniformly convergent with the derivatives to the fifts order in the set $E_2^+\times E_2^+$ and the functions

$cr_i K_1(cr_i)$ $(i = 1, 2, 3)$ are solutions of the equation (1), ([1], p. 8) then the function $G(X, Y)$ satisfies the equation (1).

Applying to the function $G(X, Y)$ the formulae ([2], p. 117)

$$(4)\qquad D_z(z^{-n}K_n(z)) = -z^{-n}K_{n+1}(z), \qquad D_z(z^n K_n(z)) = -z^n K_{n-1}(z)$$

we get

$$(5)\quad D_{y_2}G(X, Y) = \frac{c^2}{h}[K_0(cr_1)(x_2-y_2)-K_0(cr_2)(x_2+y_2)]-2cr_2K_1(cr_2)+ \\ -2h\,I(X, Y).$$

By (5) we get

$$D_{y_2}G(X, Y)+hG(X, Y) = 0 \quad \text{for} \quad y_2 = 0.$$

Further

$$\Delta_Y G(X, Y) = \frac{c^2}{h}((cr_1K_1(cr_1)-2K_0(cr_1)+cr_2K_1(cr_2)-2K_0(cr_2))+ \\ +2c^2\int_0^\infty e^{hv}(cr_3K_1(cr_3)-2K_0(cr_3))dv$$

and

$$D_{y_2}\Delta_Y G(X, Y)+h\Delta_Y G(X, Y) = 0 \quad \text{for} \quad y_2 = 0.$$

Lemma 1. *If the function f is continuous at the point x_0 and $\int_{-\infty}^{\infty}|f(y_1)|dy_1<\infty$, then*

$$V(X) = \int_{-\infty}^{\infty} f(y_1)[D_{x_2}G(X, Y)+hG(X, Y)]_{y_2=0}dy_1 \to 0 \quad \textit{as} \quad X\to(x_0, 0^+).$$

Proof. By (3) and (5) we have

$$G(X, Y)|_{y_2=0} = \frac{2}{h}c\varrho K_1(c\varrho)+2\int_0^\infty e^{hv}cRK_1(cR)dv = -\frac{2}{h}\int_0^\infty e^{hv}D_v(cRK_1(cR))dv$$

and

$$D_{x_2}G(X, Y)|_{y_2=0} = -\frac{2c}{h}\int_0^\infty e^{hv}D_v^2(RK_1(cR))dv = x_2I(\varrho)+2c\int_0^\infty e^{hv}D_v(RK_1(cR))dv$$

where

$$I(\varrho) = -\frac{2}{h}c^2K_0(c\varrho).$$

Thus

$$D_{x_2}G(X, Y)+hG(X, Y) = -\frac{2}{h}c^2 K_0(c\varrho)x_2 \quad \text{for} \quad y_2 = 0 .$$

By ([3], p. 276, 372)

$$(6) \qquad \int_0^{\infty} K_p(\alpha\sqrt{x^2+z^2})\frac{x^{2q+1}}{\sqrt{(x^2+z^2)^p}}dx = \frac{2^q\gamma(q+1)}{\alpha^{q+1}z^{p-q-1}}K_{p-q-1}(\alpha z)$$

and

$$(7) \qquad K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$$

we get

$$(8) \qquad \int_{-\infty}^{\infty} K_0(c\varrho)dy_1 = \int_{-\infty}^{\infty} K_0(c\sqrt{t^2+x_2^2})dt = \frac{\pi}{c}e^{-cx_2} .$$

Let

$$V(X) = x_2 \int_{-\infty}^{\infty} f(y_1)I(\varrho)dy_1 = A_1(X)+A_2(X) ,$$

where

$$A_1(X) = x_2 \int_{-\infty}^{\infty} (f(y_1)-f(x_0))I(\varrho)dy_1$$

and

$$A_2(X) = x_2 \int_{-\infty}^{\infty} f(x_0)I(\varrho)dy_1$$

Let ε be an arbitrary positive number. By (8) and by the inequality $K_0(c\varrho)\geqslant 0$ we have

$$|A_2(X)|<\frac{\varepsilon}{4} \quad \text{for} \quad 0<x_2<\eta_1 ,$$

η_1 being a sufficiently small positive number. From the continuity of the function f at the point x_0 it follows that there exists a positive number δ, such that

$$(9) \qquad |f(y_1)-f(x_0)|<\frac{\varepsilon}{4} \quad \text{for} \quad |y_1-x_0|<\delta .$$

Let

$$A_1(X) = B_1(X)+B_2(X)+B_3(X) ,$$

where

$$B_1(X) = x_2 \int_{|y_1-x_0|<\delta} (f(y_1)-f(x_0))I(\varrho)dy_1 ,$$

$$B_2(X) = x_2 \int_{|y_1-x_0|\geq\delta} f(y_1)I(\varrho)dy_1\,,$$

$$B_3(X) = -x_2 \int_{|y_1-x_0|\geq\delta} f(x_0)I(\varrho)dy_1\,.$$

By (8), (9) we get

$$|B_1(X)| \leqslant x_2 \frac{2\varepsilon}{4|h|} \int_{-\infty}^{\infty} K_0(c\varrho)dy_1 \leqslant \frac{\varepsilon}{4} \quad \text{for} \quad 0<x_2<\eta_2\,,$$

η_2 being a sufficiently small positive number.

$$\text{If } |x_1-x_0|<\frac{\delta}{2}, \quad \text{then} \quad \{y_1:\ |y_1-x_0|\geqslant\delta\}\subset\left\{y_1:\ |y_1-x_1|\geqslant\frac{\delta}{2}\right\}.$$

Hence

$$|B_2(X)| \leqslant x_2 \int_{|y_1-x_1|<\delta/2} |f(y_1)I(\varrho)|\,dy_1 \leqslant M_1 x_2 \int_{-\infty}^{\infty} |f(y_1)|\,dy_1 < \frac{\varepsilon}{4} \quad \text{for} \quad 0<x_2<\eta_3\,,$$

where

$M_1 = \sup_{|y_1-x_1|\geqslant\delta/2} K_0(c\varrho)c^2\left|\frac{2}{h}\right|$ and η_3 is a positive number sufficiently small.

Similarly

$$|B_3(X)| \leqslant |f(x_0)|\,x_2 \int_{-\infty}^{\infty} |I(\varrho)|\,dy_1 = x_2 \frac{2\pi}{|ch|} c^2 |f(x_0)| e^{-cx_2} < \frac{\varepsilon}{4}$$

for

$0<x_2<\eta_4$, η_4 being a positive number sufficiently small.

If $\eta = \min\{\eta_1, \eta_2, \eta_3, \eta_4\}$, $0<x_2<\eta$ and $|x_1-x_0|<\delta/2$,

then

$$|A_1(X)|+|A_2(X)|<\varepsilon\,.$$

LEMMA 2. *If the function f is continuous at the point x_0 and*

$$\int_{-\infty}^{\infty} |f(y_1)|\,dy_1 < \infty\,,$$

then

$$a_1 \int_{-\infty}^{\infty} f_1(y_1)[D_{x_2}\Delta_Y G(X, Y) + h\Delta_Y G(X, Y)]_{y_2=0} \to f(x_0) \quad \textit{as} \quad X\to(x_0, 0^+)\,,$$

where

$$a_1 = (4\pi c^3)^{-1} h\,.$$

Proof.

$$[D_{x_2}\Delta_Y G(X, Y)+h\Delta_Y G(X, Y)]_{y_2=0} = c^2 x_2 I(\varrho)+x_2 J(\varrho),$$

where

$$J(\varrho) = \frac{4c^4}{h} K_1(c\varrho)\frac{1}{c\varrho}.$$

Let

$$V_1(X) = a_1 \int_{-\infty}^{\infty} f(y_1)[D_{x_2}\Delta_Y G(X, Y)+h\Delta_Y G(X, Y)]_{y_2=0}dy_1 = C_1(X)+C_2(X),$$

where

$$C_1(X) = a_1 c^2 x_2 \int_{-\infty}^{\infty} f(y_1)I(\varrho)ay_1$$

and

$$C_2(X) = a_1 x_2 \int_{-\infty}^{\infty} f(y_1)J(\varrho)dy_1 .$$

By lemma 1 $C_1(X)\to 0$ as $X\to(x_0, 0^+)$.
By the formula (6) and (7) we have

$$a_1 x_2 \int_{-\infty}^{\infty} J(\varrho)dy_1 = e^{-cx_2}. \tag{10}$$

Let

$$C_2(X) = I_1(X)+I_2(X),$$

where

$$I_1(X) = a_1 x_2 \int_{-\infty}^{\infty} f(x_0)J(\varrho)dy_1,$$

$$I_2(X) = a_1 x_2 \int_{-\infty}^{\infty} (f(y_1)-f(x_0))J(\varrho)dy_1 .$$

From (10) we deduce $I_1(X)\to f(x_0)$ as $X\to(x_0, 0^+)$.
We shall prove that $I_2(X)\to 0$ as $X\to(x_0, 0^+)$.
Let ε be an arbitrary positive number. Since f is continuous at the point x_0, thus $|f(y_1)-f(x_0)|<\varepsilon/2$ for $|y_1-x_0|<\delta$, where δ is a positive number sufficiently small.
Let

$$I_2(X) = W_1(X)+W_2(X),$$

where

$$W_1(X) = a_1 x_2 \int_{|y_1-x_0|<\delta} (f(y_1)-f(x_0))J(\varrho)dy_1,$$

$$W_2(X) = a_1 x_2 \int_{|y_1-x_0|\geq\delta} (f(y_1)-f(x_0))J(\varrho)dy_1 .$$

By (10) we get

$$|W_1(X)| \leqslant |a_1| x_2 \frac{\varepsilon}{2} \int_{-\infty}^{\infty} |J(\varrho)| \, dy_1 = \frac{\varepsilon}{2} \cdot e^{-cx_2} \leqslant \frac{\varepsilon}{2}.$$

If $|x_1 - x_0| < \frac{\delta}{2}$, then $\{y_1 : |y_1 - x_0| \geqslant \delta\} \subset \left\{y_1 : |y_1 - x_1| \geqslant \frac{\delta}{2}\right\}$.

Hence

$$|W_2(X) \leqslant |a_1| x_2 \int_{|y_1 - x_1| \geqslant \delta/2} |f(y_1) - f(x_0)| \, |J(\varrho)| \, dy_1 .$$

Applying the formula ([3], p. 365)

$$K_p(xz) = \frac{\gamma(p+1)(2z)^p}{x^p \gamma(\frac{1}{2})} \int_0^{\infty} \frac{\cos(xt)}{(t^2 + z^2)^{p+1/2}} \, dt \quad (p \geqslant -\tfrac{1}{2} \wedge x > 0 \wedge z > 0)$$

we get the inequality

(11) $$K_n(z) \leqslant B z^{-n},$$

where n is a positive integer and B a positive constant.
Whence

$$|W_2(X)| \leqslant K \frac{4c^4}{|h|} |a_1| \int_{|y_1 - x_1| \geqslant \delta/2} |f(y_1) - f(x_0)| \frac{x_2}{(c\varrho)^2} \, dy_1 \leqslant B_1 x_2 \int_{-\infty}^{\infty} |f(y_1)| \, dy_1 +$$

$$+ B_2 x_2 \int_{|y_1 - x_1| \geqslant \delta/2} \frac{1}{(c\varrho)^2} \, dy_1 \leqslant \frac{\varepsilon}{2} \quad \text{for} \quad 0 < x_2 < \eta_1 ,$$

where

$$B_1 = B \frac{4c^4}{|h|} |a_1| \sup_{|y_1 - x_1| \geqslant \delta/2} \frac{1}{(c\varrho)^2}, \qquad B_2 = 2B \frac{4c^4}{|h|} |a_1 \cdot f(x_0)| ,$$

η_1 is a positive number sufficiently small.
Hence

$$|I_2(X)| \leqslant |W_1(X)| + |W_2(X)| < \varepsilon \quad \text{for} \quad 0 < x_2 < \eta_1 \quad \text{and} \quad |x_1 - x_0| < \frac{\delta}{2}.$$

Now we shall prove the following

THEOREM 2. *If functions f_i $(i = 1, 2)$ are continuous at the point x_0 and*

$$\int_{-\infty}^{\infty} |f_i(y_1)| \, dy_1 < \infty \qquad (i = 1, 2)$$

then the function

$$(12)\qquad U(X) = a_1 \int_{-\infty}^{\infty} [f_1(y_1)[\Delta_Y G(X, Y) - 2c^2 G(X, Y)]_{y_2=0} + f_2(y_1) G(X, Y)|_{y_2=0}]\, dy_1$$

is the solution of the boundary problem (1), (2).

Proof.

Let

$$U(X) = J_1(X) + J_2(X) + J_3(X),$$

where

$$J_1(X) = a_1 \int_{-\infty}^{\infty} f_1(y_1) \Delta_Y G(X, Y)|_{y_2=0}\, dy_1,$$

$$J_2(X) = -a_1 \int_{-\infty}^{\infty} f_1(y_1) 2c^2 G(X, Y)|_{y_2=0}\, dy_1,$$

$$J_3(X) = a_1 \int_{-\infty}^{\infty} f_2(y_1) G(X, Y)|_{y_2=0}\, dy_1.$$

By lemma 1 we have

$$[D_{x_2}(J_2(X) + J_3(X)) + h(J_2(X) + J_3(X))] \to 0 \quad \text{as} \quad X \to (x_0, 0^+).$$

By lemma 2 we obtain

$$[D_{x_2} J_1(X) + h J_1(X)] \to f_1(x_0) \quad \text{as} \quad X \to (x_0, 0^+).$$

By changing the derivation and integration we get

$$\Delta J_1(X) = a_1 \int_{-\infty}^{\infty} f_1(y_1) \Delta_X \Delta_Y G(X, Y)|_{y_2=0}\, dy_1,$$

$$\Delta J_2(X) = a_1 \int_{-\infty}^{\infty} f_1(y_1) \Delta_X (-2c^2 G(X, Y))|_{y_2=0}\, dy_1,$$

$$\Delta J_3(X) = a_1 \int_{-\infty}^{\infty} f_2(y_1) \Delta_X G(X, Y)|_{y_2=0}\, dy_1,$$

$$\Delta_X \Delta_Y G(X, Y)|_{y_2=0} = \frac{1}{h} c^4 (2c\varrho K_1(c\varrho) - 8K_0(c\varrho)) + 2c^4 \int_0^{\infty} e^{hv} (cRK_1(cR) - 4K_0(cR))\, dv =$$

$$= -\frac{2}{h} c^4 \int_0^{\infty} e^{hv} D_v (cRK_1(cR) - 4K_0(cR))\, dv$$

and

$$[D_{x_2}\Delta_X\Delta_Y G(X, Y)+h\Delta_X\Delta_Y G(X, Y)]_{y_2=0} = c^4x_2 I(\varrho)+2c^2x_2 J(\varrho)$$

By lemma 1 and lemma 2 we obtain

$$J_1(X)\to 2c^2f_1(x_0) \quad \text{as} \quad X\to(x_0, 0^+).$$

By the identity

(13) $$\Delta_X G(X, Y)|_{y_2=0} = \Delta_Y G(X, Y)|_{y_2=0}$$

and lemma 2 we get

$\lim J_2(X) = -2c^2f_1(x_0)$, $\lim J_3(X) = f_2(x_0)$ as $X\to(x_0, 0^+)$ and finally

$$[D_{x_2}\Delta U(X)+h\Delta U(X)]\to f_2(x_0) \quad \text{as} \quad X\to(x_0, 0^+).$$

Thus theorem 2 is proved.

References

[1] F. Barański: *On the Lauricelli problem for the equation* $(\Delta-c^2)^2u = 0$ *in the half-plane*. Rocznik Nauk.-Dydak. WSP w Krakowie, Prace Mat. VII (1974).

[2] N. N. Lebiediew: *Funkcje specjalne i zastosowania*. Warszawa 1957.

[3] I. M. Ryżyk, I. M. Gradsztejn: *Tablice całek, sum, szeregów i iloczynów*. Warszawa 1964.