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On the Lauricelli problem for the equation $(\Delta+c^2)^2 u(x_1, x_2, x_3) = 0$ in the half-space $x_3>0$

1. In this paper we shall construct the solution of the equation

$$(\Delta+c^2)^2u(X) = \Delta^2u(X)+2c^2\Delta u(X)+c^4u(X) = 0\,, \tag{1}$$

where $X = (x_1, x_2, x_3)$ and c is a positive constant in the half-space $x_3>0$ satisfying the boundary conditions

$$u(x_1, x_2, 0) = f_1(x_1, x_2)\,, \tag{2}$$

$$D_{x_3}u(x_1, x_2, 0) = f_2(x_1, x_2)\,. \tag{3}$$

We shall use the convenient Green function to the construction of the solution of this problem.

2. Let $X \neq Y$ denote the points of the space R_3 for which $x_3>0$, $y_3\geqslant 0$ and let

$$r^2 = \sum_{i=1}^{3}(x_i-y_i)^2\,,$$

$$r_1^2 = \sum_{i=1}^{2}(x_i-y_i)^2+(x_3+y_3)^2,$$

$$R^2 = \sum_{i=1}^{2}(x_i-y_i)^2+x_3^2\,.$$

Let $J_\nu(cr)$ denote the Bessel function of the first kind with the index $\nu = \frac{3}{2}$. Now we shall prove the following

LEMMA 1. *The function*

$$u(r) = (cr)^{\frac{1}{2}}J_{\frac{1}{2}}(cr)$$

is the fundamental solution of the equation (1).

From ([2], p. 109) and by formula

$$\Delta u(r) = D^2 u(r) + \frac{2}{r} Du(r)$$

we get

$$(\Delta + c^2)^2 [(cr)^{\frac{1}{2}} J_{\frac{1}{2}}(cr)] = 0 .$$

Now we shall prove the following

THEOREM 1. *The function*

$$G(X, Y) = I_1(cr) - 2c^2 x_3 y_3 I_2(cr) - I_1(cr_1) ,$$

where

$$I_1(cr) = (cr)^{\frac{1}{2}} J_{\frac{1}{2}}(cr) , \quad I_2(cr) = (cr)^{-\frac{1}{2}} J_{-\frac{1}{2}}(cr)$$

is the Green function for the problem (1), (2), (3) *with the pole* X *for the half-space* $y_3 > 0$.

Proof. By lemma 1 the function $I_1(cr) - I_1(cr_1)$ is the solution of the equation (1) with respect to the point Y for $y_3 > 0$. The function $I_2(cr)$ is the solution of the equation $(\Delta + c^2)u = 0$.

Since

$$\Delta_Y [2c^2 x_3 y_3 I_2(cr)] = 4c^2 x_3 D_{y_3} I_2(cr) - 2c^4 x_3 y_3 I_2(cr)$$

thus

$$(\Delta_Y + c^2)^2 2c^2 x_3 y_3 I_2(cr) = 0 \quad \text{for} \quad y_3 > 0 .$$

If $y_3 = 0$ then $r = r_1 = R$ and $G(X, Y) = 0$ for $y_3 = 0$. Similarly we get

$$D_{y_3} G(X, Y) = 0 \quad \text{for} \quad y_3 = 0 .$$

3. Let $X' = (x_1, x_2)$, $\quad Y' = (y_1, y_2)$

and

(4) $$u(X) = a_1 \int_{E_2} f_1(Y') D_{y_3} \Delta_Y G(X, Y)|_{y_3=0} dY' + a_1 \int_{E_2} f_2(Y') \Delta_Y G(X, Y)|_{y_3=0} dY' ,$$

where a_1 is a convenient constant and f_1, f_2 are given functions.

We have

$$\Delta_Y G(X, Y) = -4c^4 x_3^2 (cr)^{-\nu} J_{-\nu}(cR) \quad \text{for} \quad y_3 = 0$$

and

$$D_{y_3} \Delta_Y G(X, Y) = 4c^6 x_3^3 (cr)^{-\nu-1} J_{-\nu-1}(cR) \quad \text{for} \quad y_3 = 0 .$$

Consequently we get

(5) $$u(X) = a_1 \int_{E_2} [f_1(Y') 4c^6 x_3^3 (cR)^{-\nu-1} J_{-\nu-1}(cR) - f_2(Y') 4c^4 x_3^2 (cR)^{-\nu} J_{-\nu}(cR)] dY' .$$

LEMMA 2. *If* $\int\limits_{E_2} |f_i(Y')|\,dY' < \infty \quad (i = 1, 2)$,

then the function $u(X)$ *defined by formulae* (4) *or* (5) *is of class* C^4 *for* $x_3 > 0$ *and*

$$D_{x_1^{p_1} x_2^{p_2} x_3^{p_3}} u(X) = a_1 \int\limits_{E_2} [f_1(Y') D_{y_3} \Delta_Y D_{x_1^{p_1} x_2^{p_2} x_3^{p_3}} G(X, Y)|_{y_3=0} + \\ + f_2(Y') \Delta_Y D_{x_1^{p_1} x_2^{p_2} x_3^{p_3}} G(X, Y)|_{y_3=0}]\, dY',$$

for arbitrary non negative integers p_i $(i = 1, 2, 3)$ $\left(\sum_{i=1}^{3} p_i \leqslant 4\right)$.

Proof. Let $a > 0$ and $E_3^a = \{x: |x_i| < \infty \ i = 1, 2, x_3 \geqslant a\}$. By ([2], p. 132) we get

$$\text{(6)} \qquad \begin{cases} \sup\limits_{X \in E_3^a} |D_{x_1^{p_1} x_2^{p_2} x_3^{p_3}} (cR)^{-\nu} J_{-\nu}(cR)| < \infty\,, \\ \sup\limits_{X \in E_3^a} |D_{x_1^{p_1} x_2^{p_2} x_3^{p_3}} (cR)^{-\nu-1} J_{-\nu-1}(cR)| < \infty \quad \text{for} \quad \sum\limits_{i=1}^{3} p_i \leqslant 4\,. \end{cases}$$

From (6) the integral defined by formula (4) is of class C^4 in the set $\{X: x_3 > 0\}$. Now we shall prove the following

LEMMA 3. *For arbitrary negative real number* $s \leqslant -1$

$$J = \int\limits_{E_2} (cR)^s J_s(cR)\, dY' = -2\pi c^{-2} (cx_3)^{s+1} J_{s+1}(cx_3)\,.$$

Proof. Let

$$\text{(7)} \qquad y_1 = x_1 + r\cos t\,, \quad y_2 = x_2 + r\sin t\,,$$

where

$$r \in [0, \infty)\,, \quad t \in [0, 2\pi]\,.$$

Applying the change of variables (7) to the integral J we get

$$J = 2\pi \int\limits_0^\infty (c\sqrt{r^2 + x_3^2})^s J_s(c\sqrt{r^2 + x_3^2})\, r\, dr\,.$$

By the transformation

$$\text{(8)} \qquad u = c\sqrt{r^2 + x_3^2}$$

we obtain the thesis of lemma 3.

Let

$$K_1(X, Y') = a_1 4c^6 x_3^3 (cR)^{-\nu-1} J_{-\nu-1}(cR)\,,$$

$$K_2(X, Y') = -a_1 4c^4 x_3^2 (cR)^{-\nu} J_{-\nu}(cR)\,,$$

$$K_3(X, Y') = a_1 4c^8 x_3^4 (cR)^{-\nu-2} J_{-\nu-2}(cR)\,,$$

where

$$a_1 = (8\sqrt{2\pi c})^{-1}\,.$$

From formulae ([3], p. 371) we have

(9) $$\begin{cases} J_{-\frac{1}{2}}(u) = q\cos u, \quad J_{-\frac{3}{2}}(u) = q(-\sin u - u^{-1}\cos u) \\ J_{-\frac{5}{2}}(u) = q[3u^{-1}\sin u + (3u^{-2}-1)\cos u] \\ J_{-\frac{7}{2}}(u) = q[\sin u + 6u^{-1}\cos u - 15u^{-2}\sin u - 15u^{-3}\cos u], \end{cases}$$

where

$$q = \sqrt{\frac{2}{\pi u}}.$$

By lemma 3 we obtain

(10) $$\int_{E_2} K_1(X, Y')dY' = \cos(cx_3) + cx_3\sin(cx_3),$$

(11) $$\int_{E_2} K_2(X, Y')dY' = x_3\cos(cx_3),$$

(12) $$\int_{E_2} K_3(X, Y')dY' = -x_3^{-1}3\cos(cx_3) - 3c\sin(cx_3) + c^2x_3\cos(cx_3).$$

Using the formulae (9) we get

LEMMA 4. *If a is an arbitrary positive number, then*

$$\sup_{z\geq a}|J_s(z)| < \infty, \quad s = -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{7}{2}.$$

LEMMA 5. *If the function f_1 is absolutely integrable in E_2 and continuous at the point $X_0' = (x_1^0, x_2^0)$, then*

$$A_1(X) = \int_{E_2} f_1(Y')K_1(X, Y')dY' \to f_1(X_0') \quad as \quad X \to (x_1^0, x_2^0, 0).$$

Proof. Let

$$A_1(X) = V_1(X) + V_2(X),$$

where

$$V_1(X) = \int_{E_2} (f_1(Y') - f_1(X_0'))K_1(X, Y')dY'$$

$$V_2(X) = \int_{E_2} f_1(X_0')K_1(X, Y')dY'.$$

From (10) we obtain

$$V_2(X) \to f_1(X_0') \quad \text{as} \quad X \to (x_1^0, x_2^0, 0).$$

We shall prove that $V_1(X) \to 0$ as $X \to (x_1^0, x_2^0, 0)$.
Let ε be an arbitrary positive number. From the continuity of the function $f_1(Y')$ at the point X_0' it follows that there exists a positive number $\delta_1(\varepsilon)$, such that

$$|f_1(Y') - f_1(X_0')| < \frac{\varepsilon}{4} \quad \text{for} \quad (\overline{X_0', Y'})^2 < \delta_1^2,$$

where

$$(\overline{X'_0 Y'})^2 = (x_1^0 - y_1)^2 + (x_2^0 - y_2)^2 .$$

Let

$$x_3 \in \left(0, \frac{\pi}{8c}\right) \quad \text{and} \quad \delta = \min\left\{\delta_1, \frac{\pi}{8c}\right\}.$$

By (9) we get

$$J_{-\frac{5}{2}}(cR) \geq 0 \quad \text{for} \quad R \in (0, 4\delta) . \tag{13}$$

Further let

$$V_1(X) = P_1(X) + P_2(X) ,$$

where

$$P_1(X) = \int\limits_{K(X'_0, \delta)} [f_1(Y') - f_1(X'_0)] K_1(X, Y') dY' ,$$

$$P_2(X) = \int\limits_{CK(X'_0, \delta)} [f_1(Y') - f_1(X'_0)] K_1(X, Y') dY' ,$$

$K(X'_0, \delta)$ being the circle with radius δ centred at the point X'_0 and

$$CK(X'_0, \delta) = E_2 \setminus K(X'_0, \delta) .$$

Let

$$Y' \in K(X'_0, \delta) \quad \text{and} \quad \overline{X'_0 X'} < \frac{\delta}{2} .$$

We get

$$\overline{Y'X'} \leq \overline{Y'X'_0} + \overline{X'_0 X'} \leq \delta + \frac{\delta}{2} = \frac{3}{2}\delta = d , \quad cR \leq 4\delta \leq \frac{\pi}{2}$$

and

$$K(X'_0, \delta) \subset K(X', d) .$$

Applying the change of variables (7), (8) in the integral

$$\int\limits_{K(X', d)} |K_1(X, Y')| dY'$$

and by (13) we obtain

$$P_1(X) \leq \frac{\varepsilon}{4} \int\limits_{K(X', d)} |K_1(X, Y')| dY' \leq 2\pi a_1 \frac{\varepsilon}{4} \int\limits_{cx_3}^{c\sqrt{d^2 + x_3^2}} 4c^4 x_3^3 u^{-\frac{3}{2}} J_{-\frac{5}{2}}(u) du.$$

By lemma 3 we have

$$|P_1(X)| \leq \frac{\varepsilon}{4} [\cos(cx_3) + cx_3 \sin(cx_3)] \leq \frac{\varepsilon}{4}(1 + cx_3) \leq \frac{\varepsilon}{2} \quad \text{for} \quad x_3 \in \left(0, \frac{\pi}{8c}\right).$$

If $\overline{X_0'X'} < \frac{\delta}{2}$, $Y' \in CK(X_0', \delta)$ then $\overline{Y'X'} \geqslant \frac{\delta}{2}$ and $CK(X_0', \delta) \subset CK\left(X' \frac{\delta}{2}\right)$.

Hence by lemma 4 we get

$$|P_2(X)| \leqslant \int_{CK\left(X', \frac{\delta}{2}\right)} |f_1(Y')K_1(X, Y')|\,dY' + \int_{CK\left(X', \frac{\delta}{2}\right)} |f_1(X_0')K_1(X, Y')|\,dY' \leqslant$$

$$\leqslant x_3 M_1 \int_{E_2} |f_1(Y')|\,dY' + x_3^3 M_2 \int_{CK\left(X', \frac{\delta}{2}\right)} (cR)^{-\frac{5}{2}}\,dY',$$

where

$$M_1 = \sup_{Y' \in CK\left(X', \frac{\delta}{2}\right)} |a_1 4c^6 (cR)^{-\nu-1} J_{-\nu-1}(cR)|,$$

$$M_2 = \sup_{Y' \in CK\left(X', \frac{\delta}{2}\right)} |a_1 f(X_0') 4c^6 J_{-\nu-1}(cR)|.$$

If $0 < x_3 < \eta$ for convenably $\eta > 0$, then $|P_2(X)| < \frac{\varepsilon}{2}$.

For $\eta^* = \min\left\{\eta, \frac{\pi}{8c}\right\}$ and $0 < x_3 < \eta^*$ we obtain

$$|V_1(X)| \leqslant |P_1(X)| + |P_2(X)| < \varepsilon.$$

Now we shall prove the following

LEMMA 6. *If the function f_1 is absolutely integrable in E_2 and continuous at the point X_0', then*

$$A_2(X) = \int_{E_2} f_2(Y')K_2(X, Y')\,dY' \to 0 \quad \textit{as} \quad X \to (x_1^0, x_2^0, 0).$$

Proof. From the continuity of the function f_2 at the point X_0' it follows that there exists a circle $K(X_0', \delta_1)$, such that

(14) $$|f_2(Y')| \leqslant M \quad \text{for} \quad Y' \in K(X_0', \delta_1).$$

Let

$$x_3 \in \left(0, \frac{\pi}{8c}\right) \quad \text{and} \quad \delta = \min\left\{\delta_1, \frac{\pi}{8c}\right\}.$$

By formula (9) we have

(15) $$J_{-\nu}(cR) \leqslant 0 \quad \text{for} \quad R \in (0, 4\delta).$$

Further, let

$$A_2(X) = B_1(X) + B_2(X),$$

where

$$B_1(X) = \int_{K(X_0', \delta)} f_1(Y')K_2(X, Y')\,dY', \quad B_2(X) = \int_{CK(X_0', \delta)} f_2(Y')K_2(X, Y')\,dY'.$$

Let

$$Y' \in K(X'_0, \delta) \quad \text{and} \quad \overline{X'_0, X} < \frac{\delta}{2}.$$

We have then

$$\overline{Y'X'} \leqslant \frac{3}{2}\delta = d, \; cR \leqslant 4\delta \leqslant \frac{\pi}{2} \quad \text{and} \quad K(X'_0, \delta) \subset K(X', d).$$

Applying the change of variables (7), (8) in the integral $B_1(X)$ and by (14), (15) we get

$$|B_1(X)| \leqslant -M \int_{K(X',d)} K_2(X, Y')dY' = -a_1 M 8\pi c^2 x_3^2 \int_{cx_3}^{c\sqrt{d^2+x_3^2}} (u)^{\frac{1}{2}} J_{-\frac{3}{2}}(u)du.$$

By lemma 3 and formula (9) we have

$$|B_1(X)| \leqslant M_1 x_3.$$

Hence, if $x_3 < \eta_1 \leqslant \frac{\pi}{8c}$, then $|B_1(X)| < \frac{\varepsilon}{2}$. If $\overline{X'_0, X'} < \frac{\delta}{2}$ then $\overline{X'Y'} \geqslant \frac{\delta}{2}$ for $Y' \in CK(X'_0, \delta)$ and by lemma 4 we get

$$|B_2(X)| \leqslant a_1 4c^4 x_3^2 \sup_{Y' \in CK\left(X', \frac{\delta}{2}\right)} [(cR)^{-\nu} |J_{-\nu}(cR)|] \int_{E_2} |f_2(Y')| dY' \leqslant M_2 x_3^2.$$

Hence, if

$$x_3 < \eta_2, \quad \text{then} \quad |B_2(X)| < \frac{\varepsilon}{2}.$$

Finally

$$|A_2(X)| \leqslant |B_1(X)| + |B_2(X)| < \varepsilon \quad \text{for} \quad 0 < x_3 < \min\{\eta_1, \eta_2\}.$$

Let

$$K(X, Y') = 3x_3^{-1} K_1(X, Y') + K_3(X, Y').$$

LEMMA 7. *If the function f_1 is absolutely integrable in E_2 and f_1 is of class C^1 in the circle $K(X'_0, \delta^*)$, $(\delta^* > 0)$, then*

$$W_1(X) = \int_{E_2} f_1(Y') K(X, Y') dY' \to 0 \quad \textit{as} \quad X \to (x_1^0, x_2^0, 0).$$

Proof. By formulae (10), (12) we get

(16) $$\int_{E_2} K(X, Y') dY' \to 0 \quad \text{as} \quad X \to (x_1^0, x_2^0, 0).$$

Let

$$F_1(X', Y') = f_1(Y') - f_1(X') - \sum_{i=1}^{2} (y_i - x_i) D_{x_i} f_1(X')$$

and

$$F_2(X', Y') = \sum_{i=1}^{2} (y_i - x_i) D_{x_i} f_1(X').$$

We get

$$W_1(X) = \int_{E_2} F_1(X', Y')K(X, Y')dY' + \int_{E_2} F_2(X', Y')K(X, Y')dY' +$$
$$+ \int_{E_2} f_1(X')K(X, Y')dY' .$$

By (16) we obtain

$$\int_{E_2} f_1(X')K(X, Y')dY' \to 0 \quad \text{as} \quad X \to (x_1^0, x_2^0, 0) .$$

Introducing the change of variables (7) in the integral $\int_{E_2} F_2(X', Y')K(X, Y')dY'$ we have

$$\int_{E_2} F_2(X', Y')K(X, Y')dY' = 0 .$$

Let ε be an arbitrary positive number and $\overline{X_0', X'} < \frac{\delta^*}{2}$. By the assumption of lemma 7 there exists a number $\delta_1(\varepsilon) > 0$ such that

$$\left| \frac{F_1(X', Y')}{\overline{X'Y'}} \right| < \frac{4\varepsilon}{33} \quad \text{for} \quad Y' \in K(X', \delta_1) .$$

By formula (9) we get

$$\text{(17)} \qquad J_{-\frac{5}{2}}(cR) \geqslant 0 , \; J_{-\frac{7}{2}}cR) \leqslant 0 \quad \text{for} \quad x_3 \in \left(0, \frac{\pi}{4c}\right), \; R \in (0, \delta) ,$$

where

$$\delta = \min\left\{\delta_1, \frac{\pi}{4c}\right\} .$$

Let

$$R(X) = \int_{E_2} F_1(X', Y')K(X, Y')dY' = R_1(X) + R_2(X) ,$$

where

$$R_1(X) = \int_{K(X', \delta)} F_1(X', Y')K(X, Y')dY' ,$$

$$R_2(X) = \int_{CK(X', \delta)} F_1(X', Y')K(X, Y')dY' .$$

By formula (17) we obtain

$$|R_1(X)| \leqslant 16\varepsilon(33)^{-1} a_1 c^5 [\int_{K(X', \delta)} 3x_3^2 (cR)^{-\nu} J_{-\nu-1}(cR)dY' +$$
$$- \int_{K(X', \delta)} c^2 x_3^4 (cR)^{-\nu-1} J_{-\nu-2}(cR)dY'] .$$

By an application of transformation (7), (8) we get

$$
(18)\qquad \begin{cases} \int\limits_{K(X',\delta)} (cR)^{-\nu}J_{-\nu-1}(cR)\,dY' = 2\pi c^{-2}\int\limits_{cx_3}^{t} u^{-\nu+1}J_{-\nu-1}(u)\,du \\ \int\limits_{K(X',\delta)} (cR)^{-\nu-1}J_{-\nu-2}(cR)\,dY' = 2\pi c^{-2}\int\limits_{cx_3}^{t} u^{-\nu}J_{-\nu-2}(u)\,du\,, \end{cases}
$$

where

$$t = c\sqrt{\delta^2+x_3^2}\,.$$

In virtue of (9) we obtain

$$
(19)\qquad \begin{cases} \int\limits_{cx_3}^{t} (u)^{-\nu+1}J_{-\nu-1}(u)\,du = 3\,\dfrac{\cos(cx_3)}{2(cx_3)^2} + o(x_3^{-2}) \\ \int\limits_{cx_3}^{t} (u)^{-\nu}J_{-\nu-2}(u)\,du = 15\,\dfrac{\cos(cx_3)}{4(cx_3)^4} + o(x_3^{-4}) \end{cases}
$$

By formulae (18), (19) we get

$$\lim|R_1(X)|\leqslant\varepsilon \quad \text{as} \quad X\to(x_1^0, x_2^0, 0)\,.$$

Now we shall prove that

$$R_2(X)\to 0 \quad \text{as} \quad X\to(x_1^0, x_2^0, 0)\,.$$

We have

$$R_2(X) = \int\limits_{CK(X',\delta)} (f_1(Y')-f_1(X'))K(X, Y')\,dY' + \\ + \int\limits_{CK(X',\delta)} F_2(X', Y')K(X, Y')\,dY'\,.$$

Introducing in the integral

$$\int\limits_{CK(X',\delta)} F_2(X', Y')K(X, Y')\,dY'$$

the change of variables (7) we get

$$(20)\qquad \int\limits_{CK(X',\delta)} F_2(X', Y')K(X, Y')\,dY' = 0\,.$$

By (20) we obtain

$$R_2(X) = \sum_{i=1}^{4} L_i(X)\,,$$

where

$$L_1(X) = \int\limits_{CK(X',\delta)} f_1(Y')3x_3^{-2}K_1(X, Y')\,dY'\,,$$

$$L_2(X) = \int_{CK(X',\delta)} f_1(Y')K_3(X, Y')dY' ,$$

$$L_3(X) = -\int_{CK(X',\delta)} f_1(X')3x_3^{-2}K_1(X, Y')dY' ,$$

$$L_4(X) = -\int_{CK(X',\delta)} f_1(X')K_3(X, Y')dY' .$$

We shall prove that $L_i(X) \to 0$ $(i = 1, 2, 3, 4)$ as $X \to (x_1^0, x_2^0, 0)$.
By lemma 4 we get

$$|L_1(X)| \leqslant a_1 4c^6 3x_3^2 \sup_{Y' \in CK(X',\delta)} |(cR)^{-\nu-1}J_{-\nu-1}(cR)| \int_{E_2} |f_1(Y')dY' \leqslant \frac{\varepsilon}{4}$$

for

$$0 < x_3 < \eta_1 .$$

Similarly

$$|L_3(X)| \leqslant \frac{\varepsilon}{4} \quad \text{for} \quad 0 < x_3 < \eta_3 .$$

By analogy we can verify that there exist numbers η_2, η_4 such that

$$|L_i(X)| < \frac{\varepsilon}{4} \quad \text{for} \quad x_3 < \eta_i \; i = 2, 4$$

and finally,

$$|R_2(X)| \leqslant \varepsilon \quad \text{for} \quad 0 < x_3 < \min\{\eta_i \colon i = 1, 2, 3, 4\} .$$

LEMMA 8. *If the function f_2 is absolutely integrable in E_2 and continuous at the point X_0', then*

$$W_2(X) = \int_{E_2} f_2(Y')[2x_3^{-1}K_2(X, Y') - K_1(X, Y')]dY' \to f_2(X_0') \quad as \quad X \to (x_1^0, x_2^0, 0) .$$

By lemma 5 we obtain

$$-\int_{E_2} f_2(Y')K_1(X, Y')dY' \to -f_2(X_0') \quad \text{as} \quad X \to (x_1^0, x_2^0, 0) .$$

Similarly as in the proof lemma 5 we can prove that

$$\int_{E_2} f_2(Y')2x_3^{-1}K_2(X, Y')dY' \to 2f_2(X_0') \quad \text{as} \quad X \to (x_1^0, x_2^0, 0) .$$

Finally we get

$$W_2(X) \to f_2(X_0') \quad \text{as} \quad X \to (x_1^0, x_2^0, 0) .$$

From lemma 2 and theorem 1 we obtain

LEMMA 9. *If the functions f_i $(i = 1, 2)$ are absolutely integrable in E_2, then the function $u(X)$ defined by formulae* (4) *or* (5) *satisfies equation* (1) *for $x_3 > 0$.*
By lemmas 5, 6, 7, 8, 9 we obtain the fundamental

THEOREM 2. *If functions f_1, f_2 are absolutely integrable in E_2, continuous at the point X_0' and the function f_1 is of class C^1 at the circle $K(X_0', \delta^*)$ $(\delta^* > 0)$, then the function*

$$U(X) = \int_{E_2} [f_1(Y')K_1(X, Y') + f_2(Y')K_2(X, Y')]\,dY'$$

satisfies the equation (1) *and boundary conditions* (2), (3).

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