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On some homomorphisms in Ehresmann groupoids

Introduction. In the first section of this paper we consider the extensibility of homomorphisms defined on some subsystems of Ehresmann groupoids. We generalize the results of J. Aczél, J. A. Baker, D. Ž. Djoković, P. Kannappan, F. Rado, given in [2]. In [1] J. Aczél solved the functional equation

$$K[F(x,z), F(y,z)] = F(x,y),$$

where K is the operation inverse to the group operation. In the second part of this note we generalize this result solving the functional equation

$$F(x) \circ_2^{-1} F(y) = F(x \circ_1^{-1} y),$$

where $\circ_1^{-1}$, $\circ_2^{-1}$ are operations inverse to operations in Ehresmann groupoids. Besides in the second part of this paper we solved the equation

$$F(x) \bullet F(y) = F(x \circ^{-1} y),$$

where $\bullet$ is the group operation and $\circ^{-1}$ is the operation inverse to the operation in Ehresmann groupoid.

Preliminary. In [5] W. Waliszewski gave the following definitions of Ehresmann groupoid and Brandt groupoid: The pair $(E, \bullet)$, where E is a non-empty set and $\bullet$ is a binary interior operation defined for some pairs $(x, y) \in E \times E$, will be called Ehresmann groupoid if the following axioms are satisfied

(a) If in the equation

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

one of its sides or both of the products $y \cdot z$, $x \cdot y$ are defined, then both sides of this equation are defined and the equality holds,

(b) For every element $x \in E$ there exists exactly one left unit f_x and exactly one right unit e_x such that

$$f_x \cdot x = x \cdot e_x = x,$$

(c) If the product $x \cdot y$ is defined, then $e_x = f_y$,

(d) For every element $x \in E$ there exists exactly one element x^{-1} (inverse to x) such that

$$x \cdot x^{-1} = f_x, \, x^{-1} \cdot x = e_x .$$

An Ehresmann groupoid $(E, \bullet)$ will be called a Brandt groupoid, if the following condition holds.

(e) For every two elements $x, y \in E$ there exists such an element $z \in E$ that the products $x \cdot z$, $z \cdot y$ are defined.

Let $(E, \bullet)$ be an Ehresmann groupoid. If the product $x_0 \cdot \ldots \cdot x_n$ is defined, then we write $x_0 \cdot \ldots \cdot x_n = \prod_{i=0}^{n} x_i$. If $S_i \subset E$ for $i = 0, 1, 2, \ldots, n$, then $\prod_{i=0}^{n} S_i$ denotes the set of all product $\prod_{i=0}^{n} x_i$, where $x_i \in S_i$. If $S \subset E$, then S^{-1} denotes the set of all elements inverse to the elements of the set S. E^0 denotes the set of all elements $e \in E$ such that the product $e \cdot e$ is defined and $e \cdot e = e$. Moreover, we call S the subsystem of $(E, \bullet)$, if $S \cdot S \subset S$.

Let $(E_1, \bullet)$, $(E_2, \circ)$ be Ehresmann groupoids. We say, that the function F: $E_1 \to E_2$ is a homomorphism of $(E_1, \bullet)$ into $(E_2, \circ)$ if for arbitrary $x, y \in E_1$ such that the product $x \cdot y$ is defined the product $F(x) \circ F(y)$ is defined and the equality

$$F(x) \circ F(y) = F(x \cdot y)$$

holds. We mean the same, when we say, that the function F satisfies the above functional equation.

Let A be an arbitrary non-empty set, G an arbitrary group. In the set $A \times A \times G$ we define the operation $*$ as follows: The product $(a, b, \alpha) * (c, d, \beta)$ is defined iff $b = c$, and then

$$(a, b, \alpha) * (c, d, \beta) = (a, d, \alpha\beta) .$$

It is easy to verify, that the set $A \times A \times G$ with such an operation is a Brandt groupoid. This groupoid will be called a product Brandt groupoid.

A. Nijenhuis has proved the theorem which can be formulated in the following way ([4], p. 11): Every Brandt groupoid is isomorphic to some product Brandt groupoid.

On some extensions of homomorphisms in Ehresmann groupoids

LEMMA 1. *Let E_1, E_2 be Ehresmann groupoids, let S be a subsystem of E_1 such that $E_1^0 \subset S$, let h be a homomorphism of S into E_2, let n be an arbitrary positive integer such that*

$$E_1 = \prod_{i=0}^{n} S^{(-1)^i} \tag{1}$$

and let k be an arbitrary non-negative integer. Moreover for arbitrary sequences $(x_0, \ldots, x_{n-1}) \in S^n$, $(y_0, \ldots, y_{n-1}) \in S^n$ such that the products $\prod_{i=0}^{n-1} x_i^{(-1)^i}$, $\prod_{i=0}^{n-1} y_i^{(-1)^i}$ are *defined and the equality*

(2)
$$\prod_{i=0}^{n-1} x_i^{(-1)^i} = \prod_{i=0}^{n-1} y_i^{(-1)^i}$$

holds, let the products $\prod_{i=0}^{n-1} (h(x_i))^{(-1)^i}$, $\prod_{i=0}^{n-1} (h(y_i))^{(-1)^i}$ *be defined and the equality*

(3)
$$\prod_{i=0}^{n-1} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n-1} (h(y_i))^{(-1)^i}$$

holds. Then for arbitrary sequences $(x_0, \ldots, x_n) \in S^{n+1}$, $(y_0, \ldots, y_{n+k}) \in S^{n+k+1}$ *such that the products* $\prod_{i=0}^{n} x_i^{(-1)^i}$, $\prod_{i=0}^{n+k} y_i^{(-1)^i}$ *are defined and the equality*

(4)
$$\prod_{i=0}^{n} x_i^{(-1)^i} = \prod_{i=0}^{n+k} y_i^{(-1)^i}$$

holds the products $\prod_{i=0}^{n} (h(x_i))^{(-1)^i}$, $\prod_{i=0}^{n+k} (h(y_i))^{(-1)^i}$ *are defined and the equality*

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n+k} (h(y_i))^{(-1)^i}$$

holds.

Proof. We shall proceed by induction on k. Let $k = 0$ and let $(x_0, \ldots, x_n) \in S^{n+1}$, $(y_0, \ldots, y_n) \in S^{n+1}$ be arbitrary sequences such that the products $\prod_{i=0}^{n} x_i^{(-1)^i}$, $\prod_{i=0}^{n} y_i^{(-1)^i}$ are defined and the equality

(5)
$$\prod_{i=0}^{n} x_i^{(-1)^i} = \prod_{i=0}^{n} y_i^{(-1)^i}$$

holds. We can write equality (5) in the form

(6)
$$x_0 \cdot \prod_{i=1}^{n} x_i^{(-1)^i} \, . \, [y_n^{(-1)^n}]^{-1} = \prod_{i=0}^{n-1} y_i^{(-1)^i}$$

In virtue of (1) it follows, that there exists a sequence $(z_0, \ldots, z_n) \in S^{n+1}$ such that the product $\prod_{i=0}^{n} z_i^{(-1)^i}$ is defined and

(7)
$$\prod_{i=1}^{n} x_i^{(-1)^i} \cdot [y_n^{(-1)^n}]^{-1} = \prod_{i=0}^{n} z_i^{(-1)^i}.$$

Hence

(8)
$$\prod_{i=2}^{n} x_i^{(-1)^i} \cdot [y_n^{(-1)^n}]^{-1} \cdot [z_n^{(-1)^n}]^{-1} = x_1 \cdot z_0 \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i} .$$

From (6) and (7) we obtain

$$x_0 \cdot \prod_{i=0}^{n} z_i^{(-1)^i} = \prod_{i=0}^{n-1} y_i^{(-1)^i},$$

whence

(9) $$x_0 \cdot z_0 \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i} = \prod_{i=0}^{n-2} y_i^{(-1)^i} \cdot y_{n-1}^{(-1)^{n-1}} \cdot [z_n^{(-1)^n}]^{-1}.$$

We have

(10) $$[y_n^{(-1)^n}]^{-1} \cdot [z_n^{(-1)^n}]^{-1} = \begin{cases} (z_n \cdot y_n)^{-1} & \text{for even number } n, \\ y_n \cdot z_n & \text{for odd number } n, \end{cases}$$

and

(11) $$y_{n-1}^{(-1)^{n-1}} \cdot [z_n^{(-1)^n}]^{-1} = \begin{cases} (z_n \cdot y_{n-1})^{-1} & \text{for even number } n, \\ y_{n-1} \cdot z_n & \text{for odd number } n. \end{cases}$$

For (8) and (9) by (10) and (11) we obtain for even number n

(12) $$\prod_{i=2}^{n} x_i^{(-1)^i} \cdot (z_n \cdot y_n)^{-1} = x_1 \cdot z_0 \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i}$$

and

(13) $$x_0 \cdot z_0 \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i} = \prod_{i=0}^{n-2} y_i^{(-1)^i} \cdot (z_n \cdot y_{n-1})^{-1}.$$

From (12), (13) and the assumption of the lemma it follows, that for the sequences $(x_2, \dots, x_n, z_n \cdot y_n)$, $(x_1 \cdot z_0, z_1, \dots, z_{n-1})$ and the sequences $(x_0 \cdot z_0, z_1, \dots, z_{n-1})$, $(y_0, \dots, y_{n-2}, z_n \cdot y_{n-1})$ the products below are defined and the equalities

(14) $$\prod_{i=2}^{n} (h(x_i))^{(-1)^i} \cdot (h(z_n \cdot y_n))^{-1} = h(x_1 \cdot z_0) \cdot \prod_{i=1}^{n-1} (h(z_i))^{(-1)^i}$$

and

$$h(x_0 \cdot z_0) \cdot \prod_{i=1}^{n-1} (h(z_i))^{(-1)^i} = \prod_{i=0}^{n-2} (h(y_i))^{(-1)^i} \cdot (h(z_n \cdot y_{n-1}))^{-1}$$

hold.

Since h is the homomorphism from S into E_2, therefore from (14) and (15) we obtain, respectively

(16) $$\prod_{i=1}^{n} (h(x_i))^{(-1)^i} \cdot (h(y_n))^{-1} = \prod_{i=0}^{n} (h(z_i))^{(-1)^i}$$

and

(17) $$h(x_0) \cdot \prod_{i=0}^{n} (h(z_i))^{(-1)^i} = \prod_{i=0}^{n-1} (h(y_i))^{(-1)^i}.$$

From (16) and (17) we get

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n} (h(y_i))^{(-1)^i} .$$

Thus for $k = 0$ and the even number n the lemma is true. Let now n be an odd number. Similarly as above, from (8), (9), (10), (11) we obtain

$$\prod_{i=2}^{n} x_i^{(-1)^i} \cdot (y_n \cdot z_n) = (x_1 \cdot z_0) \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i} \tag{17'}$$

and

$$(x_0 \cdot z_0) \cdot \prod_{i=1}^{n-1} z_i^{(-1)^i} = \prod_{i=0}^{n-2} y_i^{(-1)^i} (y_{n-1} \cdot z_n) . \tag{18}$$

In virtue of (17′), (18) and the assumption of the lemma it follows, that for the sequences $(x_2, \ldots, x_n, y_n \cdot z_n)$, $(x_1 \cdot z_0, z_1, \ldots, z_{n-1})$ and the sequences

$$(x_0 \cdot z_0, z_1, \ldots, z_{n-1}) , \; (y_1, \ldots, y_{n-2}, y_{n-1} \cdot z_n)$$

the products below are defined and the equalities

$$\prod_{i=2}^{n} (h(x_i))^{(-1)^i} \cdot h(y_n \cdot z_n) = h(x_1 \cdot z_0) \cdot \prod_{i=1}^{n-1} (h(z_i))^{(-1)^i} , \tag{19}$$

$$h(x_0 \cdot z_0) \cdot \prod_{i=1}^{n-1} (h(z_i))^{(-1)^i} = \prod_{i=0}^{n-2} (h(y_i))^{(-1)^i} \cdot h(y_{n-1} \cdot z_n) \tag{20}$$

hold. h is a homomorphism from S into E_2, thus from (19) and (20) we get

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n} (h(y_i))^{(-1)^i} ,$$

which completes the proof in the case when $k = 0$. Let now lemma 1 be true for some positive integer k. Let $(x_0, \ldots, x_n) \in S^{n+1}$, $(y_0, \ldots, y_n, y_{n+1}, \ldots, y_{n+k+1}) \in$ $\in S^{n+k+2}$ be arbitrary sequences such that the below products are defined and the equality

$$\prod_{i=0}^{n} x_i^{(-1)^i} = \prod_{i=0}^{n+k+1} y_i^{(-1)^i} \tag{21}$$

holds. From (1) it follows, that there exists a sequence $(z_0, \ldots, z_n) \in S^{n+1}$ such that the product $\prod_{i=0}^{n} z_i^{(-1)^i}$ is defined and

$$\prod_{i=0}^{n} z_i^{(-1)^i} = \prod_{i=1}^{n+k+1} y_i^{(-1)^i} . \tag{22}$$

Hence

$$(y_1 \cdot z_0) \cdot \prod_{i=1}^{n} z_i^{(-1)^i} = \prod_{i=2}^{n+k+1} y_i^{(-1)^i} (e_{y_{n+k+1}^{(-1)^{n+k+1}}})^{(-1)^{n+k+2}} . \tag{23}$$

From (23) it follows, that for the sequences

$$(y_1 \cdot z_0, z_1, \ldots, z_n), \ (y_2, \ldots, y_{n+k+1}, e_{y_{n+k+1}^{(-1)^{n+k+1}}})$$

the inductive assumption is fulfilled.
Thus

$$(24) \qquad h(y_1 \cdot z_0) \cdot \prod_{i=1}^{n} (h(z_i))^{(-1)^i} = \prod_{i=2}^{n+k+1} (h(y_i))^{(-1)^i} \cdot [h(e_{y_{n+k+1}^{(-1)^{n+k+1}}})]^{(-1)^{n+k+2}}.$$

Since h is the homomorphism from S into E_2, thus $h(e_{y_{n+k+1}^{(-1)^{n+k+1}}}) \in E_2^0$ and from (24) we obtain

$$(25) \qquad \prod_{i=0}^{n} (h(z_i))^{(-1)^i} = \prod_{i=1}^{n+k+1} (h(y_i))^{(-1)^i}.$$

From (21) and (22) we get

$$(26) \qquad \prod_{i=0}^{n} x_i^{(-1)^i} = (y_0 \cdot z_0) \cdot \prod_{i=1}^{n} z_i^{(-1)^i}.$$

We have proved above, that lemma 1 is true for $k = 0$. Therefore by (26) we get, that the products below are defined and the equality

$$(27) \qquad \prod_{i=0}^{n} (h(x_i))^{(-1)^i} = h(y_0 \cdot z_0) \cdot \prod_{i=1}^{n} (h(z_i))^{(-1)^i}$$

holds.
Since h is a homomorphism, thus by (27) we have

$$(28) \qquad \prod_{i=0}^{n} (h(x_i))^{(-1)^i} = h(y_0) \cdot \prod_{i=0}^{n} (h(z_i))^{(-1)^i}.$$

Comparing (28) with (25) we see that the equality

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n+k+1} (h(y_i))^{(-1)^i}$$

holds, which completes the proof of lemma 1.

Remark 1. One can prove, that from (1), (2) and (3) it follows, that every homomorphism from S (not necessary, such that $E_1^0 \subset S$) can be uniquelly extended on the set $S \cup E_1^0$. Thus the assumption $E_1^0 \subset S$ is not essential in lemma 1.

THEOREM 1. *Let E_1, E_2 be Ehresmann groupoids, let n be an arbitrary positive integer, let S be a subsystem of E_1 such that $E_1^0 \subset S$ and*

$$(1) \qquad E_1 = \prod_{i=0}^{n} S^{(-1)^i},$$

and let h be a homomorphism from S into E_2. Then the homomorphism h can be extended to a homomorphism H from E_1 into E_2 iff the following condition is fulfilled: for arbi-

trary sequences $(x_0, \ldots, x_{n-1}) \in S^n$, $(y_0, \ldots, y_{n-1}) \in S^n$ *such that the products* $\prod_{i=0}^{n-1} x_i^{(-1)^i}$, $\prod_{i=0}^{n-1} y_i^{(-1)^i}$ *are defined and the equality*

$$\prod_{i=0}^{n-1} x_i^{(-1)^i} = \prod_{i=0}^{n-1} y_i^{(-1)^i} \tag{2}$$

holds, the products $\prod_{i=0}^{n-1} (h(x_i))^{(-1)^i}$, $\prod_{i=0}^{n-1} (h(y_1))^{(-1)^i}$ *are defined and the equality*

$$\prod_{i=0}^{n-1} (h(x_i))^{(-1)^i} = \prod_{i=0}^{n-1} (h(y_i))^{(-1)^i} \tag{3}$$

holds.

If the extension H exists, then it is unique.

Proof. Let H be the extension of h onto the set E_1. Let the products $\prod_{i=0}^{n-1} x_i^{(-1)^i}$, $\prod_{i=0}^{n-1} y_i^{(-1)^i}$ be defined for the sequences $(x_0, \ldots, x_{n-1}) \in S^n$, $(y_0, \ldots, y_{n-1}) \in S^n$ and let equality (2) hold. H is a homomorphism from E_1 into E_2, thus the products $\prod_{i=0}^{n-1} (H(x_i))^{(-1)^i}$, $\prod_{i=0}^{n-1} (H(y_i))^{(-1)^i}$ are defined and the equality

$$\prod_{i=0}^{n-1} (H(x_i))^{(-1)^i} = \prod_{i=0}^{n-1} (H(y_i))^{(-1)^i} \tag{29}$$

holds. H is an extension of h, therefore from (29) we get equality (3), which completes the first part of the proof. Let us put

$$H(x) = \prod_{i=0}^{n} (h(x_i))^{(-1)^i} \quad \text{for } x \in E_1 , \tag{30}$$

where $(x_0, \ldots, x_n) \in S^{n+1}$ is an arbitrary sequence such that the product $\prod_{i=0}^{n} x_i^{(-1)^i}$ is defined and $x = \prod_{i=0}^{n} x_i^{(-1)^i}$. From lemma 1 (the case when $k = 0$) it follows, that H is a function from E_1 into E_2. Let x, y, z be arbitrary elements of the set E_1 such that

$$x \cdot y = z . \tag{31}$$

In virtue of (1) it follows, that there exist sequences $(x_0, \ldots, x_n) \in S^{n+1}$, $(y_0, \ldots, y_n) \in S^{n+1}$, $(z_0, \ldots, z_n) \in S^{n+1}$ such that the products below are defined and the equalities

$$x = \prod_{i=0}^{n} x_i^{(-1)^i}, \; y = \prod_{i=0}^{n} y_i^{(-1)^i}, \; z = \prod_{i=0}^{n} z_i^{(-1)^i} \tag{32}$$

hold. By (31), (32) we get

$$\prod_{i=0}^{n} x_i^{(-1)^i} \cdot \prod_{i=0}^{n} y_i^{(-1)^i} = \prod_{i=0}^{n} z_i^{(-1)^i} .$$

Concerning lemma 1 it is easy to show, that the products

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} \cdot \prod_{i=0}^{n} (h(y_i))^{(-1)^i}, \prod_{i=0}^{n} (h(z_i))^{(-1)^i}$$

are defined and the equality

$$\prod_{i=0}^{n} (h(x_i))^{(-1)^i} \cdot \prod_{i=0}^{n} (h(y_i))^{(-1)^i} = \prod_{i=0}^{n} (h(z_i))^{(-1)^i}$$

holds. Thus $H(x)\cdot H(y) = H(x\cdot y)$, i.e. H is a homomorphism from E_1 into E_2. Now we shall show, that H is the extension of h. Let $x \in S$. Since $E^0 \subset S$, thus for $x \in S$ we have

$$x = x\cdot \prod_{i=0}^{n} e_x^{(-1)^i} . \tag{33}$$

Since H is a homomorphism and by (30) we get

$$H(x) = h(x)\cdot \prod_{i=1}^{n} (h(e_x))^{(-1)^i} = h(x),$$

thus H is an extension of h. From (30) it follows, that H as defined by (30) is the unique extension of h, which completes the proof of theorem 1.

Remark 2. By remark 1 we can conclude, that the assumption $E_1^0 \subset S$ is not essential in theorem 1. In particular cases, when Ehresmann groupoids E_1, E_2 in theorem 1 are groups and $n = 1, 2, 3, 4$, we obtain theorems 2, 3, 4, 5 from the note [2], respectively.

On the solutions of the equation $F(x, z, \alpha)\cdot F(y, z, \beta) = F(x, y, \alpha\cdot\beta^{-1})$

J. Aczél has proved ([1], p. 41–42) the following theorem

THEOREM 2. *Let A be an arbitrary set, $(B, \bullet)$ an arbitrary group and K an operation in B defined as follows*

$$K(x, y) = x\cdot y^{-1} \quad \textit{for} \quad x, y \in B .$$

Then the function F: $A \times A \to B$ is the solution of the equation

$$K[F(x, z), F(y, z)] = F(x, y)$$

iff it has the form $\quad F(x, y) = f(x)\cdot(f(y))^{-1} \quad$ *for* $\quad x, y \in A$,

where f is an arbitrary function mapping A into B.

We are going to generalize this result. Let $(E, \bullet)$ be an Ehresmann groupoid. We call the operation K the operation inverse to the operation $\bullet$, iff the following conditions are fulfilled:

(34)
a) $K(x, y)$ is defined iff $x\cdot y^{-1}$ is defined,
b) if $K(x, y)$ is defined, than $K(x, y) = x\cdot y^{-1}$.

THEOREM 3. *Let* $(E_1, \cdot_1)$, $(E_2, \cdot_2)$ *be Ehresmann groupoids, let* K_1, K_2 *be operations inverse to the operations* $\cdot_1$, $\cdot_2$, *respectively. Then the function* $F\colon E_1 \to E_2$ *is the solution of the equation*

$$K_2[F(x), F(y)] = F[K_1(x, y)] \tag{35}$$

iff it satisfies the equation

$$F(x) \cdot_2 F(y) = F(x \cdot_1 y)\,. \tag{36}$$

Proof. Let the function F satisfy equation (35) and let $e \in E_1^0$. We obtain

$$F(e)\cdot[F(e)]^{-1} = K_2[F(e), F(e)] = F\big(K_1(e, e)\big) = F(e\cdot e^{-1}) = F(e) \qquad \text{i.e.}$$

$$F(e) \in E_2^0\,. \tag{37}$$

Let us replace x by $f_{y^{-1}}$ in (35). By (34) we get

$$F(f_{y^{-1}}) \cdot_2 [F(y)]^{-1} = F(y^{-1})\,,$$

and hence, by (37)

$$[F(y)]^{-1} = F(y^{-1})\,. \tag{38}$$

In virtue of (34) and (38) we can conclude, that equations (35) and (36) are equivalent, which completes the proof.

THEOREM 4. *Let* $(A\times A\times G, *)$ *be an arbitrary product Brandt groupoid, let* $(E, \cdot)$ *be an arbitrary Ehresmann groupoid. Then the function* $F\colon A\times A\times G \to E$ satisfies the equation

$$F(x, y, \alpha)\cdot F(y, z, \beta) = F(x, z, \alpha\cdot\beta) \tag{39}$$

iff it has the following form

$$F(x, y, \alpha) = f(x)\cdot g(\alpha)\cdot[f(y)]^{-1} \quad \textit{for} \quad (x, y, \alpha) \in A\times A\times G \tag{40}$$

where f *is an arbitrary mapping from* A *into* E, g *is an arbitrary homomorphism of the group* $(G, \cdot)$ *into* $(E, \cdot)$ *and for arbitrary* $(x, y, \alpha) \in A\times A\times G$ *the product* $f(x)\cdot g(\alpha)\cdot[f(y)]^{-1}$ *is defined (thus the functions of the form* (40) *are the only homomorphisms of the product Brandt groupoid* $(A\times A\times G, *)$ *into the Ehresmann groupoid* $(E, \cdot)$*).*

We omit the proof of the above theorem, because it is quite similar to the proof of this theorem in the case, when $(E, \cdot)$ is a group, which was given by A. Grząślewicz ([3], p. 16). In virtue of theorems 3 and 4 we obtain the following

THEOREM 5. *Let* $(A\times A\times G, *)$ *be an arbitrary product Brandt groupoid and let* $(E, \cdot)$ *be an arbitrary Ehresmann groupoid. The function* $F\colon A\times A\times G \to E$ *satisfies the equation*

$$F(x, z, \alpha)\cdot F(y, z, \beta^{-1}) = F(x, y, \alpha\cdot\beta^{-1}) \tag{41}$$

iff it has the form (40).

It is easy to see, that in a particular case, when $(G, \bullet)$ is an one-element group we obtain from theorem 5 the result obtained by J. Aczél (theorem 1).

Let now $(E, \circ)$ be an arbitrary Ehresmann groupoid, let $(H, \bullet)$ be an arbitrary group with the unit e' and let $\circ_1$ be the operation inverse to the operation $\circ$. We shall consider the functional equation

$$F(x)\cdot F(y) = F(x\circ_1 y)\,. \tag{42}$$

THEOREM 6. *The function F: $E \to H$ is the solution of equation* (42) *iff it is the solution of the equation*

$$F(x)\cdot F(y) = F(x\circ y) \tag{43}$$

and

$$[F(x)]^2 = e' \quad \textit{for all} \quad x \in E\,.$$

Proof. Let F: $E \to H$ be the solution of equation (42) and let $e \in E^0$. By (42) we have

$$F(e)\cdot(F(e) = F(e)\,,$$

i.e.

$$F(e) = e'\,. \tag{44}$$

Let x be an arbitrary element belonging to the set E. Then $x \circ_1 x$ is defined and from (42) we obtain

$$F(x)\cdot F(x) = F(x\circ x^{-1}) = F(f_x)\,,$$

whence, by (44),

$$[F(x)]^2 = e'\,. \tag{45}$$

Moreover, for $x \in E$ we get

$$F(x^{-1}) = F(e_x\circ x^{-1}) = F(e_x\circ_1 x) = F(e_x)\cdot F(x)\,,$$

whence, by (44)

$$F(x^{-1}) = F(x)\,. \tag{46}$$

Let now the product $x \circ y$ by defined. Then the product $x \circ_1 y^{-1}$ is defined and by (42), (46) we get $F(x\circ y) = F(x\circ_1 y^{-1}) = F(x)\cdot F(y^{-1}) = F(x)\cdot F(y)$, which completes, the first part of the proof. Let us consider the function F: $E \to H$ satisfying equation (43), let $[F(x)]^2 = e'$ for every $x \in E$ and let $x \circ_1 y$ be defined. It is easy to see, that then $F(y^{-1}) = F(y)$ and by (43) and (45) we obtain $F(x\circ_1 y) = F(x\circ y^{-1}) = F(x)\cdot F(y^{-1}) = F(x)\cdot[F(y)]^{-1}\cdot F(y)\cdot F(y) = F(x)\cdot F(y)$, which completes the proof.

In the particular case, when in theorem 6 $(E, \bullet)$ is the product Brandt groupoid $(A\times A\times G, *)$, equation (42) has the form

$$F(x, z, \alpha)\cdot F(y, z, \beta) = F(x, y, \alpha\cdot\beta^{-1})\,. \tag{47}$$

Corollary 1. *The function* $F: A\times A\times G\to H$ *satisfies the equation* (47) *iff it has the form*

$$F(x, y, \alpha) = k(x)\cdot h(\alpha)\cdot k(y), \tag{48}$$

where k *is an arbitrary function mapping* A *into* H *such that* $[k(x)]^2 = e'$ *and* $k(x)\cdot k(y) = k(y)\cdot k(x)$ *for all* $x, y\in A$, h *is an arbitrary homomorphism from the group* $(G, \bullet)$ *into the group* $(H, \bullet)$ *such that* $[g(\alpha)]^2 = e'$ *for every* $\alpha\in G$ *and* $k(x)\cdot g(\alpha) = g(\alpha)\cdot k(x)$ *for all* $\alpha\in G$, $x\in A$.

Proof. Let $F: A\times A\times G\to H$ satisfy equation (47). In virtue of theorem 6 F satisfies the equation

$$F(x, y, \alpha)\cdot F(y, z, \beta) = F(x, z, \alpha\cdot\beta) \tag{49}$$

and $[F(x, y, \alpha)]^2 = e'$ for $(x, y, \alpha)\in A\times A\times G$. By theorem 4 F has the form (40). Thus for $\alpha\in G$, $x\in A$ we obtain

$$e' = [F(x, x, \alpha)]^2 = f(x)\cdot g(\alpha)\cdot [f(x)]^{-1}\cdot f(x)\cdot g(\alpha)\cdot [f(x)]^{-1} = \\ = f(x)\cdot [g(\alpha)]^2\cdot [f(x)]^{-1}$$

whence

$$(g(\alpha))^2 = e'. \tag{50}$$

For $\alpha = e$ and $x, y\in A$

we get

$$e' = [F(x, y, e)]^2 = f(x)\cdot [f(y)]^{-1}\cdot f(x)\cdot [f(y)]^{-1},$$

whence

$$f(x)\cdot [f(y)]^{-1} = f(y)\cdot [f(x)]^{-1}. \tag{51}$$

Let us put

$$k(x) = f(x)\cdot [f(a)]^{-1} \quad \text{for} \quad x\in A,$$
$$h(\alpha) = f(a)\cdot g(\alpha)\cdot [f(a)]^{-1} \quad \text{for} \quad \alpha\in G,$$

where a is a fixed element in A. For α, $\beta\in G$ we have

$$h(\alpha)\cdot h(\beta) = f(a)g(\alpha)\cdot [f(a)]^{-1}\cdot f(a)\cdot g(\beta)[f(a)]^{-1} = \\ = f(a)g(\alpha\cdot\beta)[f(a)]^{-1} = h(\alpha\cdot\beta)$$

and by (45)

$$[h(\alpha)]^2 = [f(a)\cdot g(\alpha)\cdot (f(a))^{-1}]^2 = [F(a, a, \alpha)]^2 = e'.$$

By (51), (46) and (50) we obtain

$$h(\alpha)\cdot k(x) = f(a)\cdot g(\alpha)\cdot [f(a)]^{-1}\cdot f(x)\cdot [f(a)]^{-1} = \\ = f(a)\cdot g(\alpha)\cdot [f(a)]^{-1}\cdot f(a)\cdot [f(x)]^{-1} = f(a)\cdot g(\alpha)\cdot [f(x)]^{-1} = \\ = F(a, x, \alpha) = F(x, a, \alpha^{-1}) = f(x)\cdot g(\alpha^{-1})\cdot [f(a)]^{-1} = \\ = f(x)\cdot [f(a)]^{-1}\cdot f(a)\cdot g(\alpha)\cdot [f(a)]^{-1} = k(x)\cdot h(\alpha)$$

for $x \in A$, $\alpha \in G$.
Moreover

$$k(x)\cdot k(y) = f(x)\cdot[f(a)]^{-1}\cdot f(y)\cdot[f(a)]^{-1} = f(x)\cdot[f(a)]^{-1}\cdot f(a)\cdot[f(y)]^{-1} =$$
$$= f(x)\cdot[f(y)]^{-1} = f(y)\cdot[f(x)]^{-1} = f(y)\cdot[f(a)]^{-1}\cdot f(a)\cdot[f(x)]^{-1} =$$
$$= f(y)\cdot[f(a)]^{-1}\cdot f(x)\cdot[f(a)]^{-1} = k(y)\cdot k(x)$$

and

$$k(x)\cdot h(\alpha)\cdot k(y) = f(x)\cdot[f(a)]^{-1}\cdot f(a)\cdot g(\alpha)\cdot[f(a)]^{-1}\cdot f(y)\cdot[f(a)]^{-1} =$$
$$= f(x)\cdot g(\alpha)\cdot[f(a)]^{-1}\cdot f(a)\cdot[f(y)]^{-1} = f(x)\cdot g(\alpha)\cdot[f(y)]^{-1} =$$
$$= F(x, y, \alpha),$$

which completes the first part of the proof. It is easy to verify, that every function F of the form (48) satisfies equation (47). It completes the proof.

References

[1] J. Aczél: *Miszellen über Funktionalgleichungen. II.* Mathemathische Nachrichten 23. Band, Heft 1, 1961, Akademie-Verlag, Berlin.

[2] J. Aczél, J. A. Baker, D. Ž. Djoković, P. Kannappan, F. Rado: *Extension of certain homomorphisms of semigroups to homomorphisms of groups.* Aequat. Math. 6 (1971), p. 263–271.

[3] A. Grząślewicz: *O pewnych homomorfizmach i homomorfizmach ciągłych grupoidu Brandta.* Rocznik Nauk.-Dydak. WSP w Krakowie, Prace Mat. VI, 41, (1970).

[4] A. Nijenhuis: *Theory of the geometric object.* Amsterdam 1952.

[5] W. Waliszewski: *Categories, groupoids, pseudogroups and analitical structures.* Rozprawy Mat. 45, Warszawa 1965.