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# On some homomorphisms in product Brandt groupoids

**Basic notations.** By $f\colon A \twoheadrightarrow B$ we shall denote the function (called partial function) the domain of which is contained in the set $A$ and the range of which is contained in the set $B$. The domain of the function $f$ will be denoted by $D_f$ and the range will be denoted by $\mathcal{Q}_f$. If $F$ is a function of the form

$$F\colon A \times B \times G \twoheadrightarrow K,$$

then we shall denote:

$$D_F^1 = \{x \in A\colon \bigvee_{y \in B} \bigvee_{\alpha \in G} ((x, y, \alpha) \in D_F)\},$$

$$D_F^2 = \{y \in B\colon \bigvee_{x \in A} \bigvee_{\alpha \in G} ((x, y, \alpha) \in D_F)\},$$

$$D_F^3 = \{\alpha \in G\colon \bigvee_{x \in A} \bigvee_{y \in B} ((x, y, \alpha) \in D_F)\}.$$

Let $B$ be an arbitrary non-empty set and let $\bullet$ be an arbitrary partial mapping of the set $B \times B$ in the set $B$. We shall call the pair $(B, \bullet)$ the m u l t i p l i c a t i v e s y s t e m. When $S \subset B$ and $S$ with the operation $\bullet$ restricted to the set $S$ is a multiplicative system, then we call $S$ the s u b s y s t e m of the multiplicative system $(B, \bullet)$. The multiplicative system $(B, \bullet)$ will be called a s s o c i a t i v e, if the following condition is satisfied:
If in the equation

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

one of its sides or both of the products $y \cdot z$, $x \cdot y$ are defined, then both sides of the equation are defined and the equality holds. If in the associative multiplicative system $(B, \bullet)$ the following conditions are satisfied:

a) For every element $x$ of $B$ there exists exactly one left unit $f_x$ and exactly one right unit $e_x$ such that $f_x \cdot x = x \cdot e_x = x$,

b) If the product $x \cdot y$ is defined, then $e_x = f_y$,

c) For every element $x$ of $B$ there exists exactly one element $x^{-1}$ (inverse to $x$) such that $x \cdot x^{-1} = f_x$, $x^{-1} \cdot x = e_x$,

d) For every of the element $x$, $y$ there exists such an element $z$ that both the products $x \cdot z$, $z \cdot y$ are defined, then we call this system the Brandt groupoid ([2]).

Let $A$ be an arbitrary non-empty set and $G$ an arbitrary group. In the set $A \times A \times G$ we define the operation $*$ as follows:
The product $(a, b, \alpha) * (c, d, \beta)$ is defined iff $b = c$, and then

$$(a, b, \alpha) * (c, d, \beta) = (a, d, \alpha \cdot \beta).$$

It is easy to verify that the set $A \times A \times G$, with such an operation $*$ is a Brandt groupoid. This groupoid is called a product Brand groupoid ([3]). A. Nijenhuis has proved the theorem which can be formulated in the following way ([1], p. 11): Every Brandt groupoid is isomorphic to some product Brandt groupoid.

Let $(A, \bullet)$, $(B, \circ)$ be multiplicative systems. We shall say, that the function $F: A \twoheadrightarrow B$ is a solution of the equation

$$\text{(i)} \qquad F(x) \circ F(y) = F(x \cdot y)$$

if for every $x \in D_F$, $y \in D_F$, $(x, y) \in D_\bullet$, $x \cdot y \in D_F$ the product $F(x) \circ F(y)$ is defined and the equality (i) holds. We mean the same when we say that the function $F$ is a homomorphism of $(A, \bullet)$ into $(B, \circ)$.

We shall say that the function $\bar{F}: A \twoheadrightarrow B$ satisfying (i) is the extension of the solution $F$ of equation (i), if $\bar{F}|_{D_F} = F$. We mean the same when we say, that $\bar{F}$ is the extension of $F$ onto the set $D_F$.

**Extensions of some homomorphisms in the product Brandt groupoids.** Let $(A \times A \times G, *)$ be an arbitrary product Brandt groupoid and let $(K, \bullet)$ be an arbitrary associative multiplicative system.

Definition 1. *We will denote $\Gamma_1$ the set of all functions $F: A \times A \times G \twoheadrightarrow K$ such that the following conditions*

$$\text{(1)} \qquad \bigwedge_{(x,y,\alpha) \in A \times A \times G} \ \bigwedge_{\beta \in G} [(x, y, \alpha) \in D_F \Rightarrow (x, y, \beta) \in D_F],$$

$$\text{(2)} \qquad \bigvee_{a \in A} \ \bigwedge_{(x,y,\alpha) \in D_F} [(x, a, e) \in D_F \wedge (a, y, e) \in D_F] \text{ hold.}$$

Theorem 1. *Every solution $F \in \Gamma_1$ of the equation*

$$\text{(3)} \qquad F(x, y, \alpha) \cdot F(y, z, \beta) = F(x, z, \alpha \cdot \beta)$$

*can be uniquely extended onto the set $D_F^1 \times D_F^2 \times G$ and this extension belongs to $\Gamma_1$.*

Proof. Let $F \in \Gamma_1$ be an arbitrary solution of equation (3), let a be an arbitrary element satisfying (2) and let $(x, y, \alpha) \in D_F^1 \times D_F^2 \times G$. Then $(x, a, e) \in D_F$, $(x, a, \alpha) \in D_F$, $(a, y, e) \in D_F$, $(a, y, \alpha) \in D_F$, $(a, a, \alpha) \in D_F$. Since $F$ is a homomorphism then the products $F(x, a, e) \cdot F(a, a, \alpha)$ and $F(a, a, \alpha) \cdot F(a, y, e)$ are defined. Let us put $\bar{F}(x, y, \alpha) = F(x, a, e) \cdot F(a, a, \alpha) \cdot F(a, y, e)$ for $(x, y, \alpha) \in D_F^1 \times D_F^2 \times G$. It is easy to see that $\bar{F}|_{D_F} = F$.

Let $(x, y, \alpha) \in D_{\bar{F}}$ and $(y, z, \beta) \in D_{\bar{F}}$. We have

$$\bar{F}(x, y, \alpha) \cdot \bar{F}(y, z, \beta) = F(x, a, e) \cdot F(a, a, \alpha) \cdot (F(a, y, e) \cdot F(y, a, e)) F(a, a, \beta) \cdot$$
$$\cdot F(a, z, e) = F(x, a, e) \cdot (F(a, a, \alpha) \cdot F(a, a, e) \cdot$$
$$\cdot F(a, a, \beta)) F(a, z, e) = F(x, a, e) \cdot F(a, a, \alpha \cdot \beta) \cdot$$
$$\cdot F(a, z, e) = \bar{F}(x, z, \alpha \cdot \beta) .$$

Thus $\bar{F}$ is an extension of $F$. It is easy to see that $\bar{F} \in \Gamma_1$ and that $\bar{F}$ is the unique extension of $F$.

In virtue of theorem 1 we can restrict our considerations concerning the extensibility to these functions of the set $\Gamma_1$ for which the following condition

(4) $$D_F = D_F^1 \times D_F^2 \times G \text{ holds.}$$

DEFINITION 2. *We will denote by $\Gamma$ the set of all functions $F$: $A \times A \times G \twoheadrightarrow K$ such that conditions* (2) *and* (4) *are fulfilled.*

THEOREM 2. *The solution $F \in \Gamma$ of equation* (3) *can be extended on the set $(D_F^1 \cup D_F^2) \times (D_F^1 \cup D_F^2) \times G$ iff the equation*

(5) $$F(a, x, e) \cdot \mu_x = F(a, a, e)$$

*has a solution for every $x \in D_F^2$ and the equation*

(6) $$v_x \cdot F(x, a, e) = F(a, a, e)$$

*has a solution for every $x \in D_F^1$ ($a \in A$ is an arbitrary, fixed element satisfying* (2)).

Proof. Let $\bar{F}$ be an arbitrary extension of the solution $F \in \Gamma$ of equation (3) on the set $(D_F^1 \cup D_F^2) \times (D_F^1 \cup D_F^2) \times G$. It is easy to verify that $\mu_x = \bar{F}(x, a, e)$ and $v_x = \bar{F}(a, x, e)$ are solutions of equations (5) and (6), respectively. Now we go to prove the sufficiency of conditions (5), (6). Let us put

(7) $$\bar{F}(x, y, \alpha) = \mu_x \cdot F(a, a, \alpha) \cdot v_y \quad \text{for} \quad x, y \in D_F^1 \cup D_F^2, \alpha \in G$$

where $\mu_x$ and $v_x$ are solutions of equations (5) and (6), when $x \in D_F^2$ and $y \in D_F^1$ respectively and however $x \in D_F^1$ we put

(8) $$\mu_x = F(x, a, e) \text{ and when } y \in D_F^2 \text{ we put}$$

(9) $$v_y = F(a, y, e) .$$

It is easy to verify, that the function $\bar{F}$ is defined on the set $(D_F^1 \cup D_F^2) \times (D_F^1 \cup D_F^2) \times G$. From (8) and (9) it follows that $\bar{F}|_{D_F} = F$. Now we shall show, that $\bar{F}$ satisfies equation (3). Let $x$, $y$, $z$ be arbitrary elements from the set $D_F^1 \cup D_F^2$ and let $\alpha, \beta \in G$. If $y \in D_F^1$ we obtain

$$\bar{F}(x, y, \alpha) \cdot \bar{F}(y, z, \beta) = (\mu_x \cdot F(a, a, \alpha) \cdot v_y) \cdot (F(y, a, e) \cdot F(a, a, \beta) \cdot v_z) =$$
$$= u_x \cdot F(a, a, \alpha) \cdot (v_y \cdot F(y, a, e)) \cdot F(a, a, \beta) v_z =$$
$$= \mu_x \cdot F(a, a, \alpha) F(a, a, e) F(a, a, \beta) \cdot v_z =$$
$$= \mu_x \cdot F(a, a, \alpha \cdot \beta) \cdot v_z = \bar{F}(x, z, \alpha \cdot \beta) .$$

If $y \in D_F^2$ we have

$$\bar{F}(x, y, \alpha) \cdot \bar{F}(y, z, \beta) = \mu_x F(a, a, \alpha) \cdot F(a, y, e)(\mu_y \cdot F(a, a, \beta) \cdot v_z) =$$
$$= u_x \cdot F(a, a, \alpha) \cdot (F(a, y, e) \cdot u_y) \cdot F(a, a, \beta) \cdot v_z =$$
$$= \mu_x \cdot F(a, a, \alpha) \cdot F(a, a, e) \cdot F(a, a, \beta) \cdot v_z =$$
$$= \mu_x \cdot F(a, a, \alpha \cdot \beta) \cdot v_z = \bar{F}(x, z, \alpha \cdot \beta) .$$

Thus $\bar{F}$ satisfies equation (3), which completes the proof.

COROLLARY. From theorem 2 it follows, that if $(K, \cdot)$ is a group, then every solution $F \in \Gamma$ of equation (3) can be extended in an unique way on the set $(D_F^1 \cup D_F^2) \times (D_F^1 \cup D_F^2) \times G$.

THEOREM 3. *If the function $F \in \Gamma$ is a solution of equation* (3) *such that*

$$D_F = M \times M \times G \tag{10}$$

*then it can be extended on the set $A \times A \times G$.*

Proof. Let $F \in \Gamma$ be a solution of equation (3) such that condition (10) holds and let a be an arbitrary, fixed element from the set $M$.
Let us put

$$\bar{F}(x, y, \alpha) = \begin{cases} F(x, y, \alpha) & \text{for} \quad (x, y, \alpha) \in D_F \\ F(x, a, \alpha) & \text{for} \quad x \in M, y \in A \setminus M, \alpha \in G, \\ F(a, y, \alpha) & \text{for} \quad x \in A \setminus M, y \in M, \alpha \in G, \\ F(a, a, \alpha) & \text{for} \quad (x, y, \alpha) \in (A^2 \times G) \setminus D_F . \end{cases}$$

We shall show, that the function $\bar{F}$ defined above is the extension of $F$. From the definition of the function $\bar{F}$ it follows that $\bar{F}|_{D_F} = F$. The following cases are possible:

$1^0$. $x \in M$, $y \in M$, $z \in M$,

$2^0$. $x \in M$, $y \in M$, $z \notin M$,

$3^0$. $x \in M$, $y \notin M$, $z \in M$,

$4^0$. $x \in M$, $y \notin M$, $z \notin M$,

$5^0$. $x \notin M$, $y \in M$, $z \in M$,

$6^0$. $x \notin M$, $y \in M$, $z \notin M$,

$7^0$. $x \notin M$, $y \notin M$, $z \in M$,

$8^0$. $x \notin M$, $y \notin M$, $z \notin M$.

It is easy to see, that in cases $1^0$ and $8^0$ the corresponding equalities hold. In case $3^0$ we have

$$\bar{F}(x, y, \alpha) \cdot \bar{F}(y, z, \beta) = F(x, a, \alpha) \cdot F(a, z, \beta) = F(x, z, \alpha \cdot \beta) = \bar{F}(x, z, \alpha \cdot \beta) .$$

In case 4° we have

$$\bar{F}(x, y, \alpha)\cdot\bar{F}(y, z, \beta) = F(x, a, \alpha)\cdot F(a, a, \beta) = F(x, a, \alpha\cdot\beta) = \bar{F}(x, z, \alpha\cdot\beta).$$

It is easy to verify that in all the other cases the corresponding equalities hold. Thus $\bar{F}$ satisfies equation (3), which completes the proof.

THEOREM 4. *A function $F \in \Gamma$ satisfies equation* (3) *iff it has the following form*

$$F(x, y, \alpha) = f(x)\cdot g(\alpha)\cdot k(y),$$

*where:*

1°. *$g$ is an arbitrary homomorphism of the group $(G, \cdot)$ into $(K, \cdot)$,*

2°. *$f$ and $k$ are arbitrary partial mappings of the set $A$ into $K$ such that $D_f \cap D_k \neq \emptyset$, the product $k(x)\cdot f(x)$ is defined and $k(x)\cdot f(x) = g(e)$ for $x \in D_f \cap D_k$,*

3°. *The products $f(x)\cdot g(\alpha)$, $g(\alpha)\cdot k(y)$ are defined and the equalities $f(x)\cdot g(e) = f(x)$, $g(e)\cdot k(y) = k(y)$ hold for $x \in D_f$, $y \in D_k$, $\alpha \in G$.*

Proof. Let $F \in \Gamma$ satisfy equation (3) and let a be an arbitrary element from the set $A$ for which condition (2) is satisfied. Let us put

$$g(\alpha) = F(a, a, \alpha) \quad \text{for} \quad \alpha \in G,$$

$$f(x) = F(x, a, e) \quad \text{for} \quad x \in D_F^1,$$

$$k(x) = F(a, x, e) \quad \text{for} \quad x \in D_F^2.$$

From the definition of the functions $g$, $f$, $k$ it follows immediately, that conditions 1°, 2°, 3° of theorem 4 are satisffed. For $(x, y, \alpha) \in D_F$ we have $F(x, y, \alpha) = F(x, a, e)\cdot F(a, a, \alpha)\cdot F(a, y, e) = f(x)\cdot g(\alpha)\cdot k(y)$, which completes the first part of the proof.

Let now conditions 1°, 2°, 3° of theorem 4 be fulfilled for $f$, $k$, $g$. Let us put $F(x, y, \alpha) = f(x)\cdot g(\alpha)\cdot k(y)$ for $(x, y, \alpha) \in D_f \times D_k \times G$. It is easy to verify, that $F \in \Gamma$. Let $(x, y, \alpha) \in D_F$ and $(y, z, \beta) \in D_F$. Then $(x, y, \alpha\cdot\beta) \in D_F$ and by 2° and 1° we have

$$\begin{aligned} F(x, y, \alpha)\cdot F(y, z, \beta) &= f(x)\cdot g(\alpha)\cdot k(y)\cdot f(y)\cdot g(\beta)\cdot k(z) = \\ &= f(x)\cdot g(\alpha)\cdot g(e)\cdot g(\beta)\cdot k(z) = \\ &= f(x)\cdot g(\alpha\cdot\beta)\cdot k(z) = F(x, z, \alpha\cdot\beta), \end{aligned}$$

which completes the proof.

If in the above theorems we replace $(K, \cdot)$ by the semigroup of all partial mappings of $a$ set $X$ (with the superposition of mappings) and we put

$$\varphi(\mu, x, y, \alpha) = [F(x, y, \alpha)](\mu) \quad \text{for} \quad (x, y, \alpha) \in D_F, \mu \in D_{F(x,y,\alpha)}$$

then the function $\varphi$ satisfies the translation equation

$$\varphi(\varphi(\mu, y, z, \beta), x, y, \alpha) = \varphi(\mu, x, z, \alpha \cdot \beta) .$$

Thus we can use the above theorems in the theory of translation equation.

## References


[1] A. Nijenhuis: *Theory of the geometric object*. Amsterdam 1952.

[2] W. Waliszewski: *Categories, groupoids, pseudogroups and analitical structures*. Rozprawy Mat. 45, Warszawa 1965.

[3] J. Tabor: *The translation equation and algebraic objects*. Ann. Polon. Math., XXVII (1973).