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On the Green function and Cauchy-Dirichlet problem for polyparabolic equation

1. In the present paper we shall solve the limit problem for the equation

(1) $$L^m u(X) = F(X), \quad X = (x, t) = (x_1, x_2, \ldots, x_n, t)$$

in the unbounded strip

$$\Omega = \{X:\ -\infty < x_i < \infty,\ |x_n| < c,\ t > 0,\ i = 1, 2, \ldots, n-1\},$$

where

$$L = \sum_{i=1}^{n} (a^2 D^2_{x_i} + a_i D_{x_i}) - D_t - b,\ L^m = L(L^{m-1}),$$

a and c being positive constants, b non negative constant, a_i $(i = 1, \ldots, n)$ arbitrary real numbers and m an positive integer.

Let H denote the set of the functions $u(X)$, continuous with the derivatives $D_t^{\beta} D_x^{|\alpha|} u(X)$, $|\alpha| + 2\beta \leqslant 2m$ satisfying the equation (1) in Ω, $|\alpha|$ being an multiindex.

We shall construct the function $u(X) \in H$ satisfying the initial conditions

(2) $$L^k u(x, 0) = f_k(x)$$

and boundary conditions

(3) $$L_k u(X) = h_{q,k}(X') \quad \text{for} \quad x_n = (-1)^{q+1} c,$$

where

$$X' = (x_1, x_2, \ldots, x_{n-1}, t),\ k = 0, 1, \ldots, m-1,\ q = 1, 2.$$

2. Let us consider the sets

$$S_0 = \{X:\ -\infty < x_i < \infty,\ |x_n| < c,\ t = 0\},$$

and

$$S^{(q)} = \{X:\ -\infty < x_i < \infty,\ x_n = (-1)^{q+1} c,\ t > 0\},$$

where

$i = 1, 2, \ldots, n-1$, $q = 1, 2$. Let $Y = (y, s) = (y_1, y_2, \ldots, y_n, s)$.

By ([2], p. 456) the function

$$U(X, Y) = \begin{cases} (t-s)^{-\frac{n}{2}} \exp\{[(4a^2)(s-t)]^{-1}|y-x|^2\} & \text{for} \quad s<t, \\ 0 & \text{for} \quad s \geqslant t,\ X \neq Y, \end{cases}$$

is the fundamental solution of the heat conduction equation.

Let us consider equations

(4) $$Lu(X) = 0$$

and

(5) $$L^* u(X) = 0, \quad L^* = \sum_{i=1}^{n} (a^2 D_{x_i}^2 + a_i D_{x_i}) + D_t - b.$$

Let

$$V(X; Y) = \begin{cases} N(X; Y) U(X; Y) & \text{for} \quad s<t, \\ 0 & \text{for} \quad s \geqslant t,\ X \neq Y, \end{cases}$$

where

$$N(X; Y) = \exp\left[-\left(b + \sum_{i=1}^{n} \frac{a_i^2}{4a^2}\right)(t-s) - \sum_{i=1}^{n} \frac{a_i}{2a^2}(x_i - y_i)\right].$$

It is easy to verify that the function $V(X; Y)$ satisfies the equation (4) with respect to X and the equation (5) with respect to Y.

We shall solve the problem (1), (2), (3) using the convenient Green function for the equation (4) and the domain Ω.

Let X denote an arbitrary point of Ω. Let $X_1^{(1)}$ be the symmetric image of the point X with respect to the plane $S^{(1)}$ and $X_{2p}^{(1)}$ the symmetric image of the point $X_{2p-1}^{(1)}$ with respect to $S^{(2)}$ and $X_{2p+1}^{(1)}$ the symmetric image of the point $X_{2p}^{(1)}$ with respect to $S^{(1)}$, $p = 1, 2, \ldots$ Further let $X_1^{(2)}$ be the symmetric image of the point X with respect to $S^{(2)}$ and $X_{2p}^{(2)}$ the symmetric image of the point $X_{2p-1}^{(2)}$ with respect to $S^{(1)}$ and $X_{2p+1}^{(2)}$ the symmetric image of the point $X_{2p}^{(2)}$ with respect to $S^{(2)}$, $p = 1, 2, \ldots$ Let

$$X_p^{(q)} = (x_1, x_2, \ldots, x_{n-1}, x_{n,p}^{(q)}, t), \quad (q = 1, 2, p = 1, 2, \ldots).$$

It easy to verify by induction, that

$$x_{n,p}^{(q)} = (-1)^p [x_n + (-1)^q 2pc].$$

Let us consider the function

(6) $$G(X; Y) = \begin{cases} N(X; Y) g(X; Y) & \text{for} \quad s<t, \\ 0 & \text{for} \quad s \geqslant t,\ X \neq Y, \end{cases}$$

where

$$(7)\qquad g(X;\,Y) = U(X;\,Y)+\sum_{p=1}^{\infty}(-1)^p[U(X_p^{(1)};\,Y)+U(X_p^{(2)};\,Y)]\,,$$
$$X\in\Omega,\quad Y\in\Omega\cup S^{(1)}\cup S^{(2)}\,.$$

Let

$$(8)\qquad Q^{(q)}(X;\,Y) = \sum_{p=2}^{\infty}|D^{|\alpha|}_{x,y,t,s}U(X_p^{(q)};\,Y)|\quad (q = 1,2)$$

and

$$\Omega^{(\lambda+1)} = \{X\colon\ |x_i|\leqslant c_i,\, |x_n|\leqslant 2^\lambda c,\, 0\leqslant t\leqslant T,\ i = 1,2,\dots,n-1\}\,,$$

where $\lambda = 0, 1$, c_i and T being positive constants. In the sequel we shall use

LEMMA 1. [3] *The series defined by formula* (8) *are uniformly convergent for* $X\in\Omega^{(2)}$, $Y\in\Omega^{(1)}$, $X\neq Y$.

Now we shall prove

THEOREM 1. *The function* $G(X;\,Y)$ *defined by formula* (6) *is the Green function with the pole* X *for the equation* (4), *domain* Ω *and Dirichlet boundary condition.*

Proof. The function $N(X;\,Y)$ is analytic and the function $V(X;\,Y)$ has the analogous properties as the function $U(X;\,Y)$. The remaining terms of the function $G(X;\,Y)$ are regular in the domain Ω. The function $g(X,\,Y)$ may be written in the form:

$$g(X;\,Y) = U(X;\,Y)-U(X_1^{(1)};\,Y)+\sum_{p=1}^{\infty}\{[U(X_{2p}^{(1)};\,Y)-U(X_{2p+1}^{(1)};\,Y)]+$$
$$+[U(X_{2p}^{(2)};\,Y)-U(X_{2p-1}^{(2)};\,Y)]\}\,,\qquad p\in N\,.$$

Since

$$U(X;\,Y)-U(X_1^{(1)};\,Y) = U(X_{2p}^{(1)};\,Y)-U(X_{2p+1}^{(1)};\,Y) =$$
$$= U(X_{2p}^{(2)};\,Y)-U(X_{2p-1}^{(2)};\,Y) = 0$$

for

$$Y\in S^{(1)},\quad Y\neq X,\ (p = 1,2,\dots)\,,$$

thus

$$G(X;\,Y) = 0\qquad\text{for}\qquad Y\in S^{(1)},\ X\neq Y\,.$$

Similarly we can verify that

$$G(X;\,Y) = 0\qquad\text{for}\qquad Y\in S^{(2)},\ X\neq Y\,.$$

3. We shall denote by Ω_t, $S_t^{(1)}$, $S_t^{(2)}$ the subdomains of the domain Ω, $S^{(1)}$, $S^{(2)}$ lying under the characteristic $s = t$. Let $dy = dy_1dy_2\dots dy_n$, $dY' = dy_1dy_2\dots dy_{n-1}ds$ and $dY = dy_1dy_2\dots dy_nds$.

Let

$$v_j(X) = \frac{(-1)^j t^j A}{j!} \int\limits_{S_0} f_j(y) G(X;\, Y)|_{s=0} dy\,,$$

and

$$w_{q,j}(X) = (-1)^{j+1} a^2 A \int\limits_{S_t^{(q)}} h_{q,j}(Y') \frac{(t-s)^j}{j!} N(X;\, Y) D_{y_n} g(X;\, Y)|_{y_n=(-1)^{q+1}} dY'$$

where

$$A = (2a\sqrt{\pi})^{-1},\ q = 1, 2,\ j = 0, 1, \ldots, m-1\,.$$

Let us consider the functions

$$v(X) = \sum_{j=1}^{m-1} v_j(X)\,, \tag{9}$$

$$w(X) = \sum_{j=0}^{m-1} [w_{1,j}(X) + w_{2,j}(X)]\,, \tag{10}$$

$$z(X) = (-1)^m A \int\limits_{\Omega_t} F(Y) \frac{(t-s)^{m-1}}{(m-1)!} G(X;\, Y) dY\,. \tag{11}$$

We shall prove that the function

$$u(X) = v(X) + w(X) + z(X) \tag{12}$$

is the solution of the problem (1), (2), (3) in the domain Ω.

We can easy verify by induction the following

LEMMA 2. *If the function $w(X;\, Y)$ satisfies the equation* (4) *for arbitrary Y, then*

$$L_X^k\left(\frac{(t-s)^j}{j!} w(X;\, Y)\right) = \begin{cases} \dfrac{(-1)^k (t-s)^{j-k}}{(j-k)!} w(X;\, Y) & \text{for} \quad k \leqslant j\,, \\ 0 & \text{for} \quad k > j\,, \end{cases}$$

where k, j denote arbitrary positive integers.

Let

$$J(X) = A \int\limits_{E_n} f(y, t) V(x, t; y, 0) dy\,.$$

We shall prove

LEMMA 3. *If the function $f(y, t)$ is bounded and measurable in the strip $E_n \times [0, t]$ and continuous at the point $X_0 = (\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_n, 0)$ then $\lim J(X) = f(X_0)$ as $X \to X_0$, $X \in E_n \times (0, T)$.*

Proof. We have

$$\int\limits_{E_n} V(X; y, 0) dy = e^{-Bt} \prod_{i=1}^{n} P_i$$

where

$$B = b + \sum_{i=1}^{n} \frac{a_i^2}{4a^2} \quad \text{and} \quad P_i = \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} \exp\left(-\frac{(y_i - x_i)^2}{4a^2 t} + \frac{a_i}{2a^2}(y_i - x_i)\right) dy_i .$$

Using in integrals the change of variables

$$z_i = (y_i - x_i)(2at^{\frac{1}{2}})^{-1} \quad (i = 1, \dots, n)$$

by ([1], p. 499) we get

$$\frac{1}{2a\sqrt{\pi}} P_i = \exp\left(\frac{a_i^2 t}{4a^2}\right) \quad (i = 1, \dots, n) .$$

Hence

$$J(X) = f(X_0) \exp\left(\frac{t}{4a^2} \sum_{i=1}^{n} a_i^2\right) + A \int_{E_n} [f(y, t) - f(X_0)] V(X; y, 0) dy .$$

Since the first term of the sum tends to $f(X_0)$ as $X \to X_0$. It is enough to prove that

$$\int_{E_n} [f(y, t) - f(X_0)] V(X; y, 0) dy \to 0 \quad \text{as} \quad X \to X_0 .$$

Let

$$K_\delta = \{y: |y_i - \mathring{x}_i| \leqslant \delta, \quad i = 1, 2, \dots, n\} , \quad \delta > 0 .$$

We choose the number δ such that

$$|f(y, t) - f(X_0)| \leqslant \frac{\varepsilon}{4} \text{ and } \exp\left(\frac{t}{4a^2} \sum_{i=1}^{n} a_i^2\right) \leqslant 2 \quad \text{for} \quad y \in K_\delta \; 0 \leqslant t \leqslant \delta \leqslant T$$

ε being arbitrary positive number.

We have

$$A \int_{E_n} [f(y, t) - f(X_0)] V(X; y, 0) dy = A \int_{K_\delta} [f(y, t) - f(X_0)] V(X; y, 0) dy + $$
$$+ A \int_{E_n - K_\delta} [f(y, t) - f(X_0)] V(X; y, 0) dy$$

and

$$A \int_{K_\delta} |f(y, t) - f(X_0)| V(X; y, 0) dy \leqslant \frac{\varepsilon}{2} .$$

We can easy verify that

$$A \int_{E_n - K_\delta} |f(y, t) - f(X_0)| V(X; y, 0) dy \leqslant \frac{\varepsilon}{2}$$

and finally we get the inequality

$$A\int_{E_n}|f(y,t)-f(X_0)|\,V(X;y,0)\,dy\leqslant\varepsilon\,.$$

Let

$$\tilde{\Omega}\;=\{X:\ -\infty<x_i<\infty,\ |x_n|\leqslant c,\ t\geqslant 0,\ i=1,2,\ldots,n-1\},$$

$$\Omega^{(3)}=\{x:\ -\infty<x_i<\infty,\ |x_n|\leqslant c,\ i=1,2,\ldots,n-1\},$$

$$\Omega^{(4)}=\{X':\ -\infty<x_i<\infty,\ t\geqslant 0,\ i=1,2,\ldots,n-1\}$$

and let

$$J_q(X)=A\int_{-\infty}^{t}\int_{E_{n-1}}h(Y')\frac{c-(-1)^q x_n}{t-s}\,V(X;\,Y)|_{y_n=(-1)^{q+1}c}\,dY'\quad(q=1,2)\,.$$

We shall prove

LEMMA 4. *If the function $h(Y')$ is bounded and measurable for $Y'\in\Omega^{(4)}$ and continuous at the point $X_0'=(\mathring{x}_1,\mathring{x}_2,\ldots,\mathring{x}_{n-1},t_0)$ and $X\in\Omega$,*

$$X_0=(\mathring{x}_1,\mathring{x}_2,\ldots,\mathring{x}_{n-1},c,t_0)\in S^{(1)},$$

then $\lim J_1(X)=h(X_0')$ as $X\to X_0$.

Proof. Introducing to the integrals

(13) $$P^i=\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{t-s}}\exp\left(-\frac{(y_i-x_i)^2}{4a^2(t-s)}+\frac{a_i(y_i-x_i)}{2a^2}\right)dy_i$$

the change of variables

$$z_i=(y_i-x_i)\big(2a(t-s)^{\frac{1}{2}}\big)^{-1}\quad(i=1,2,\ldots,n-1)$$

and by ([1], p. 499) we get

$$P^i=\exp\left(\frac{a_i^2(t-s)}{4a^2}\right)\quad(i=1,2,\ldots,n-1)\,.$$

From (13) and definition of the function $N(X;\,Y)$ we get

$$A\int_{-\infty}^{t}\frac{c-x_n}{t-s}\,V(X;\,Y)|_{y_n=c}\,dY'=\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{t}\frac{c-x_n}{(t-s)^{3/2}}\exp\left(-\frac{(c-x_n)^2}{4a^2(t-s)}-b(t-s)\right)ds$$

After the change of variables

$$z=(c-x_n)\big(2a(t-s)^{\frac{1}{2}}\big)^{-1}$$

and by ([1], p. 500) we obtain

$$A \int_{-\infty}^{t} \int_{E_{n-1}} \frac{c-x_n}{t-s} V(X;\, Y)|_{y_n=c} dY' = \exp\left(-\frac{b(c-x_n)}{a}\right).$$

Hence

$$J_1(X) = h(X_0')\exp\left(-\frac{b(c-x_n)}{a}\right) + J_3(X)$$

where

$$J_3(X) = A \int_{-\infty}^{t} \int_{E_{n-1}} [h(Y')-h(X_0')] \frac{c-x_n}{t-s} V(X;\, Y)|_{y_n=c} dY' .$$

Similarly as in [3] we have $J_3(X)\to 0$ as $X \to X_0$. This implics the thesis of lemma 4.

LEMMA 5. *If the function* $h(Y')$ *is bounded and measurable for* $Y' \in \Omega^{(4)}$ *and continuous at the point* $X_0' = (\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_{n-1}, t_0)$ *and* $X \in \Omega$,

$$X_0 = (\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_{n-1}, -c, t_0) \in S^{(2)},$$

then $\lim J_2(X) = h(X_0')$ *as* $X \to X_0$.

The proof of lemma 5 is analogous to the proof of lemma 4.

LEMMA 6. *If the functions* $F(Y)$, $D_{y_i}F(Y)$ *are bounded and continuous in the set* $\tilde{\Omega}$, *then the function* $z(X)$ *defined by formula* (11) *belongs to the class* H.

The proof of Lemma 6 is similar to the proof of the convenient lemma in the paper [3].

LEMMA 7. *If the function* $F(X)$. $D_{y_i}F(Y)$ $(i = 1, 2, \ldots, n)$ *are bounded and continuous in the set* $\tilde{\Omega}$ *and the function* $f_j(y)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$ $(j = 0, 1, \ldots, m-1)$ *are bounded and measurable respectively in the sets* $\Omega^{(3)}$ *and* $\Omega^{(4)}$, *then the function* $u(X)$ *belongs to the class* H.

Proof. Similarly as in [2] and [4] we can verify that the integrals in the formula (12) and its derivatives are almost uniformly convergent in Ω. Hence the function $u(X)$ and its derivatives $D_t^{\beta} D_x^{|\alpha|} u(X)$, $|\alpha| + 2\beta \leqslant 2m$, are continuous in Ω. Moreover by lemmas 2 and 3 follows that the function $u(X)$ satisfies the equation (1).

4. We shall prove that the function $u(X)$ defined by formula (12) satisfies the limit conditions (2), (3).

LEMMA 8. *If the functions* $f_j(y)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$, $F(Y)$ *are bounded and measurable respectively in the sets* $\Omega^{(3)}$, $\Omega^{(4)}$ *and* $\tilde{\Omega}$, *and the functions* $f_j(y)$ $(j = 0, 1, \ldots, m-1)$ *are continuous at the point* $x_0 = (\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_n)$ *then the function* $u(X)$ *defined by formula* (12) *satisfies the initial conditions* (2).

Proof. The function $v(X)$ defined by formula (9) is the sum of the integrals

$$I_1(X) = \sum_{j=0}^{m-1} \frac{(-1)^j t^j A}{j!} \int_{S_0} f_j(y) V(X;\, Y)|_{s=0} dy$$

and

$$I_2(X) = \sum_{j=0}^{m-1} \frac{(-1)^j t^j A}{j!} \int_{S_0} f_j(y) N(X;\, Y) \sum_{p=1}^{\infty} (-1)^p [U(X_p^{(1)};\, Y) +$$

$$+ U(X_p^{(2)};\, Y)|_{s=0}\, dy\,.$$

Let

$$\bar{f}_j(y) = \begin{cases} f_j(y) & \text{for} \quad y \in \Omega^{(1)}\,, \\ 0 & \text{for} \quad y \in E_n - \Omega^{(1)}\,. \end{cases}$$

Hence

$$I_1(X) = \sum_{j=0}^{m-1} \frac{(-1)^j t^j A}{j!} \int_{E_n} \bar{f}_j(y)\, V(X;\, Y)|_{s=0} dy\,.$$

By lemmas 2 and we have:

$$\lim L^k \left(\frac{(-1)^j t^j A}{j!} \int_{E_n} \bar{f}_j(y)\, V(X;\, Y)|_{s=0}\, dy \right) = \begin{cases} f_k(x_0) & \text{for} \quad j = k \\ 0 & \text{for} \quad j \neq k \end{cases}$$

as $X \to X_0 \in S_0$, $X \in \Omega$ $(j, k = 0, 1, \dots, m-1)$.
Now we get

$$\lim L^k I_1(X) = f_k(x_0) \quad \text{as} \quad X \to X_0 \in S_0,\ X \in \Omega\,.$$

By similar estimations as in ([4], p. 132) follows that the function $Ct^\mu(C, \mu$ being positive constants) is the common majorant for the functions $|L^k I_2(X)|$, $|L^k w(X)|$, $|L^k z(X)|^{(k=0,1,\dots,m-1)}$. Hence the integrals $L^k I_2(X)$, $L^k w(X)$, $L^k z(X)$ tends to zero as $X \to X_0 \in S_0$.

Now we shall prove

Lemma 9. *If the functions $h_1, j(Y')$, $h_{2,j}(Y')$ are bounded and measurable in the set $\Omega^{(4)}$ and the functions $h_{1,j}(Y')$ are continuous at the point*

$$X_0' = (\mathring{x}_1, \mathring{x}_2, \dots, \mathring{x}_{n-1}, t_0)\,,$$

then the function $w(X)$ defined by formula (10) *satisfies the boundary conditions* $\lim L^k w(X) = h_{1,k}(X_0')$ *as* $X \to X_0 \in S^{(1)}$, $X \in \Omega$.

Proof. By formula (10) we get

$$w(X) = w_1(X) + w_2(X)\,,$$

where

$$w_1(X) = \sum_{j=0}^{m-1} w_{1,j}(X)\,, \qquad w_2(X) = \sum_{j=0}^{m-1} w_{2,j}(X)\,.$$

Using lemma 2 to the functions $w_{1,j}$ we get

$$L^k w_{1,j}(X) = \begin{cases} J_{j,k}(X) & \text{for} \quad j \geqslant k \quad (j = 0, \dots, m-1) \\ 0 & \text{for} \quad j < k \end{cases}$$

where

$$J_{j,k}(X) = (-1)^{j+k+1}a^2A\int\limits_{S_t^{(1)}} h_{1,j}(Y')\frac{(t-s)^{j-k}}{(j-k)!}N(X;\,Y)\,D_{y_n}g(X;\,Y)|_{y_n=c}\,dY'$$

By definition of function $g(X, Y)$ we have:

$$J_{j,k}(X) = J_{j,k}^{(1)}(X)+J_{j,k}^{(2)}(X)\,,$$

where

$$J_{j,k}^{(1)}(X) = (-1)^{j+k+1}a^2A\int\limits_{S_t^{(1)}} h_{1,j}(Y')\frac{(t-s)^{j-k}}{(j-k)!}N(X;\,Y)\,D_{y_n}[U(X;\,Y)+$$

$$-U(X_1^{(1)};\,Y)|_{y_n=c}\,dY'\,,$$

and

$$J_{j,j}^{(2)}(X) = (-1)^{j+k+1}a^2A\int\limits_{S_t^{(1)}} h_{1,j}(Y')\frac{(t-s)^{j-k}}{(j-k)!}N(X;\,Y)\sum_{p=1}^{\infty}(-1)^p\times$$

$$\times D_{y_n}[U(X_p^{(2)};\,Y)-U(X_{p+1}^{(1)};\,Y)]|_{y_n=c}\,dY'\,.$$

It easy to verify that

$$D_{y_n}[U(X;\,Y)-U(X_1^{(1)};\,Y)]|_{y_n=c} = -\frac{c-x_n}{a^2(t-s)}\,U(X;\,Y)|_{y_n=c}$$

and

$$J_{j,k}^{(1)}(X) = (-1)^{j+k}A\int\limits_{S_t^{(1)}} h_{1,j}(Y')\,\frac{(t-s)^{j-k}(c-x_n)}{(j-k)!(t-s)}\,V(X;\,Y)|_{y_n=c}\,dY'$$

Let

$$\bar{h}_{1,k}(Y') = \begin{cases} h_{1,k}(Y') & \text{for} \quad Y'\in\Omega^{(4)} \\ 0 & \text{for} \quad Y'\in E_n-\Omega^{(4)}\,. \end{cases}$$

Applying lemma 4 to the integral $J_{j,k}(X)$ as $j = k$ we obtain

$$J_{k,k}^{(1)}(X) = A\int\limits_{-\infty}^{t}\int\limits_{E_{n-1}} \bar{h}_{1,k}(Y')\,\frac{c-x_n}{t-s}\,V(X;\,Y)|_{y_n=c}\,dY'\to h_{1,k}(X_0') \tag{14}$$

as $X\to X_0\in S^{(1)}$ $X\in\Omega$,
when $j>k$ we have the inequalite:

$$J_{j,k}^{(1)}(X)\leqslant C(c-x_n)\int\limits_0^t (t-s)^{\tau}ds\,,$$

C being an positive constant and $\tau > -1$. Hence

(15) $$\lim J_{j,k}^{(1)}(X) = 0 \quad \text{as} \quad X \to X_0 \in S^{(1)},\ X \in \Omega,\ j > k\,.$$

The integral $J_{j,k}^{(2)}(X)$ is uniformly convergent in every set

$$\Omega^{(5)} = \{X:\ |x_i| \leqslant c_i,\ |x_n| \leqslant 2c,\ 0 \leqslant t \leqslant T,\ i = 1, 2, \ldots, n-1\}\,,$$

where c_i, T are positive real numbers. Hence the function $J_{j,k}^{(2)}(X)$ is continuous at the point $X_0 \in S^{(1)}$. By the conditions

$$D_{y_n}[U(X_p^{(2)};\ Y) - U(X_{p+1}^{(1)};\ Y)]_{y_n = c} = 0 \quad \text{for} \quad X = X_0 \in S^{(1)}\ (p = 1, 2, \ldots)$$

we get

(16) $$\lim J_{j,k}^{(2)}(X) = 0 \quad \text{as} \quad X \to X_0 \in S^{(1)},\ X \in \Omega\,.$$

Finally we obtain in view of (14), (15), (16), that

$$\lim L^k w_1(X) = h_{1,k}(X_0') \quad \text{as} \quad X \to X_0 \in S^{(1)},\ X \in \Omega\ (k = 0, 1, \ldots, m-1)\,.$$

By lemma 2 we get

$$L^k w_{2,j}(X) = \begin{cases} I_{j,k}(X) & \text{for} \quad j \geqslant k\,, \\ 0 & \text{for} \quad j < k\,, \end{cases}$$

where

$$I_{j,k}(X) = (-1)^{j+k} a^2 A \int\limits_{S_t^{(2)}} h_{2,j}(Y') \frac{(t-s)^{j-k}}{(j-k)!} N(X;\ Y)\{D_{y_n}[U(X;\ Y) +$$

$$- U(X_1^{(1)};\ Y) + \sum_{p=1}^{\infty} (-1)^p D_{y_n}[U(X_p^{(2)};\ Y) - U(X_{p+1}^{(1)};\ Y)]\}|_{y_n = -c} dY'\,.$$

The integral $I_{j,k}(X)$ is uniformly convergent in every set

$$\Omega^{(6)} = \{X:\ |x_i| \leqslant c_i,\quad d_1 \leqslant x_n \leqslant 2c,\quad 0 \leqslant t \leqslant T\}\,,$$

c_i, d_1, T being arbitrary numbers for which $c_i > 0$, $-c < d_1 < 2c$, $T > 0$. Hence the function $I_{j,k}(X)$ is continuous at the point $X_0 \in S^{(1)}$. Using the conditions

$$D_{y_n}[U(X;\ Y) - U(X_1^{(1)};\ Y)]|_{y_n = -c} = 0 \quad \text{for} \quad X = X_0 \in S^{(1)}\,,$$

and

$$D_{y_n}[U(X_p^{(2)};\ Y) - U(X_{p+1}^{(1)};\ Y)]|_{y_n = -c} = 0 \quad \text{for} \quad X = X_0 \in S^{(1)}\ (p = 1, 2, \ldots)$$

we get

$$\lim L^k w_2(X) = 0 \quad \text{as} \quad X \to X_0 \in S^{(1)},\ X \in \Omega,\ (k = 0, 1, \ldots, m-1)\,.$$

Lemma 9 is thus proved.

LEMMA 10. *If the functions* $h_{1,j}(Y')$, $h_{2,j}(Y')$ *are bounded and measurable in the set* $\Omega^{(4)}$ *and the function* $h_{2,j}(Y')$ *are continuous at the point* $X_0' = (\mathring{x}_1, \mathring{x}, \ldots, \mathring{x}_{n-1}, t_0)$, *then the function* $w(X)$ *defined by formula* (10) *satisfies the boundary conditions*

$$\lim L^k w(X) = h_{2,k}(X_0') \quad as$$
$$X \to X_0 \in S^{(2)},\ X \in \Omega,\ X_0 = (\mathring{x}_1, \mathring{x}_2, \ldots, \mathring{x}_{n-1}, -c, t_0)\,.$$

The proof of lemma 10 is similar to the proof of lemma 9.
We shall prove

LEMMA 11. *If the functions* $f_j(y)$ $(j = 0, 1, \ldots, m-1)$ *are bounded and measurable in the set* $\Omega^{(3)}$, *then the function* $v(X)$ *defined by formula* (9) *satisfies the conditions*

$$\lim L^k v(X) = 0 \quad as \quad X \to X_0 \in S^{(1)} \cup S^{(2)},\ X \in \Omega\,.$$

Proof. By definition of function $v(X)$ we have

$$v(X) = \sum_{j=0}^{m-1} \frac{(-1)^j t^j A}{j!} \int_{s_0} f_j(y) N(X;\,Y)\{[U(X;\,Y) - U(X_1^{(1)};\,Y)] +$$
$$+ \sum_{p=1} (-1)^p [U(X_p^{(2)};\,Y) - U(X_{p+1}^{(1)};\,Y)]\}|_{s=0}\, dy\,.$$

The integrals

$$K_{j,k}(X) = \int_{s_0} f_j(y) L_X^k \{N(X;\,Y)[U(X;\,Y) - U(X_1^{(1)};\,Y) +$$
$$+ \sum_{p=1}^{\infty} (-1)^p (U(X_p^{(2)};\,Y)) - U(X_{p+1}^{(1)};\,Y)]\}|_{s=0} dy$$

are uniformly convergent in every set

$$\Omega^{(7)} = \{X\colon |x_i| \leqslant c_i, \quad |x_n| \leqslant 2c, 0 < T_0 \leqslant t \leqslant T, \quad i = 1, 2, \ldots, n-1\}\,,$$

where c_i, T_0, T being an positive constants. Hence the functions $L^k v(X)$ $(k = 0, 1, \ldots, m-1)$ are continuous at the point $X_0 \in S^{(1)}$. By the conditions

$$[U(X;\,Y) - U(X^{(1)}_1;\,Y)]|_{s=0} = 0 \quad \text{for} \quad X = X_0 \in S^{(1)}\,,$$

and

$$[U(X_p^{(2)};\,Y) - U(X_{p+1}^{(1)};\,Y)]|_{s=0} \quad \text{for} \quad X = X_0 \in S^{(1)}\ (p = 1, 2, \ldots)$$

we get

$$\lim L^k v(X) = 0 \quad \text{as} \quad X \to X_0 \in S^{(1)},\ X \in \Omega\,.$$

Similarly we can prove that

$$\lim L^k v(X) = 0 \quad \text{as} \quad X \to X_0 \in S^{(2)},\ X \in \Omega\,.$$

LEMMA 12. *If the function $F(Y)$ bounded and measurable in the set $\tilde{\Omega}$, then the function $z(X)$ given by formula* (11) *satisfies the conditions*

$$\lim L^k z(X) = 0 \quad as \quad X \to X_0 \in S^{(1)} \cup S^{(2)},\ X \in \Omega,\ (k = 0, 1, \dots, m-1)\,.$$

The proof of lemma 12 is analogous to the proof of lemma 11. By lemmas 7—12 we get the fundamental

THEOREM 2. *Let the functions $f_j(y)$, $h_{1,j}(Y')$, $h_{2,j}(Y')$, $F(Y)$ and $D_{y_i}F(Y)$ $(i = 1, 2, \dots, n;\ j = 0, 1, \dots, m-1)$ be bounded and continuous respectively in the sets $\Omega^{(3)}$, $\Omega^{(4)}$, $\tilde{\Omega}$, then the function $u(X)$ defined by formula* (12) *belong to the class H and satisfies the limit conditions* (2), (3).

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