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Markoff's chain discribing a certain experimental pattern

Introduction. The subject of our considerations is an experimental pattern which is described by a sequence of stochastic variables. It is shown that the sequence is a homogeneous Markoff's chain; later the conditions required for a chain to become a stationary chain are established.

At the end of the considerations there are some comments concerning the ergodicity introduced by the above pattern of the chain, especially the different ergodicity formulations and related conceptions which are found in older literature about Markoff's chains.

An experimental pattern is discribed below: $(k+1)$ urns, $U_0, U_1, U_2, \ldots, U_k$, contain balls numbered $1, 2, \ldots, k$. Let us suppose that in urn U_i there are s_{ij} balls with number j, and that

a) the urns, $U_0, U_1, \ldots, U_k$ are not empty, so that $\sum_{j=1}^{k} s_{ij} = s_i > 0$,

b) $s_{0j} > 0$ for $j = 1, 2, \ldots, k$.

We draw (always returning the extracted ball to the urn) in conformity with the pattern below.

Let us start by drawing at random a ball from the urn U_0; this is the preliminary or zero stage. If the drawn ball has number j, then the next drawing will be made from the urn U_j $(j = 1, 2, \ldots, k)$. This random drawing is stage one.

Then the second stage consists in drawing a ball from the urn with the number of the ball drawn in the first stage; this operation is continued and the stage n will consist in drawing a ball from the urn which has the same number as the ball in drawing in stage $(n-1)$. The space of elementary events E describing this pattern is a set of sequences (x_n) (limite or infinite — since in theory it is possible to continue indefinitely the drawing of the balls), where x_n for $n = 0, 1, 2, \ldots$ denotes a ball which is drawn in the stage n and for $n > 1$ arises from the urn having a number equal to the number of the ball drawn in the stage $(n-1)$.

Let us denote by A_j^m a stochastic event consisting in extracting of a ball with number j in the drawing m: $A_j^0 = \{(x_n): x_0 \text{ is a ball with number } j \text{ from urn } U_0\}$; $A_j^1 = \{(x_n): x_1 \text{ is a ball with number } j \text{ drawn at random from the urn whose number is equal to the number of the ball drawn from the urn } U_0\}$.

Let ξ_n denote the number of the ball drawn from an urn in stage n. For $n = 0, 1, 2, \ldots$ the functionals ξ_n map the space E into a set $\{1, 2, \ldots, k\}$, and it is a simple matter to show that they are measurable in a probabilistic system. In this way we obtain a sequence of stochastic variables $\xi_0, \xi_1, \xi_2, \ldots$

For each $n \geqslant 0$ we have $\xi_n(E) \subset \{1, 2, \ldots, k\}$.

Let us proceed to the construction of the distribution of these stochastic variables.

$P(\xi_0 = j) = P(A_j^0) = \dfrac{s_{0j}}{s_0}$, in view of theorem of classical probability.

Let further $p_j(0) = P(A_j^0)$. Thus we obtain the vector $\vec{m}_0 = [p(0), p_2(0), \ldots, p_k(0)]$, whose components are the probabilities in the distribution of the stochastic variable ξ_0. Thus the distribution of the stochastic variable ξ_0 is the set

$$\{(j, p_j(0)) \colon j = 1, 2, \ldots, k\}.$$

Further we have $P(\xi_1 = j) = P(A_j^1)$. Let denote by analogy $p_j(1) = P(\xi_1 = j)$ and $\vec{m}_1 = [p_1(1), p_2(1), \ldots, p_k(1)]$.

Thus $p_j(1)$ is a probability of drawing a ball number j in the first stage. Let us find the distribution of the random variable ξ_1.

The events $A_1^0, A_2^0, \ldots, A_k^0$ form a complete system of events; in view of the theorem of complete probability $\left(P(A_j^0) > 0\right)$ we have

$$P(A_j^1) = \sum_{i=1}^{k} P(A_i^0) P(A_j^1 \mid A_i^0). \tag{1}$$

The conditional probability $P(A_j^1 \mid A_i^0)$ is a probability that the random variable ξ_1 will have the value j, when the variable ξ_0 has been taken the value i; thus it is the probability that in the first stage the ball j was drawn on the condition that it was drawn from urn U_i. Let us denote this probability by p_{ij}.

The numbers p_{ij} form a square matrix of order k; it is denoted by $Q(1)$. We shall prove that for $i = 1, 2, \ldots, k$

$$\sum_{j=1}^{k} p_{ij} = 1$$

and that this relation is independent of the distribution of the balls in the urns.

The following symbols are introduced in order to simplify the notations: A_{ij}^n denotes a random event consisting in drawing a ball with number j in the n-th stage, on the condition that in the preceding stage a ball with number i has been drawn, i.e. a ball drawn from the urn U_i. Thus the random event A_{ij}^n is a set of sequences $(x_n) \in E$ in which x_n is a ball with number j, and x_{n-1} a ball with number i. It is obvious that $A_{ij}^n = A_j^n \cap A_i^{n-1}$.

The random events $A_{i1}^n, A_{i2}^n, \ldots, A_{ik}^n$ are two by two disjoint, and $\bigcup_{l=1}^{k} A_{il}^n$ is the set including those sequences $(x_n) \in E$ in which x_{n+1} is a ball with number i.

Hence

$$\bigcup_{l=1}^{k} A_{il}^{n} = A_i^{n-1} \quad \text{for} \quad n \geqslant 1 .$$

Thus we have for $n = 1$ (and not only in this case), for any distribution of the balls in the urns, that in the urn U_0 the numbers of balls of all kinds must be different from zero:

$$\sum_{j=1}^{k} p_{ij} = \sum_{j=1}^{k} P(\xi_1 = j \mid \xi_0 = i) = \sum_{j=1}^{k} P(A_j^1 \mid A_i^0) = \sum_{j=1}^{k} \frac{P(A_j^1 \cap A_i^0)}{P(A_i^0)} =$$

$$= \sum_{j=1}^{k} \frac{P(A_{ij}^1)}{P(A_i^0)} = \frac{1}{P(A_i^0)} \sum_{j=1}^{k} P(A_{ij}^1) = \frac{1}{P(A_i^0)} P\Big(\bigcup_{j=1}^{k} A_{ij}^1\Big) = \frac{1}{P(A_i^0)} P(A_i^0) = 1.$$

Thus matrix $Q(1)$ is a stochastic matrix; we call it the transition matrix after the first step.

We shall prove that in our experimental scheme the transition matrix after every step is just the matrix $Q(1)$. We have, by virtue of (1), that

$$p_i(1) = P(\xi_1 = i) = \sum_{l=1}^{k} P(\xi_0 = l) P(\xi_1 = i \mid \xi_0 = l) = \sum_{l=1}^{k} p_l(0) p_{li};$$

hence it follows that the distribution of the random variable ξ_1 is the set

$$\{(j, \sum_{l=1}^{k} p_l(0) p_{lj}) \colon j = 1, 2, \ldots, k\};$$

this means that

(2) $$\vec{m}_1 = m_0 Q(1) .$$

In turn we shall prove that the distribution of the random variable ξ_l depends only on the distribution of the random variable ξ_{l-1}, but it is independent of the distribution of the random variables in the sequences in question occurring "earlier".

According to the prescription of our procedure, $P(A_j^l \mid A_i^{l-1}) = P(A_j^1 \mid A_i^0)$ for $l = 2, 3, \ldots$, whence it follows that matrix $Q(1)$ is a transition matrix after any step (after one step in any stage). In a way similar to that used previously in the construction of the random variable ξ_1 leads to the result:

$$p_j^{(l)} = P(\xi_l = j) = \sum_{i=1}^{k} P(\xi_{l-1} = i) P(\xi_l = j \mid \xi_{l-1} = i) = \sum_{i=1}^{k} p_i(l-1) p_{ij} .$$

For $l > 1$ the distribution of the random variable ξ_l is the set

$$\{(j, \sum_{i=1}^{k} p_i(l-1) p_{ij}) \colon j = 1, 2, \ldots, k\} .$$

hence

(3) $$\vec{m}_l = m_{l-1}Q(1); \text{ q.d.e.}$$

Every step of our procedure (of our experimental scheme) is described by a stochastic variable.

The sequence $\xi_0, \xi_1, \xi_2, \ldots$ is a special sequence, in which the distribution of every variable (the variable ξ_0 excepted) depends only on the distribution of the variable immediately preceding it in the sequence.

$$P(\xi_n = i_n | \xi_{n-1} = i_{n-1}, \ldots, \xi_0 = i_0) = P(\xi_n = i_n | \xi_{n-1} = i_{n-1})$$

for $n = 2, 3, \ldots,$ where $i_n, i_{n-1}, \ldots, i_0 = 1, 2, \ldots, k$.

This specific relation is the factor linking these variables in a chain.

The sequence of the random variables constructed in the above fashion is a model of a homogeneous Markoff's chain.

The stochastic variables $\xi_0, \xi_1, \xi_2, \ldots$ take the same values; however, the distributions of the variables do not need to be identical. For instance assuming that the initial mesure is $\vec{m}_0 = [\frac{1}{6}, \frac{2}{6}, \frac{3}{6}]$

$$Q(1) = \begin{bmatrix} \frac{2}{6} & \frac{1}{6} & \frac{3}{6} \\ \frac{1}{6} & 0 & \frac{5}{6} \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

(it is easy to establich in this case the composition of the urns U_0, U_1, U_2, U_3), then $\vec{m}_1 = [\frac{13}{36}, \frac{7}{36}, \frac{16}{36}]$.

In this case not only the random variables ξ_0 and ξ_1, but also the subsequent variables are different.

We may ask the question whether it is possible to select the composition of the balls of different kinds in the urn U_0 in a way ensuring the identical distribution of the random variables ξ_0 and ξ_1; in other words ensuring the equality $\vec{m}_0 Q(1) = \vec{m}_0$.

Thus the problem consists in deciding whether for a given transition matrix $Q(1)$, after one step, it is possible to establish the initial measure

$$\vec{m}_0 = [p_1(0), p_2(0), \ldots, p_k(0)]$$

in such a way leading to the realization of the relation below:

$$[p_1(0), p_2(0), \ldots, p_k(0)]\, Q(1) = [p_1(0), p_2(0), \ldots, p_k(0)]\,.$$

Because of (3) the theorem below is valid:

If $\vec{m}_1 = \vec{m}_0$, then the distribution of all the random variables in the sequence (ξ_n) are is identical. Vector $\vec{m}_0$, which for a given transition matrix is an object of our research is known under the designation of a invariant measure.

Thus our problem is reduced to the demonstration of the existence (for a given stochastic matrix) of positive numbers $p_1(0), p_2(0), \dots, p_k(0)$ fulfilling the relations

$$\text{(4a)} \qquad \sum_{i=1}^{k} p_i(0) p_{ij} = p_j(0) \quad \text{for} \quad j = 1, 2, \dots, k$$

and

$$\text{(4b)} \qquad \sum_{i=1}^{k} p_i(0) = 1 .$$

The relations (4a) describe a homogeneous system of equations, whose principal matrix has the form $[q_{ij}]$,
where

$$q_{ij} = \begin{cases} p_{ij} & \text{for} \quad i \neq j \\ p_{ij} = 1 & \text{for} \quad i = j \end{cases} \left([p_{ij}] = Q(1)\right).$$

$p_{i1} + p_{i2} + \dots + p_{ik} = 1$; hence it follows that if we add to the first column all the other columns we obtain a matrix, whose first column is equal to zero; this means that the system (4a) has non zero solutions. In [4] it is shown that all the non zero components of each non zero solution of system (4a) have the same sign; hence it follows that after normalisation of the vector which is a non zero solution of (4a) this vector will fulfill condition (4b).

It follows also from the theorem proved in [4] that every matrix $Q(1) > 0$ is characteried by an unique invariant measure; in the case $Q(1) \geqslant 0$ there exists at least one invariant measure. In the example above there is only one invariant measure, namely the vector

$$\vec{m}_0 = [\tfrac{20}{54}, \tfrac{11}{54}, \tfrac{23}{54}] .$$

In case of matrix

$$Q(1) = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} .$$

There is a great number of invariant measures: they have the form

$$\vec{m}_0 = [\tfrac{3}{7}(1-\alpha), \alpha, \tfrac{4}{7}(1-\alpha)] ,$$

where $0 < \alpha < 1$.

If the composition of the balls in urn U_0 is such that the vector

$$[p_1(0), p_2(0), \dots, p_k(0)]$$

is an invariant measure (it is always possible to make such a choice of the balls), then the experimental scheme introduced in this study describes a sequence of stochastic variables which is a stationary chain of Markoff.

Assuming the existence of such an l that for $i = 1, 2, \dots, k$ the condition $s_{il} > 0$ will be fulfilled (this means that the number of balls with number 1 is a positive number in each of the urns $U_1, U_2, \dots, U_k$) then the sequence (ξ_n) is an ergodic chain.

Let $Q(n) = [p_{ij}(n)]$ be a transition matrix after n steps. It is easy to show that in the case of our chain $Q(n) = [Q(1)]^n$. It follows from the ergodic theorem that for $i = 1, 2, ..., k$ $\lim_{n\to\infty} p_{ij}(n) = p_j$. Matrix $Q(n)$ tends to a matrix having the same elements in the columns. If for $i, j = 1, 2, ..., k$ we have $s_{ij}>0$, then by the ergodic theorem we shall have $p_j>0$ for $j = 1, 2, ..., k$.

We shall modify our experimental scheme in following way: Besides urn U_0 we have the sequence of U_{ij} urns ($i = 1, 2, ..., k$; $j = 1, 2, ...$). As before we draw a ball from urn U_0. This is the initial stage. If i is the number of the balls extracted at that stage then the following drawing (this is first stage) is made from the urn U_{i1}. The urns $U_{11}, U_{21}, ..., U_{k1}$ serve to accomplish the drawing at first stage. If the ball drawn at first stage is a ball with number m, then the subsequent drawing (stage number 2) is made from urn U_{m2} and so on. We shall consider — as previously — the sequence of random variables ξ_n, where ξ_n is the number of balls drawn at n stage.

The transition matrix $Q(0, 1)$ from the initial to the first stage is determinated by the distribution of the balls in the urns $U_{12}, U_{22}, ..., U_{k2}$. In this case a sufficient condition is the existence of such m and l from the set $\{1, 2, ..., k\}$ that the probability of drawing a ball with number l from the urn U_{m1} should be different from the probability of drawing a ball with the same number from the urn U_{m2}, i.e. $Q(0, 1) \neq Q(1, 2)$. In the case of such a collection of urns the sequence ξ_n is am nonhomogeneous Markoff's chain. It is easy to verify that

$$\vec{m}_1 = \vec{m}_0 \cdot Q(0, 1),$$

$$\vec{m}_2 = \vec{m}_1 \cdot Q(1, 2)$$

or more generally

$$\vec{m}_n = \vec{m}_{n-1} \cdot Q(n-1, n).$$

Finally let us apply the definition of a regular chain, of a positive and regular chain and of a completely regular chain — defined in [2] — to our case. According to [4] a regular chain is an ergodic chain. A regular positive chain is a chain for which all the limit probabilities are positive. On the base of the already mentioned theorem in [2] every ergodic chain in the sense of Markoff [2] is a positive regular chain in the sense of Romanowski [4].

According to [4] a completely regular chain is a positive regular chain, in which all the limit probabilities p_j are equal (in our case this means that if in each urn the number of balls of all kinds were positive, then the limit probabilities would all assume the value $1/k$). If $\lambda_0 = 1$ is a simple eigenvalue of the matrix Q and all the other eigenvalues are smaller than unity — then the matrix Q is called in [4] a regular matrix.

From [4] the conclusion follows that if the matrix $Q(1)$ is a regular one in the sense given above — then a homogeneous Markoff's chain defined by its help is an ergodic chain.

It follous from the theorem (4) in [4] (page 16) that every positive is a regular one.

References

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