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Some remarks on congruences in multiplicative systems

1. Introduction

1.1. It is well known that in a groupoid $(G, \cdot)$ every relation $\varrho_0 \subset G \times G$ generates a congruence (one side congruence) in $(G, \cdot)$. In [1] (see p. 37, 38) we find the construction of such generated congruence in a semigroup $(G, \cdot)$ for an arbitrary ϱ_0.

The equivalence classes of a congruence ϱ in $(G, \cdot)$ constitute the partition of G which is invariant under the operation $\cdot$, i.e. if A is an equivalence class of ϱ, $x \in G$, then $x \cdot A$ and $A \cdot x$ are contained in an equivalence class.

The invariant (one-side invariant) partitions of more general systems are used in the theory of the equation of translation and in the theory of algebraic objects (see [2], [3], [4]).

In this paper we consider congruences in multiplicative system $(S, \circ)$, where the domain $D_\circ$ of the operation "$\circ$" is a nonvoid subset of $S \times S$. We first examine the connections between congruences and homomorphisms of such systems. Later we construct certain one side congruences in a category.

1.2. Let $(S, \circ)$ be a multiplicative system. The domain of the operation $\circ$ will be denoted by $D_\circ$. For arbitrary nonvoid $A, B \subset S$ we denote, as usual

$$A \circ B = \{x \circ y \colon x \in A \wedge y \in B\} = \{s \in S \colon \bigvee_{x, y \in S} (x, y) \in D_\circ \wedge s = x \circ y\}.$$

By analogy to [1] (see p. 21) we define a left [right] ideal of $(S, \circ)$ as a nonempty subset A of S such that $S \circ A \subset A$ [$A \circ S \subset A$].

It is evident that every left [right] ideal of $(S, \circ)$ is closed under the operation $\circ$, i.e. $A \circ A \subset A$. If $A \circ A \neq \emptyset$ then the left [right] ideal A can be treated as the multiplicative system $(A, *)$, $*$ being the restriction of $\circ$ to A.

1.3. Let $(S_i)_{i \in J}$ be a given partition of S, i.e. $S = \bigcup_{i \in J} S_i$, $S_i \neq \emptyset$ for $i \in J$, $S_i \cap S_j = \emptyset$ for $i \neq j$, $i, j \in J$. After [2] we recall the notion of the invariant partitions. $(S_i)_{i \in J}$ is said to be left invariant under $\circ$ (shortly 1-invariant) in $(S, \circ)$ provided it satisfies the condition

$$\bigwedge_{i \in J} \bigwedge_{s \in S} \bigvee_{j \in J} s \circ S_i \subset S_j.$$

An r-invariant partition is defined by the condition

$$\bigwedge_{i\in J}\bigwedge_{s\in S}\bigvee_{j\in J} S_i\circ s\subset S_j .$$

Let now A be a left ideal of $(S,\circ)$ and $(A_t)_{t\in T}$ a given partition of A. $(A_t)_{t\in T}$ is said to be left invariant (l-invariant) in $(S,\circ)$ partition of A provided it fulfils

$$\bigwedge_{t\in T}\bigwedge_{s\in S}\bigvee_{\tau\in T} s\circ A_t\subset A_\tau .$$

Obviously if we replace the condition $s\circ A_t\subset A_\tau$ by $A_t\circ s\subset A_\tau$ then we obtain an r-invariant in $(S,\circ)$ partition of A.

If * is the restriction of "$\circ$" to A then $(A,*)$ can have l-invariant [r-invariant] partitions which are not l-invariant [r-invariant] in $(S,\circ)$. There is evidently true the following:

LEMMA 1. *Let $(S_i)_{i\in J}$ be an l-invariant partition of $(S,\circ)$ and A be a given left ideal of this system. Denote $I=\{i\in J\colon\ S_i\cap A\neq\emptyset\}$. Then $(S_i\cap A)_{i\in I}$ is an l-invariant in $(S,\circ)$ partition of A.*

The analogous statement is true for r-invariant partitions in right ideals of $(S,\circ)$.

1.4. We introduce now the definitions of congruences in $(S,\circ)$.

DEFINITION 1. *Let $\varrho\subset S\times S$ be an equivalence relation in S.*

(1) *ϱ is said to be a left congruence (shortly l-congruence) [right congruence, (r-congruence)] in $(S,\circ)$ iff*

$$\bigwedge_{a,b,c\in S}\big(((a,b)\in D_\circ\wedge(a,c)\in D_\circ\wedge b\varrho c)\Rightarrow a\circ b\varrho a\circ c\big)$$

$$\Big[\bigwedge_{a,b,c\in S}\big(((b,a)\in D_\circ\wedge(c,a)\in D_\circ\wedge b\varrho c)\Rightarrow b\circ a\varrho c\circ a\big)\Big] .$$

(2) *ϱ is said to be a congruence in $(S,\circ)$ iff*

$$\bigwedge_{a,b,c,d\in S}\big(((a,c)\in D_\circ\wedge(b,d)\in D_\circ\wedge a\varrho b\wedge c\varrho d)\Rightarrow a\circ c\varrho b\circ d\big) .$$

(3) *ϱ is said to be the l-congruence [r-congruence, congruence] in $(S,\circ)$ generated by a given $\varrho_0\subset S\times S$ provided it is the minimal l-congruence [r-congruence, congruence] in $(S,\circ)$ containing ϱ_0. (ϱ is the intersection of all the l-congruences [r-congruences, congruences] in $(S,\circ)$ containing ϱ_0).*

If $D_\circ=S\times S$ then these definitions lead to well known definitions of congruences in a groupoid and the following statement is true: an equivalence relation ϱ is S is a congruence in $(S,\circ)$ if and only if it is both a l-congruence and a r-congruence in $(S,\circ)$. If $D_\circ\neq S\times S$ then the above mentioned statement is not true. For example put $S=\{a,b,c,d\}$ and define $\circ$ by the table (see p. 149).

Let ϱ be such the equivalence relation in S that $S/\varrho=\{\{a,b\},\{c,d\}\}$. This ϱ is both a l-congruence and a r-congruence in $(S,\circ)$ but it is not a congruence in this system. It suffices to observe that $a\varrho b$ and $c\varrho d$, $b=a\circ c$, $c=b\circ d$ and $\sim b\varrho c$.

Of course, if ϱ is a congruence in $(S,\circ)$ then it is a l-congruence and an r-congruence in this system.

o	a	b	c	d
a	×	d	b	×
b	c	×	×	c
c	×	×	×	d
d	a	×	c	×

Definition 2. *Let A be a left [right] ideal of $(S, \circ)$, ϱ an equivalence relation in A. Then ϱ is said to be a strong l-congruence [r-congruence] in A provided it fulfils the condition*

$$\bigwedge_{s\in S}\bigwedge_{a,b\in A}\big(((s, a)\in D_\circ\wedge(s, b)\in D_\circ\wedge a\varrho b)\Rightarrow s\circ a\varrho s\circ b\big)$$

$$\Big[\bigwedge_{s\in S}\bigwedge_{a,b\in A}\big(((a, s)\in D_\circ\wedge(b, s)\in D_\circ\wedge a\varrho b)\Rightarrow a\circ s\varrho b\circ s\big)\Big].$$

There is true the following

Theorem 1.

(1) *A partition $(S_i)_{i\in I}$ of S is l-invariant [r-invariant] in $(S, \circ)$ iff it is the partition into equivalence classes of certain l-congruence [r-congruence] in $(S, \circ)$.*

(2) *A partition $(A_t)_{t\in T}$ of a left [right] ideal of $(S, \circ)$ is l-invariant [r-invariant] in $(S, \circ)$ iff it is the partition into equivalence classes of certain strong l-congruence [r-congruence] in A.*

Proof. We give the proof of the statement (2); the statement (1) can be proved analogously. We are going to examine the case where A is an l-ideal of $(S, \circ)$ (the other case is analogous). Suppose that $(A_t)_{t\in T}$ is an l-invariant in $(S, \circ)$ partition of a left ideal A of $(S, \circ)$. If $x\varrho y\Leftrightarrow \bigvee_{t\in T} x, y\in A_t$ for $x, y\in A$, then ϱ is an equivalence relation in A and A_t are its equivalence classes. Let us take $s\in S$, $a, b\in A$ such that $a\varrho b$, i.e. $a, b\in A_t$ for some $t\in T$ and $(s, a)\in D_\circ$, $(s, b)\in D_\circ$. We find a $\tau\in T$ such that $s\circ A_t\subset A_\tau$. Thus $s\circ a$, $s\circ b\in A_\tau$ and $s\circ a\varrho s\circ b$. Conversely, suppose that ϱ is a strong l-congruence in A and consider the partition $(A_t)_{t\in T}$ of A into its equivalence classes. For any $s\in S$ and $t\in T$ if $s\circ A_t = \emptyset$ then $s\circ A_t\subset A_t$. Assume that $s\circ A_t\neq\emptyset$. Thus we can find an $a\in A_t$ such that $(s, a)\in D_\circ$. The element $s\circ a$ must belong to an equivalence class, say A_τ, $\tau \in T$. Let $x \in A_t$ and $(s, x) \in D_\circ$. Since $a\varrho x$ and ϱ is a strong l-congruence in A then $s \circ a\varrho s \circ x$, i.e. $s \circ x \in A_\tau$. Thus $s \circ A_t \subset A_\tau$.

1.5. We recall the notions of a homomorphism and an isomorphism of multiplicative systems.

Definition 3. *Let* $(S, \circ)$, (T, Δ) *be multiplicative systems and* f *be a mapping of* S *into* T.

(1) f *is said to be a homomorphism of* $(S, \circ)$ *into* (T, Δ) *iff it sulfils the following conditions*:

(i) $\bigwedge_{x, y \in S} (x, y) \in D_\circ \Rightarrow (f(x), f(y)) \in D_\Delta,$

(ii) $\bigwedge_{x, y \in S} (x, y) \in D_\circ \Rightarrow f(x \circ y) = f(x) \Delta f(y).$

(2) *A homomorphism* f *of* $(S, \circ)$ *into* (T, Δ) *is said to be an isomorphism of* $(S, \circ)$ *onto* (T, Δ) *if and only if it satisfies the conditions*:

(iii) f *is a bijective mapping of* S *onto* T,

(iv) $\bigwedge_{x, y \in S} (x, y) \in D_\circ \Leftrightarrow (f(x), f(y)) \in D_\Delta.$

(3) *If there exists an isomorphism of* $(S, \circ)$ *onto* (T, Δ) *then we say that the systems* $(S, \circ)$ *and* (T, Δ) *are isomorphic.*

2. Congruences and homomorphisms of multiplicative systems

2.1. Let ϱ be a congruence in $(S, \circ)$. For an arbitrary $x \in S$ the equivalence class of ϱ containing x will be denoted by $[x]_\varrho$. The operation $\circ$ in S induces the operation $\odot$ in S/ϱ, defined as follows:

(i) $D_\odot = \{(A, B) \in S/\varrho \times S/\varrho : \bigvee_{a, b \in S} A = [a]_\varrho \wedge B = [b]_\varrho \wedge (a, b) \in D_\circ\},$

(ii) if $(A, B) \in D_\odot$ and $A = [a]_\varrho$, $A = [b]_\varrho$, $(a, b) \in D_\circ$ then $A \odot B = [a \circ b]_\varrho$. The system $(S/\varrho, \odot)$ will be called the quotient system of $(S, \circ)$ towards the congruence ϱ.

It is clear that $h: S \ni x \to [x]_\varrho$ is a homomorphism of $(S, \circ)$ onto $(S/\varrho, \odot)$. This homomorphism h is said to be the natural homomorphism of $(S, \circ)$ onto $(S/\varrho, \odot)$.

2.2. The following theorem is analogous to the fundamental theorem on homomorphisms of algebraic structures.

Theorem 2. *Let* $f: S \to T$ *be a homomorphism of* $(S, \circ)$ *into* (T, Δ), $f(S) = T$, *and let be satisfied the condition*:

(H): $\bigwedge_{u, v \in T} (u, v) \in D_\Delta \Rightarrow \bigvee_{x, y \in S} ((x, y) \in D_\circ \wedge u = f(x) \wedge v = f(y)).$

Then

(1) $\varrho = f^{-1} f$ *is a congruence in* $(S, \circ)$,

(2) $(S/\varrho, \odot)$ *and* (T, Δ) *are isomorphic and there exists an isomorphism* φ *of* $(S/\varrho, \odot)$ *onto* (T, Δ) *such that* $f = \varphi h$, *where* h *is the natural homomorphism of* $(S, \circ)$ *onto the quotient system* $(S/\varrho, \odot)$.

Proof. First observe that if $x, y \in S$ then $x\varrho y \Leftrightarrow \bigvee_{u \in T} u = f(x) \wedge u = f(y) \Leftrightarrow f(x) = f(y)$ and thus ϱ is an equivalence relation in S. Later if $x\varrho y$, $z\varrho t$, $(x, z) \in D_\circ$ and $(y, t) \in D_\circ$ then $f(x) = f(y)$, $f(z) = f(t)$, $\big(f(x), f(z)\big) \in D_\Delta$, $\big(f(y), f(t)\big) \in D_\Delta$ and $f(x \circ z) = f(x) \Delta f(z)$, $f(y \circ t) = f(y) \Delta f(t)$; thus $f(x \circ z) = f(y \circ t)$ and $x \circ z \varrho y \circ t$. Hence (1) is proved.

To prove (2) observe that $A \in S/\varrho \Leftrightarrow \bigvee_{t \in T} A = f^{-1}(\{t\})$ so that if $a \in A$ then $f(a) = t$. Next for arbitrary $x \in S$ we put

$$\varphi([x]_\varrho) = f(x) .$$

φ is a bijective mapping of S/ϱ on T. Let now $A, B \in S/\varrho$ and $(A, B) \in D_\odot$. We find $a \in A$, $b \in B$ such that $(a, b) \in D_\circ$ and we obtain

$$A \odot B = [a \circ b]_\varrho, \ \varphi(A) = f(a), \ \varphi(B) = f(b) .$$

Thus $\big(f(a), f(b)\big) \in D_\Delta$ and $\varphi(A \odot B) = f(a \circ b) = f(a) \Delta f(b) = \varphi(A) \Delta \varphi(B)$. Suppose at last that $u, v \in T$ and $(u, v) \in D_\Delta$. By the hypothesis (H) we find $x, y \in S$ such that $u = f(x)$, $v = f(y)$ and $(x, y) \in D_\circ$. Hence $u = \varphi([x]_\varrho)$, $v = \varphi([y]_\varrho)$ and $([x]_\varrho, [y]_\varrho) \in D_\odot$.

Thus we have proved that φ is an isomorphism of $(S/\varrho, \odot)$ onto (T, Δ). For given $x \in S$ we immediately obtain $\varphi\big(h(x)\big) = \varphi([x]_\varrho) = f(x)$ and the proof is complete.

COROLLARY 1. The hypothesis (H) in theorem 2 is essential.

To see it consider $S = \{a, b, c, d, e\}$, $T = \{x, y\}$ and define the operations $\circ$, Δ by the tables:

$\circ$	a	b	c	d	e
a	×	×	b	×	×
b	×	×	×	×	a
c	×	×	×	e	×
d	×	a	×	×	×
e	b	×	×	×	×

Δ	x	y
x	x	x
y	x	y

Let us put $f(a) = f(b) = x, f(c) = f(d) = f(e) = y$. It is evident that f is a homomorphism of $(S, \circ)$ onto (T, Δ) which does not fulfil (H). We see that condition (2) of theorem 2 is not true, since $(x, x) \in D_\Delta$ and $(f^{-1}(\{x\}), f^{-1}(\{x\})) \notin D_\odot$.

3. Certain congruences in categories

3.1. Let $(K, \circ)$ be a category (see [2]), K^0 the set of its units. As in [2] we define for $e \in K^0$

$$K_e = \{x \in K\colon (x, e) \in D_\circ\},$$

$$_eK = \{x \in K\colon (e, x) \in D_\circ\},$$

$$_eK_e = \{x \in K\colon (e, x) \in D_\circ \wedge (x, e) \in D_\circ\}.$$

Of course the following equalities are true: $_eK = e \circ K$, $K_e = K \circ e$, $_eK_e = (e \circ K) \circ e = e \circ (K \circ e) = {_eK} \cap K_e$.

Furthermore $_eK_e$ is a subsemigroup of a category $(K, \circ)$ with the unit e, K_e is a left ideal and $_eK$ is a right ideal of $(K, \circ)$.

3.2. Let $\varrho_0 \subset K \times K$. Denote by i_K the identity relation in K. We shall construct a congruence ϱ in $(K, \circ)$ determined by ϱ_0.

DEFINITION 4. *We introduce the relations*:

$$\varrho_1 = \varrho_0 \cup \varrho_0^{-1} \cup i_K,$$

$$\varrho_2 = \{(x, y)\colon \bigvee_{a,b,c \in K} ((c, a) \in D_\circ \wedge (c, b) \in D_\circ \wedge x = c \circ a \wedge y = c \circ b \wedge a\varrho_1 b),$$

$$\varrho_3 = \varrho_2 \cup i_K,$$

$$\varrho = \bigcup_{n=1}^{\infty} \varrho_3^n, \ \varrho_3^n \textit{ being } n\textit{-th iterate of } \varrho_3.$$

LEMMA 2. *Let ϱ_3 be the relation introduced by definition* 4.
If $x, y, p \in K$, $x\varrho_3 y$ and $(p, x) \in D_\circ$ then $(p, y) \in D_\circ$.

Proof. By the definition $x\varrho_3 y \Leftrightarrow x\varrho_2 y \vee x = y$ and it suffices to prove that $x\varrho_2 y$ and $(p, x) \in D_\circ$ imply $(p, y) \in D_\circ$.

But if $x\varrho_2 y$ then there exist $a, b, c \in K$ such that $(c, a) \in D_\circ$, $(c, b) \in D_\circ$, $a\varrho_1 b$ and $x = c \circ a$, $y = c \circ b$. Thus $(p, c \circ a) \in D_\circ$ and in consequence $(p, c) \in D_\circ$. Hence by $(c, b) \in D_\circ$ we obtain $(p, c \circ b) \in D_\circ$ i.e. $(p, y) \in D_\circ$.

LEMMA 3. *If ϱ_3 is the relation from definition* 4 *then it is a reflexive and symmetric relation in K, fulfilling the condition (see def.* 1 (1)):

$$(*) \qquad \bigwedge_{x,y,p \in K} (((p, x) \in D_\circ \wedge (p, y) \in D_\circ \wedge x\varrho_3 y) \Rightarrow p \circ x\varrho_3 p \circ y).$$

Proof. By its definition ϱ_3 contains the identity relation on K, whence it is reflexive.

Since ϱ_1 is symmetric, by the definition of ϱ_2 we conclude the symmetry of ϱ_2 and consequently that of ϱ_3.

Suppose now that $x\varrho_3 y$, $(p, x) \in D_\circ$, $(p, y) \in D_\circ$, $x, y, p \in K$. Notice that $x\varrho_3 y$ implies $x = y$ or $x = c \circ a$, $y = c \circ b$, $a\varrho_1 b$ for some $a, b, c \in K$.

In the case where $x = y$ since $(p, x) \in D_\circ$ then $p \circ x = p \circ y$ and $p \circ x\varrho_3 p \circ y$.

In the other case $p \circ x = p \circ (c \circ a) = (p \circ c) \circ a$, $p \circ y = p \circ (c \circ b) = (p \circ c) \circ b$, whence $p \circ x\varrho_3 p \circ y$.

These lemmas are useful to prove the following

THEOREM 3. *The relation ϱ defined in definition 4 is a left congruence in $(K, \circ)$.*

Proof. There is $i_K \subset \varrho_3 \subset \varrho$ and ϱ is a reflexive relation in K. Let $x\varrho^{-1}y$ i.e. $y\varrho x$. Thus for some positive integer n there is $y\varrho_3^n n$ and there exist $u_1, \ldots, u_{n-1}$ such that $y\varrho_3 u_1, u_1\varrho_3 u_2, \ldots, u_{n-1}\varrho_3 x$. Since ϱ_3 is symmetric it follows that $x\varrho_3 u_{n-1}, \ldots, u_1\varrho_3 y$, whence $x\varrho y$, what proves the symmetry of ϱ.

The transitivity of ϱ is immediately seen from its definition (ϱ is the transitive closure of ϱ_3).

Suppose now that $x, y, p \in K$, $x\varrho y$, $(p, x) D_\circ$ and $(p, y) D_\circ$. There is — as above — $x\varrho_3 u_1, u_1\varrho_3 u_2, \ldots, u_{n-1}\varrho_3 y$ for suitable $u_1, \ldots, u_{n-1}$ from K. Since $(p, x) \in D_\circ$, then using lemma 2, we obtain succesively

$$(p, u_1) \in D_\circ, (p, u_2) \in D_\circ, \ldots, (p, u_{n-1}) \in D_\circ .$$

Applying lemma 3 we conclude that there is

$$p \circ x\varrho_3 p \circ u_1, p \circ u_1\varrho_3 p \circ u_2, \ldots, p \circ u_{n-1}\varrho_3 p \circ y .$$

Then $p \circ x\varrho_3^n p \circ y$ and $p \circ x\varrho p \circ y$.

Remark 1. (a) It is immediately seen that if ϱ_0 itself is a left congruence in $(K, \circ)$ then $\varrho_1 = \varrho_0$, $\varrho_2 \subset \varrho_0$, and $\varrho_3 \subset \varrho_0$, $\varrho \subset \varrho_0$. Further notice that if ϱ_0 is a left congruence in $(K, \circ)$ and every equivalence class of ϱ_0 is contained in some ideal ${}_eK$, $e \in K^0$, then $\varrho = \varrho_0$, and every left congruence in $(K, \circ)$ possesing above mentioned property (every equivalence class is contained in ${}_eK$) can be constructed in the same manner as ϱ in definition 4. (b) It follows from theorem 1 that the equivalence classes of ϱ form an l-invariant partition of $(K, \circ)$. (c) Theorem 4 and lemma 1 allow to construct strong left congruences in K_e. (d) After simple modifications of the above considerations we can obtain r-congruences in $(K, \circ)$ and strong r-congruences in ${}_eK$. (e) The following example shows that ϱ need not contain ϱ_0. We take $K = \{a, b, c, d\}$, $\varrho_0 = \{(a, b), (b, c)\}$ and define the operation $\circ$ as follows

$\circ$	a	b	c	d
a	×	d	a	×
b	c	×	×	b
c	×	b	c	×
d	a	×	×	d

Then

$$\varrho_1 = \{(a, b), (b, c), (b, a), (c, b), (a, a), (b, b), (c, c), (d, d)\},$$

$$\varrho_2 = \{(a, d), (d, a), (a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\},$$

$$\varrho_3 = \varrho_2, \varrho = \varrho_3 \text{ and } \varrho_0 \not\subset \varrho.$$

3.3. In the sequel we suppose that $e \in K^0$ and $\varrho_0 \subset {}_eK \times {}_eK$. Our purpose is to prove in this case that the relation ϱ, constructed as in definition 4, is an l-congruence in $(K, \circ)$ generated by ϱ_0.

LEMMA 4. *If* $e \in K^0$ *and* $\varrho_0 \subset {}_eK \times {}_eK$ *then* $\varrho_0 \subset \varrho$.

Proof. Let $x\varrho_0 y$. Then $x, y \in {}_eK$ i.e. $x = e \circ x$, $y = e \circ y$. Since $\varrho_0 \subset \varrho_1$ then $x\varrho_1 y$. Thus $x\varrho_2 y$ and consequently

$$\varrho_0 \subset \varrho_1 \subset \varrho_2 \subset \varrho_3 \subset \varrho\,.$$

LEMMA 5. *If* $e \in K^0$ *and* $\varrho_0 \subset {}_eK \times {}_eK$ *then* ϱ_3 *is the least reflexive and symmetric relation in* $(K, \circ)$ *containing* ϱ_0 *and sulfilling the condition* (*) *of lemma* 3.

Proof. According to lemmas 4, 3 it suffices to show that if $\gamma \subset K \times K$ is a reflexive, symmetric relation containing ϱ_0 and fulfilling the condition (*) then $\varrho_3 \subset \gamma$.

In fact if $\varrho_0 \subset \gamma$, where γ is a reflexive and symmetric relation in K then $\varrho_0^{-1} \subset \gamma^{-1} = \gamma$ and $i_K \subset \gamma$, so that $\varrho_1 \subset \gamma$.

Let now $x\varrho_3 y$ i.e. either $x = y$ or $x = p \circ a$, $y = p \circ b$, $a\varrho_1 b$ for some $a, b, p \in K$. When $x = y$ then $x\gamma y$. In the other case by the above mentioned statement we obtain $a\gamma b$ and since γ fulfils (*) then there is $x\gamma y$. Thus we have proved that $\varrho_3 \subset \gamma$.

Now we can formulate the following

THEOREM 4. *If* $e \in K^0$ *and* $\varrho_0 \subset {}_eK \times {}_eK$ *then* ϱ *constructed as in definition* 4 *is the least l-congruence in* $(K, \circ)$ *containing* ϱ_0 *(generated by* ϱ_0*).*

This theorem is an immediate consequence of lemmas 4 and 5 and theorem 3.

COROLLARY 2. (a) From the above theorem we may conclude that for every subset A of ${}_eK$ there exists an l-congruence ϱ in $(K, \circ)$ such that A is contained in one equivalence class of ϱ. (b) It is very easy to obtain the analogous results for r-congruences in $(K, \circ)$.

3.4. For the sequel suppose that $e \in K^0$, H is a subsemigroup with the unit e of the semigroup ${}_eK_e$ and $\varrho_0 = H \times H$. These assumptions we accept in the whole of this section. Let ϱ be the l-congruence in $(K, \circ)$ generated by ϱ_0.

If $(K_i)_{i \in J}$ denotes the partition of K into the equivalence classes of ϱ we obtain by theorem 1 and lemma 1, that $(K_i \cap K_e)_{i \in J^*}$, $J^* = \{i \in J\colon K_i \cap K_e \neq \emptyset\}$ is an l-invariant partition of K_e in $(K, \circ)$ and if σ is the strong l-congruence in K_e determining the above partition then $\sigma = \varrho \cap (K_e \times K_e)$.

THEOREM 5. *The above defined relation* $\sigma = \varrho \cap (K_e \times K_e)$ *is the minimal strong l-congruence in* K_e, *containing* ϱ_0.

Proof. It suffices to verify that if λ is a strong l-congruence in K_e containing ϱ_0 then $\sigma \subset \lambda$.

To prove it first observe that if $x\sigma y$ then $x\varrho y$ and $x, y \in K_e$. Hence we can find $u_1, \ldots, u_{n-1} \in K$ such that $x\varrho_3 u_1, \ldots, u_{n-1}\varrho_3 y$. When $x = u_1$ then $x\lambda u_1$. Suppose now that $x \neq u_1$. We can find $a, b, c \in K$ such that $x = c \circ a$, $y = c \circ b$ and $a\varrho_1 b$. Since $x \in K_e$ then from $x = c \circ a$ we obtain that $a \in K_e$. From the assumption $\varrho_0 = H \times H$ it follows that $\varrho_1 = H \times H \cup i_K$. Thus, since $a \in K_e$ and $a\varrho_1 b$, we can conclude that $b \in K_e$ and finally $u_1 \in K_e$. Now observe that from $a\varrho_1 b$ and $a, b \in K_e$ we may conclude that $a\varrho_0 b$ and consequently $a\lambda b$. Furthermore since λ is strong l-congruence in K_e then $c \circ a\lambda c \circ b$ i.e. $x\lambda u_1$.

The analogous considerations lead to the conclusion that $u_2, \ldots, u_{n-1} \in K_e$ and $u_1\lambda u_2, \ldots, u_{n-1}\lambda y$. Hence $x\lambda y$ and consequently $\sigma \subset \lambda$.

To determine the family of the equivalence classes of σ in K_e we first recall after [2] (see p. 34) that $\bigcup_{b \in K_e} b \circ H = K_e$ i.e. that the family $(b \circ H)_{b \in K_e}$ covers K_e.

LEMMA 6. *For every* $x, y \in K$ *the following equivalence holds*

$$x \in K_e \wedge x\varrho_3 y \Leftrightarrow \bigvee_{d \in K_e} x, y \in d \circ H\,.$$

Proof. Of course we have $x\varrho_3 y \wedge x \in K_e \Leftrightarrow x \in K_e \wedge (x\varrho_2 y \vee x = y) \Leftrightarrow x \in K_e \wedge \wedge x = y \vee x \in K_e \wedge \bigvee_{a,b,c \in K} ((c, a) \in D_\circ \wedge (c, b) \in D_\circ \wedge x = c \circ a \wedge y =$

$$= c \circ b \wedge (a, b \in H \vee a = b)).$$

Observe that $a = b$ implies $x = y$ and from $x = y \wedge x \in K_e$ we have $x = y = x \circ e \in x \circ H$. In the other case we have $x \in K_e$, $x = c \circ a$, $y = c \circ b$, where $a, b \in H$ and $c \in K$. From $H \subset {}_eK_e$ and $(c, a) \in D_\circ$, $a \in H$ we first obtain that $c \in K_e$ and then $x, y \in c \circ H$. Conversely, if $d \in K_e$ and $x, y \in d \circ H$ then $x = d \circ h_1$, $y = d \circ h_2$, where h_1, h_2 are some dements of H. Hence $h_1 \varrho_0 h_2$ and in consequence $x\varrho_3 y$. This implies $x\varrho_3 y$. Moreover, since $H \subset_e K_e$ then $d \circ H \subset K_e$ and therefore $x, y \in K_e$. Thus our lemma is proved.

LEMMA 7. *For every* $x, y \in K_e$ *we have*

$$x\sigma y \Leftrightarrow \bigvee_{d_1,\dots,d_m \in K_e} (d_i \circ H \cap d_{i+1} \circ H \neq \emptyset, i = 1, \dots, m-1 \wedge x, y \in \bigcup_{i=1}^{m} d_i \circ H).$$

Proof. First observe that $x\sigma y$ implies that $x, y \in K_e$ and there exist $u_1, \dots, u_{m-1}$ such that $x\varrho_3 u_1, \dots, u_{m-1}\varrho_3 y$. Hence by the preceding lemma we may choose $d_1, \dots, d_m \in K_e$ such that $x, u_1 \in d_1 \circ H$, $u_1, u_2 \in d_2 \circ H, \dots, u_{m-2}, y \in d_m \circ H$. These $d_1, \dots, d_m$ have all the properties stated in the lemma.

Conversely, if we have $d_1, \dots, d_m$ fulfilling all the conditions of our lemma then $x, y \in \bigcup_{i=1}^{m} d_i \circ H$, so that they belong to K_e and there are some r, s such that $x \in d_r \circ H$, $y \in d_s \circ H$. If $r = s$ then by the preceding lemma we obtain $x\varrho_3 y$ and $x\varrho y$, i.e. $x\sigma y$ because $x, y \in K_e$.

Otherwise suppose that $r<s$. From our assumption it follows that we can find $u_r, u_{r+1}, \dots, u_{s-1}$ satisfying the conditions

$$\begin{gathered} u_r \in d_r \circ H \cap d_{r+1} \circ H, \\ u_{r+1} \in d_{r+1} \circ H \cap d_{r+2} \circ H, \\ \dots\dots\dots\dots \\ u_{s-1} \in d_{s-1} \circ H \cap d_s \circ H. \end{gathered}$$

Hence $x, u_r \in d_r \circ H$, $u_r, u_{r+1} \in d_{r+1} \circ H, \dots, u_{s-1}, y \in d_s \circ H$ and by the preceding lemma $x\varrho_3 u_r, \dots, u_{s-1}\varrho_3 y$. Thus we obtain $x\varrho_3^{s-r} y$ and $x\varrho y$, i.e. $x\sigma y$. Our lemma is completely proved.

The next theorem states that K_e/σ is the maximal partition of K_e generated by the covering $(d \circ H)_{d \in K_e}$ in the sense of definition 10 from [2].

THEOREM 6.

If

(I) $A \in K_e/\delta$

then

(II) *there exists a* G, $\emptyset \neq G \subset K_e$ *possesing the properties*

(1) *if* $G' \subset G$ *and* $G' \neq \emptyset \neq G - G'$ *then* $\emptyset \neq \bigcup_{d \in G} d \circ H \cap \bigcup_{d \in G - G'} d \circ H$

(2) $A = \bigcup_{d \in G} d \circ H$

and every set A *fulfilling the condition* (II) *is contained in an equivalence class of* σ *in* K_e.

Proof. First we shall prove that (I) implies (II). Let $A \in K_e/\sigma$ and $A = [a]_\sigma$. For given $x \in [a]_\sigma$ using lemma 7 we choose $d_1, \dots, d_n \in K_e$ such that $x, a \in \bigcup_{i=1}^{n} d_i \circ H$ and $d_i \circ H \cap d_{i+1} \circ H \neq \emptyset$ for $i = 1, 2, \dots, n-1$.

Denoting $G_x = \bigcup_{i=1}^{n_x} \{d_i\}$ we see that $G_x \subset K_e$ and $x \in [a]_\sigma$ iff $x, a \in \bigcup_{d \in G_x} d \circ H$. Let $G = \bigcup_{x \in A} G_x$, then $A = \bigcup_{d \in G} d \circ H$. Suppose that $G' \subset G$ and $G' \neq \emptyset \neq G - G'$.

To prove the thesis of (1) consider first the case where $a \in \bigcup_{d \in G'} d \circ H$. From our assumptions it follows immediately that $\bigcup_{d \in G-G'} d \circ H \neq \emptyset$. Let z be an element of this set. Put $\bar{G}_z = (\bar{d}_1, \ldots, \bar{d}_s) \subset G_z$, where $\bar{d}_i \circ H \cap \bar{d}_{i+1} \circ H \neq \emptyset$ for $i = 1, \ldots, s-1$ and $z \in \bar{d}_1 \circ H$, $a \in \bar{d}_s \circ H$. If $G' \cap \bar{G}_z = \emptyset$ then $\bar{G}_z \subset G - G'$ and $\bigcup_{d \in \bar{G}_z} d \circ H \subset \bigcup_{d \in G-G'} d \circ H$. Thus $a \in \bigcup_{d \in G-G'} d \circ H$ too and (1) holds.

If $(G-G') \cap \bar{G}_z = \emptyset$ then using analogous considerations we come to conclusion that $z \in \bigcup_{d \in G'} d \circ H \cap \bigcup_{d \in G-G'} d \circ H$.

At last if $G' \cap \bar{G}_z \neq \emptyset \neq (G-G') \cap \bar{G}_z$ then we choose k, l belonging to $\{1, \ldots, s\}$ such that $\bar{d}_k \in G'$ and $\bar{d}_l \in G - G'$. Obviously $k \neq l$ and let us assume that $k < l$.

There exists an $r \in \{k, k+1, \ldots, l\}$ for which $\bar{d}_r \in G'$ and $\bar{d}_{r+1} \in G - G'$. Since $\bar{d}_r \circ H \cap \bar{d}_{r+1} \circ H \neq \emptyset$ then from above we obtain the thesis of (1). The proof in the case where a belongs to the second factor of the product in (1) is analogous.

Now we are coming to prove that if a set A satisfies (II) then there exists an equivalence class B of relation σ containing A. For, let $x, y \in A$. Then we can find a $d_x \in G$ for which $x \in d_x \circ H$. We define by recurence the sets:

$$D_0 = \{d_x\},\ D_{k+1} = \{d \in G\colon d \circ H \cap \bigcup_{p \in D_k} p \circ H \neq \emptyset\},\ k = 0, 1, 2, \ldots$$

Observe that for $d \in D_k$ holds $d \circ H \cap \bigcup_{p \in D_k} p \circ H = d \circ H \neq \emptyset$ i.e. $d \in D_{k+1}$. Thus $D_k \subset D_{k+1}$ for $k = 0, 1, 2, \ldots$

Let $D = \bigcup_{k=0}^{\infty} D_k$. D is evidently contained in G and we shall prove that $D = G$. Observe that if $p \in G$ and $p \circ H \cap \bigcup_{d \in D} d \circ H \neq \emptyset$ then we can find a $\bar{d} \in D$ such that $p \circ H \cap \bar{d} \circ H \neq \emptyset$. This $\bar{d}$ belongs to a D_i, $i \geqslant 0$ and $p \circ H \cap \bigcup_{d \in D_i} d \circ H \neq \emptyset$, so that $p \in D_{i+1} \subset D$. Hence the inequality $G - D \neq \emptyset$ implies

$$\bigcup_{d \in G-D} d \circ H \cap \bigcup_{d \in D} d \circ H = \emptyset,$$

what contradicts the hypothesis (II). Thus we have $D = G$.

Since $y \in A$ by the condition (II) (2) there exist an integer s and a $d_s \in D_s$ such that $y \in d_s \circ H$. Using the definition of the sequence $D_0, D_1, \ldots$ we can choose $d_{s-1} \in D_{s-1}$, $d_{s-2} \in D_{s-2}$, $d_0 \in D_0$ satisfying the conditions $d_i \circ H \cap d_{i+1} \circ H \neq \emptyset$ for $i = 0, 1, \ldots$ Since $x \in d_0 \circ H = d_x \circ H$ and $y \in d_s \circ H$ from the above statement by lemma 7 we obtain that $x \sigma y$. Thus x, y are the elements of the same equivalence class of σ in K_e and our theorem is completely proved.

Remark 2. (a) The analogous theorem for strong r-congruences in ${}_eK$ can be proved too. (b) From theorem 5 of the paper [2] we conclude that every l-invariant partition of K_e in $(K, \circ)$ is its decomposition into unions of equivalence classes of a strong l-congruence in K_e generated in it by $\varrho_0 = H \times H$, where H is a subsemigroup of ${}_eK_e$ with the unit e.

4. Congruences in semigroups

4.1. Since every semigroup $(S, \bullet)$ having the unit e is a category in which ${}_eS = S_e = {}_eS_e$ then we can apply directly the above obtained results to find congruences and invariant partitions of such $(S, \bullet)$ (see also [2] p. 44–46).

4.2. Suppose now that $(S, \bullet)$ is a semigroup without unit. If an $e \notin S$ then we put $S^1 = S \cup \{e\}$ and define the operation $\circ$ in S^1 as follows: $x \circ y = x \cdot y$ when $x, y \in S$ and $x \circ e = e \circ x$ when $x \in S^1$. Of course $(S^1, \circ)$ is a semigroup with the unit e.

THEOREM 7. *Let $(S_i)_{i \in J}$ be an l-invariant partition of $(S^1, \circ)$ and $e \in S_k$, $k \in J$. Then $(S_i)_{i \in J-\{k\}}$ in the case where $S_k = \{e\}$ and $(\bar{S}_i)_{i \in J}$, $\bar{S}_i = S_i$ for $i \neq k$, $\bar{S}_k = S_k - \{e\}$ in the case where $S_k \neq \{e\}$ are l-invariant partitions of $(S, \bullet)$.*

Proof. Consider first the case $S_k = \{e\}$. If $i \in J - \{k\}$ then $S_i \subset S$ and $(S_i)_{i \in J-\{k\}}$ is a partition of S. Since $x \circ y = x \cdot y \neq e$ for $x, y \in S$ then it is also an l-invariant partition of $(S, \bullet)$. Let now $S_k \neq \{e\}$ and x be an arbitrary element of S. From the supposition it follows that $x \circ S_k \subset S_l$ for a suitable $l \in J$. Hence $x \cdot (S_k - \{e\}) = x \circ (S_k - \{e\}) \subset S_l$. If $l \neq k$ then $S_l \subset S$ and S_l is a component of the mentioned partition $(\bar{S}_i)_{i \in J}$ of $(S, \bullet)$. If $l = k$ then $x \cdot (S_k - \{e\}) \subset (S_k - \{e\})$ because $x \cdot y \neq e$ for $y \in S$. Thus, in the case where $S_k \neq \{e\}$, we obtained that $x \cdot (S_k - \{e\})$ is contained in a component of the partition $(\bar{S}_i)_{i \in J}$ from the thesis.

Consider now the product $x \cdot S_j$, $j \in J$, $j \neq k$. It is evident that either $x \cdot S_j = x \circ S_j \subset S_k$ or $x \cdot S_j = x \circ S_j \subset S_i$ where $i \neq k$. The second possibility can be rewritten as follows $x \cdot \bar{S}_j \subset \bar{S}_i$. The first of them since, $S_j \subset S$ leads to $x \cdot S_j \subset S_k - \{e\}$ i.e. $x \cdot \bar{S}_j \subset \bar{S}_k$. Thus the theorem is proved.

Remark 3. The considerations from remark 2 may be applied for semigroups.

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