JÓZEF TABOR

On commutative algebraic objects over the group $GL(n, R)$

1. Let X be a non-empty set and let G be a group. A function $F: X\times G\to X$ satisfying the translation equation

$$F(F(x,\alpha),\beta) = F(x,\beta\alpha) \quad \text{for} \quad x\in X,\ \alpha,\beta\in G$$

and the identity condition

$$F(x,1) = x \quad \text{for} \quad x\in X$$

is called an algebraic object or shortly an object (cf. [6], p. 68). Let F: $X\times G\to X$ be an algebraic object and let G_1 be a subgroup of G. Then the restriction $F|X\times G_1$ of F to the set $X\times G_1$ is also an algebraic object. It is called a subobject of the object F.

Let F_1: $X\times G\to X$ and F_2: $Y\times G\to Y$ be two objects over the same group G. A function h: $X\to Y$ is called a homomorphism of the object F_1 into F_2 if the following condition holds

$$h(F_1(x,\alpha)) = F_2(h(x),\alpha) \quad \text{for} \quad x\in X,\ \alpha\in G\,.$$

If, additionaly, h maps X onto Y, then is called an epimorphism and F_2 is called a concomitant of F_1. If h is a bijection, then it is called an isomorphism and we say that F_1 and F_2 are equivalent.

Let F: $X\times G\to X$ be an object and let ϱ be an equivalence relation conformable with the object F, i.e. let ϱ satisfy the condition: If $x\varrho y$ then $F(x,\alpha)\varrho F(y,\alpha)$ for every $\alpha\in G$. Then the function F/ϱ: $X/\varrho\times G\to X/\varrho$ defined as follows

(1) $$F/\varrho([x],\alpha) = [F(x,\alpha)] \quad \text{for} \quad x\in X,\ \alpha\in G$$

is an object. It is called a factor-object of the object F.

All concomitants of a given object are determined uniquely up to the isomorphism by its factor-objects (cf. [6], p. 71). An object F: $X\times G\to X$ is called commutative (cf. [2], p. 19) if

$$F(F(x,\alpha),\beta) = F(F(x,\beta),\alpha) \quad \text{for} \quad x\in X,\ \alpha,\beta\in G\,.$$

S. Barcz, Z. Moszner and M. Siuda proved (cf. [1]) the following

LEMMA 1. *An object F: $X\times G\to X$ is commutative if and only if the derivated group $[G, G]$ of the group G is contained in the kernel of effectivity of F, i.e. if $[G, G]$ is contained in the set*

(2) $$J_F = \{\alpha\in G: \underset{x\in X}{\forall} F(x, \alpha) = x\}.$$

2. Let F: $X\times G\to X$ be an algebraic object. In the class of all concomitants of the object F there may be introduced a semiorder. We say that $F_1\geqslant F_2$ if and only if F_2 is a concomitant of F_1. This semi-order can be characterized as follows: Let F_1 and F_2 be concomitants of F and let $F_1 = F/\varrho_1$, $F_2 = F/\varrho_2$, where ϱ_1 and ϱ_2 are equivalence relations conformable to the object F. We have then (cf. [6], p. 76)

(3) $$F_1\geqslant F_2 \text{ if and only if } \varrho_1\subset\varrho_2.$$

We prove the following

THEOREM 1. *For every algebraic object F over a group there exists "the greatest" commutative concomitant, i.e. such one F^* that every commutative concomitant of F is a concomitant of F^*.*

Proof. Let F: $X\times G\to X$ be an algebraic object over a group G. We define a relation ϱ in X as follows: $x\varrho y$ if and only if there exists $\alpha\in[G, G]$ such that $y = F(x, \alpha)$. It is clear that ϱ is an equivalence relation. We prove that ϱ is conformable to the object F. Let $F(x, \alpha) = y$ with $\alpha\in[G, G]$ and let $\beta\in G$. We have then

$$F(F(x, \beta), \beta\alpha\beta^{-1}) = F(x, \beta\alpha) = F(F(x, \alpha), \beta) = F(y, \beta).$$

Obviously, $\beta\alpha\beta^{-1}\in[G, G]$. Hence $F(x, \beta)\varrho F(y, \beta)$, which proves that ϱ is conformable to the object F. To prove that the object F/ϱ is commutative we consider any elements $\alpha, \beta\in G$. We have for every $x\in X$

$$F(F(x, \alpha\beta), \beta\alpha\beta^{-1}\alpha^{-1}) = F(x, \beta\alpha).$$

Since $\beta\alpha\beta^{-1}\alpha^{-1}\in[G, G]$ this equality implies the relation

(4) $$F(x, \alpha\beta)\varrho F(x, \beta\alpha).$$

We obtain from (1) and (4)

$$F/\varrho([x]_\varrho\, \alpha\beta) = [F(x, \alpha\beta)]_\varrho = [F(x, \beta\alpha)]_\varrho = F/\varrho([x]_\varrho, \beta\alpha).$$

Hence the object F/ϱ is commutative.

We have to prove yet that F/ϱ is the greatest concomitant of F. Since every concomitant is equivalent to some factor object, we may restrict our considerations to the factor-objects only. Let ϱ_1 be an equivalence relation conformable to the object F and let the object F/ϱ_1 be commutative. Let $x\varrho y$, i.e. let there exist $\alpha_0\in[G, G]$ such that $F(x, \alpha_0) = y$. Applying Lemma 1 we obtain from the commutativity of F/ϱ_1 that

$$\alpha_0\in J_{F/\varrho_1}.$$

By (1) and (2) we have now

$$[y]_{\varrho_1} = [F(x, \alpha_0)]_{\varrho_1} = F/\varrho_1([x]_{\varrho_1}\alpha_0) = [x_0]_{\varrho_1},$$

what means that $x\varrho_1 y$. We have proved that $\varrho \subset \varrho_1$ what, in virtue of (3) completes the proof.

As an immediate consequence of Lemma 1 we obtain the following

THEOREM 2. *Let F: $X\times G \to X$ be an algebraic object such that $J_F = \{1\}$ and let F_1: $X\times G_1 \to X$ be a subobject of F. The subobject F_1 is commutative if and only if the subgroup G_1 of G is abelian.*

Proof. If F_1 is commutative then we get

$$[G_1, G_1] \subset J_{F_1} \subset J_F = \{1\},$$

what means that G_1 is obelian.

The converse implication is obvious.

3. Now we restrict our considerations to algebraic objects over subgroups of the group $GL(n, R)$ i. e. the group of nonsingular $n\times n$ matrices over a field R. The subgroup of $GL(n, R)$ consisting of all the matrices whose determinant is equal to 1 will be denoted by $SL(n, R)$. $0(n, R)$ and $D(n, R)$ will denote the subgroups of orthogonal and diagonal matrices respectively, subgroup consisting of matrices of the form sA where $s\in R$, $s\neq 0$ and $A\in 0(n, R)$ will be denoted by $P(n, R)$. Let G be a subgroup of $GL(n, R)$ and let F: $X\times G\to X$ be an algebraic object. If $F(x, A)$ for $x\in X$ and $A\in G$ depends only det A and on x i.e. if there exists a function F^* such that

$$F(x, A) = F^*(x, \det A) \quad \text{for} \quad x\in X,\ A\in G$$

then we say that F is an J — object (cf. [3], p. 83). We are going to give the characterization of the commutative objects over the group $GL(n, R)$. Namely, we shall prove the following

THEOREM 3. *Let G be a subgroup of $GL(n, R)$ such that*

$$[GL(n, R), GL(n, R)] \cap G \subset [G, G]. \tag{5}$$

An algebraic object F over G is commutative if and only if it is an J-object.

Proof. If F: $X\times G\to X$ is an J-object then we have for $x\in X$, $A, B\in G$:

$$F(x, BA) = F^*(x, \det BA) = F^*(x, \det AB) = F(x, AB),$$

which proves the commutativity of F.

Suppose now that an object F: $X\times G\to X$ is commutative. We shall prove that if $A, B\in G$ and $\det A = \det B$ then $F(x, A) = F(x, B)$ for $x\in X$. For then let $\det A^{-1}B = 1$ and hence

$$A^{-1}B\in SL(n, R). \tag{6}$$

But (cf. [4], p. 36)

$$[GL(n, R), GL(n, R)] = SL(n, R). \tag{7}$$

In consequence of (5), (6) and (7) $A^{-1}B \in [G, G]$. In virtue of lemma 1 we have now

$$F(x, A^{-1}B) = x$$

and further

$$F(x, A) = F[F(x, A^{-1}B), A] = F(x, AA^{-1}B) = F(x, B),$$

which completes the proof.

Remark. The inclusion converse to inclusion (5) is obvious. Hence in (5) the inclusion may be replaced by the equality. Condition (5) is satisfied e.g. if G is any of the groups $SL(n, R)$, $0(2, R)$, $P(2, R)$ in the group of matrices A such that $|\det A| = 1$, but it is not satisfied e.g. for $G = D(n, R)$. It would be interesting to find other convenient conditions equivalent to condition (5).

It follows immediately from Theorem 3 that W — density * and G — density are commutative objects and for $n \geqslant 2$ covariant vector, contravariant vector, definor, Pentsov object are non-commutative objects.

Non-commutative objects, of course may, have commutative subobjects. It is easy to check that for $n \geqslant 2$ the kernel of effectivity of covariant vector, contravariant vector, Pentsov object is the trivial group. Hence in virtue of theorem 4 only their subobjects over obelian subgroups are commutative.

Examples. We are going to determine all commutative objects over the group $0(2, R)$. Let F: $X \times 0(2, R) \to X$ be both transitive and commutative object. In virtue of theorem 3 F can be written in the form

$$F(x, A) = F^*(x, \det A),$$

where F^* is also an algebraic object (obviously transitive). But $\det A = \pm 1$. Hence we have two possibilities $\overline{X} = 1$ or $\overline{X} = 2$. Thus F is a scalar or a biscalar. Every algebraic object over a group is a disjoint union of transitive objects (cf. [5]). Hence every commutative object over $0(2, R)$ is a disjoint union of scalars and biscalars. On the other hand every object of this form is obviously commutative. Let us consider now an object of the form

$$F(x, A) = Ax \quad \text{for} \quad x \in R^n,\ A \in P(n, R).$$

The derivated group $[P(n, R), P(n, R)]$ consists of orthogonal matrices A such that $\det A = 1$. It is easy to check that in the case under consideration the relation ϱ (which defines the geatest commutative concomitant) has the form $x\varrho y$ if and only if $|x| = |y|$, where $|x|$ denotes the lenght of x. Every matrix from $P(n, R)$ may be written in the form sA, where s is a positive real number and $A \in 0(n, R)$.
We have

$$F/\varrho([x]_\varrho, sA) = [F(x, sA)]_\varrho = [sAx]_\varrho =$$
$$= \{y\colon |y| = |sAx|, A \in 0(n, R)\} = \{y\colon |y| = s|x|\}.$$

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* All objects occuring in our paper are defined in [3].

By identification of $[x]_\varrho$ and $|x|$ the object F/ϱ may be written in the form $F/\varrho = F_1$ where

$$F_1(x, sA) = sx \quad \text{for} \quad x \in <0, \infty), \ s \in (0, \infty) .$$

We prove now that condition (5) is essential in theorem 3. Let $G = D(n, R)$. Condition (5) is not satisfied.
We put

$$F(x, A) = Ax \quad \text{for} \quad x \in R^n, \ A \in D(n, R) .$$

This object is commutative. But from the equality $\det A = \det B$ with $A, B \in D(n, R$ the equality $Ax = Bx$ does not follow. Hence this object is not a J-object.

References

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