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On the quasi-potential for the circle and the ball

We shall consider the integral

$$V(X) = \int_D f(Y) H(X, Y) dY, \tag{1}$$

where $H(X, Y)$ is the fundamental solution of the equation

$$\Delta u(X) - c^2 u(X) = 0, \tag{2}$$

c is a positive number.

We shall call the function $V(X)$ the quasi-potential of the domain D.

In the present paper we shall evaluate the integral (1) for the circle and the ball assuming that $f(Y) = m_0 = \text{const} \neq 0$.

1. Now we shall consider the integral (1) for D being the circle

$$K = \{(s, t)\colon s^2 + t^2 < R^2, R > 0\}.$$

Let $X = (x, y)$, $Y = (s, t)$ denote two arbitrary points of the plane. Let $r^2 = (x-s)^2 + (y-t)^2$ and let $J_m(cr)$, $K_m(cr)$, $I_m(cr)$ denote the convenient Bessel functions ([4], p. 103). By [1] (p. 630), the function $K_0(cr)$ is the fundamental solution of the equation (2) in two-dimensional Euclidean space E_2 Hence $H(X, Y) = K_0(cr)$ and

$$V(X) = m_0 \int_K K_0(cr) ds dt.$$

Introducing the polar coordinates

$$x = \varrho \cos\varphi, \quad y = \varrho \sin\varphi$$

and

$$s = z\cos\psi, \quad t = z\sin\psi$$

we get

$$V(X) = m_0 \int_0^{2\pi} \int_0^R z K_0\left(c\sqrt{\varrho^2 + z^2 - 2\varrho z \cos(\varphi - \psi)}\right) dz d\psi. \tag{3}$$

Now we shall prove the following

LEMMA 1. *If* $\varrho>0$, $z>0$, *then the series* $\sum_{m=1}^{\infty}\left(\frac{z}{\varrho}\right)^m$ *is the majorant of the series*

$$\sum_{m=1}^{\infty} K_m(c\varrho)I_m(cz)\cos m\alpha .$$

Proof. By [4] (p. 149)
we get

$$K_m(c\varrho)I_m(cz) = \frac{1}{2}\int_{\ln\frac{\varrho}{z}}^{\infty} J_0(c\sqrt{2\varrho z\cosh t-\varrho^2-z^2})e^{-mt}\,dt$$

for $\varrho>0$, $c>0$, $z>0$, $m = 1, 2, \ldots$ Since $|J_0(u)|<1$ for $u>0$ and $K_m(c\varrho)>0$, $I_m(cz)>0$ for $c>0$, $\varrho>0$, $z>0$
we get

$$K_m(c\varrho)I_m(cz)\leqslant\frac{1}{2}\int_{\ln\frac{\varrho}{z}}^{\infty} e^{-mt}\,dt = \frac{1}{2m}\left(\frac{z}{\varrho}\right)^m < \left(\frac{z}{\varrho}\right)^m .$$

Now we shall evaluate the integral (3). We shall single out three cases:

I. X is exterior point with respect to K,

II. X is the interior point of the circle K,

III. $X\in\partial K$.

Ad. I. Then $\varrho>R\geqslant z$ and ([2], p. 44)

$$(4)\qquad K_0(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}) = K_0(c\varrho)I_0(cz)+ +2\sum_{m=1}^{\infty}K_m(c\varrho)I_m(cz)\cos m(\varphi-\psi) .$$

By lemma 1 for $z\in[0, R]$, $\psi\in[0, 2\pi]$ the above series is uniformly convergent and

$$V(X) = m_0\int_0^{2\pi}\int_0^R zK_0(c\varrho)I_0(cz)\,dz\,d\psi+ +2m_0\sum_{m=1}^{\infty}\int_0^{2\pi}\int_0^R zK_m(c\varrho)I_m(cz)\cos m(\varphi-\psi)\,dz\,d\psi =$$

$$= 2\pi m_0K_0(c\varrho)\int_0^R zI_0(cz)\,dz = 2\pi m_0c^{-2}K_0(c\varrho)\int_0^{cR} uI_0(u)\,du .$$

Applying to the integral on the right-hand side of last formula ([4], p. 117)

$$D_z(z^pI_p(z)) = z^pI_{p-1}(z)$$

for $p = 1$ and the condition $I_1(0) = 0$

we get

$$(5) \qquad \int_0^{cR} uI_0(u)du = cRI_1(cR)$$

and finally $V(X) = 2\pi m_0 Rc^{-1} K_0(c\varrho) I_1(cR)$.

Ad. II. Then $\varrho < R$. Let ε and δ are an arbitrary positives numbers. We have then

$$\int_K K_0(cr)\,ds\,dt = \lim_{\varepsilon\to 0} \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0\big(c\sqrt{\varrho^2 z^2 - 2\varrho z\cos(\varphi-\psi)}\big)dz\,d\psi +$$

$$+\lim_{\delta\to 0} \int_0^{2\pi} \int_{\varrho+\delta}^{R} zK_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big)dz\,d\psi .$$

By lemma 1 it follows that the series

$$\sum_{m=1}^{\infty} K_m(c\varrho) I_m(cz)\cos m(\varphi-\psi)$$

is uniformly convergent for $z \in [0, \varrho-\varepsilon]$ and $\psi \in [0, 2\pi]$. By (4) and (5) we obtain

$$\int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big)dz\,d\psi = \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0(c\varrho) I_0(cz)\,dz\,d\psi +$$

$$+2\sum_{m=1}^{\infty} \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_m(c\varrho) I_m(cz)\cos m(\varphi-\psi)\,dz\,d\psi = \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0(c\varrho) I_0(cz)\,dz\,d\psi$$

and

$$\lim_{\varepsilon\to 0} \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big)dz\,d\psi$$

$$= \lim_{\varepsilon\to 0} \int_0^{2\pi} \int_0^{\varrho-\varepsilon} zK_0(c\varrho) I_0(cz)\,dz\,d\psi = 2\pi K_0(c\varrho)c^{-2} \lim_{\varepsilon\to 0} \int_0^{c(\varrho-\varepsilon)} uI_0(u)\,du =$$

$$= 2\pi c^{-1} \varrho K_0(c\varrho) I_1(c\varrho) .$$

Similarly for $z \in (\varrho+\delta, R)$ by [2] (p. 44) we get

$$K_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big) = I_0(c\varrho)K_0(cz) +$$

$$+2\sum_{m=1}^{\infty} I_m(c\varrho) K_m(cz)\cos m(\varphi-\psi) .$$

From lemma 1 it follows that the series

$$\sum_{m=1}^{\infty} I_m(c\varrho) K_m(cz)\cos m(\varphi-\psi)$$

is uniformly convergent for $z \in [\varrho+\delta, R)$, $\psi \in [0, 2\pi]$ and

$$\int_0^{2\pi} \int_{\varrho+\delta}^{R} zK_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big)dz\,d\psi =$$

$$= \int_0^{2\pi} \int_{\varrho+\delta}^{R} zI_0(c\varrho) K_0(cz)\,dz\,d\psi = 2\pi I_0(c\varrho)c^{-2} \int_{c(\varrho+\delta)}^{cR} uK_0(u)\,du .$$

Applying the formulae ([4], p. 117)

$$D_z(uK_1(u)) = -uK_0(u)\,,$$

we get

$$\int_{c(\varrho+\delta)}^{cR} uK_0(u)\,du = c(\varrho+\delta)K_1(c\varrho+c\delta)-cRK_1(cR)\,.$$

By continuity of the function $K_1(u)$ in the interval $(0, \infty)$, we get

$$\lim_{\delta\to 0}\int_0^{2\pi}\int_{\varrho+\delta}^{R} zK_0\big(c\sqrt{\varrho^2+z^2-2\varrho z\cos(\varphi-\psi)}\big)\,dz\,d\psi =$$

$$= 2\pi I_0(c\varrho)c^{-1}[\varrho K_1(c\varrho)-RK_1(cR)]$$

and finally

$$V(X) = 2\pi m_0 c^{-1}[\varrho K_0(c\varrho)I_1(c\varrho)-RI_0(c\varrho)K_1(cR)+\varrho I_0(c\varrho)K_1(c\varrho)]\,.$$

Ad. III. Let $X = (x, y) \in \partial K$ and let K_p denote the circle with the centre $(0, 0)$ and radius p.
Then

$$V(X) = m_0 \lim_{p\to R}\int_{K_p} K_0(cr)\,ds\,dt$$

and in view of the cases I, II, we get

$$V(X) = 2\pi m_0 Rc^{-1}K_0(cR)I_1(cR)\,.$$

Finally, we get

$$(6)\qquad V(X) = \begin{cases} 2\pi m_0 c^{-1}RK_0(c\varrho)I_1(cR) & \text{for}\quad \varrho\geqslant R\,,\\ 2\pi m_0 c^{-1}[\varrho K_0(c\varrho)I_1(c\varrho)-RK_1(cR)I_0(c\varrho)+\varrho I_0(c\varrho)K_1(c\varrho)] & \\ & \text{for}\quad \varrho<R\,.\end{cases}$$

2. Now we shall consider the integral (1) for D being the ball. Let $X = (x, y, z)$, $Y = (s, t, v)$ denote two arbitrary points in three dimensional Euclidean space E_3. Let $r^2 = (x-s)^2+(y-t)^2+(z-v)^2$. Then the function $r^{-1}e^{-cr}$ is the fundamental solution of the equation (2). Let $D = \{(s, t, v)\colon s^2+t^2+v^2\leqslant R^2, R>0\}$ and let $\varrho = |\overline{0P}|$. Using a method similar to that of the method in [3] (p. 313), we get

$$(7)\qquad V(X) = m_0\int_D r^{-1}e^{-cr}\,ds\,dt\,dv =$$

$$= \begin{cases} 4\pi m_0\varrho^{-1}c^{-3}e^{-\varrho c}(Rc\cosh Rc-\sinh Rc) & \text{for}\quad \varrho\geqslant R\,,\\ 4\pi m_0[c^{-2}-\varrho^{-1}c^{-3}(Rc+1)e^{-cR}\sinh c\varrho] & \text{for}\quad \varrho<R\,.\end{cases}$$

3. Now, we shall evaluate the limit $\lim_{c\to 0} V(X)$. When $c\to 0$, then the equation (2) is the Laplace equation. We shall prove that the $\lim_{c\to 0} V(X) = V_0(X)$ for $X\in E_3$, and $V_0(X)$ is the newtonian potential. Namely, in virtue of (7) we have

$$\lim_{c\to 0} V(X) = V_0(X) = \begin{cases} \frac{4}{6}\pi m_0 R^2(3-\varrho^2 R^{-2}) & \text{for} \quad \varrho \geqslant R, \\ \frac{4}{3}\pi m_0 R^3 \varrho^{-1} & \text{for} \quad \varrho < R. \end{cases}$$

By the formula (6) we obtain $\lim\limits_{c\to 0} V(X) = \infty \neq V_1(X)$ for $X \in E_2$ and $V_1(X)$ is the logarithmic potential.

References

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