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On the Dirichlet problem for certain angular domain

1. In this paper we shall give the solution of Dirichlet problem for the domain

$$D_n = \left\{(x, y):\ 0<x<\infty, 0<y<x\operatorname{tg}\frac{\pi}{n}\right\}, \qquad n = 2, 3, \dots \tag{1}$$

We shall construct a solution $u(x, y)$ of the equation

$$\Delta u(x, y) = 0 \tag{2}$$

of class $C^{(2)}$ in D_n, satisfying the boundary conditions

$$u(x, 0) = F_1(x, 0) = f_1(x), \tag{3}$$

$$u\left(x, x\operatorname{tg}\frac{\pi}{n}\right) = F_2\left(x, x\operatorname{tg}\frac{\pi}{n}\right) = f_2(x), \qquad x>0, \tag{4}$$

where F_1, F_2 are given functions.

In order to solve the problem (2), (3), (4), we shall construct the convenient Green function using the method of symmetric images.

2. Let $X_1(x, y)$ denotes the arbitrary point of D_n. Let l_i denote the straigt lines

$$l_i:\ s\sin(i-1)\frac{\pi}{n} = t\cos(i-1)\frac{\pi}{n}, \qquad i = 1, 2, \dots \tag{5}$$

Let X_2 denote the symmetric point of X_1 with respect to l_2, X_3 denote the symmetric point of X_2 with respect to $l_3, \dots, X_n$ the symmetric point of X_{n-1} with respect to $l_2, \dots, X_{2n}$ the symmetric point of X_{2n-1} with respect to l_n. Obviously X_1 is symmetric image of X_{2n} with respect to l_1.

The coordinates of the points $X_i(x_i, y_i)$ $i = 1, 2, \dots$ are given by formulas:

$$x_i = x\cos i\frac{\pi}{n} \pm y\sin i\frac{\pi}{n} \qquad i = 2, 4, \dots, 2n$$

$$y_i = x\sin i\frac{\pi}{n} \pm y\cos i\frac{\pi}{n} \qquad i = 2, 4, \dots, 2n$$

$$(6)\qquad x_i = x\cos(i-1)\frac{\pi}{n} - y\sin(i-1)\frac{\pi}{n} \quad i = 1, 3, \ldots, 2n-1$$

$$y_i = x\sin(i-1)\frac{\pi}{n} + y\cos(i-1)\frac{\pi}{n} \quad i = 1, 3, \ldots, 2n-1$$

Let $Y(s, t)$ denotes the arbitrary point belonging to the set $D_n \cup l_1^* \cup l_2^*$, where l_i^* is defined by (5) for $s \geqslant 0$. Further let $r_i^2 = |X_i Y|^2 = (s-x_i)^2 + (t-y_i)^2 = r_i^2(x, y, s, t)$ $i = 1, 2, \ldots, 2n$.

3. Now we shall prove the following

THEOREM 1. *The function*

$$(7)\qquad G(X_1, Y) = G(x, y, s, t) = \sum_i^{2n} (-1)^{i+1} \ln r_i$$

is the Green function for the problem (2), (3), (4) *with a pole at the point* X_1.

Proof. The function $G(X_1, Y)$ is harmonic with respect to the point $Y (Y \neq X_i)$, because the functions $\ln r_i$ are harmonic. If $Y \in l_1$, then

$$r_1 = r_{2n}, r_2 = r_{2n-1}, \ldots, r_n = r_{n+1},$$

and $G(X_1, Y) = 0$. For $Y \in l_2$ we have $r_1 = r_2, r_{2n} = r_3, \ldots, r_{n+1} = r_{n+2}$, and $G(X_1, Y) = 0$.

4. Now we shall introduce any notations. Let $M = \sup_{s \geqslant 0} |f_i(s)|$, $\bar{f}_i(s) = f_i(s)$, for $s \geqslant 0$ and $\bar{f}_i(s) = 0$, for $s < 0$, $(i = 1, 2)$.

Let

$$a = \operatorname{tg}\frac{\pi}{n} \text{ and } R_i^2 = (s-x_i)^2 + y_i^2 \quad i = 1, 2, \ldots, 2n$$

$$\varrho_i^2 = (s-x_i)^2 + (as-y_i)^2 \quad i = 1, 2, \ldots, 2n.$$

Let N denotes the invard normal to $\partial D_n = l_1^* \cup l_2^*$. We shall prove, that under certain assumptions concerning the functions $f_i(s)$ the function

$$(8)\quad u(X_1) = \frac{1}{\pi}\int_0^\infty f_1(s) D_t G(X_1, Y)|_{t=0} ds + \frac{1}{\pi}\int_{s \geqslant 0} F_2(s, as) D_N G(X_1, Y)|_{t=as}\sqrt{1+a_2}\, ds$$

is the solution of the problem (2), (3), (4).

Using the formulas (7) and (8) we get

$$(9)\qquad u(X_1) = \sum_i^n (-1)^i K_i(X_1) + \sum_i^n (-1)^i H_i(X_1)$$

where

$$K_i(X_1) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \bar{f}_1(s) \frac{y_i}{R_i^2} ds; \quad H_i(X_1) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \bar{f}_2(s) \frac{y_i - ax_i}{\varrho_i^2} ds \qquad i = 1, 2, \ldots, n$$

Let

$$W_1 = \{(x, y)\colon |x| < A, 0 < c < y < C\}$$
(10)
$$W_2 = \{(x, y)\colon |x| < B, -aB - \delta < y < ax - \delta\}$$

where A, B, c, C, are abitrary positive numbers.
Let

$$K_i^{p,q}(X_1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{f}_1(s)(s)\, D^{p,q}\left(\frac{y_i}{R_i^2}\right) ds$$

$$H_i^{p,q}(X_1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{f}_2(s)\, D^{p,q}\left(\frac{y_i - ax_i}{\varrho_i^2}\right) ds$$

where $p,\ q = 0, 1, 2;\ 0 < p+q \leqslant 2;\ i = 1, 2, \ldots, n$.

5. Now we shall prove that the integrals $K^{p,q}(X_1)$, $(i = 1, 2, \ldots, n)$ are uniformly convergent in every set W_1 and the integrals $H_i^{p,q}(X_1)$ are uniformly convergent in every set W_2.
Let

$$K_i^{p,q}(R, X_1) = \frac{1}{\pi} \int_{|s| \geqslant R} \bar{f}_1(s)\, D^{p,q}\left(\frac{y_1}{R_i^2}\right) ds$$

and

$$H_i^{p,q}(R, X_1) = \frac{1}{\pi} \int_{|s| \geqslant R} \bar{f}_2(s)\, D^{p,q}\left(\frac{y_i - ax_i}{\varrho_i^2}\right) ds\,.$$

Lemma 1. *If the function* $f_1(s)$, $(s \geqslant 0)$ *is bounded and absolutely integrable, then the integrals* $K_i^{p,q}(X_1)$, $(i = 1, \ldots, n)$ *are uniformly convergent in every set* W_1.

Proof. The integrals $K_i^{pq}(R, X_1)$ have common majorant

$$MC \int_{|s| \geqslant R} \frac{ds}{(s - x_i)^2 + y_i^2}\,.$$

Since

$$\tfrac{1}{4} s^2 \leqslant (s - x_i)^2 + y_i^2 \leqslant 4s^2 \quad \text{for} \quad s \geqslant R$$

thus

$$\int_{|s| \geqslant R} \frac{ds}{(s - x_i)^2 + y_i^2} \leqslant 2 \int_{|s| \geqslant R} \frac{ds}{s^2} < \varepsilon$$

for arbitrary $\varepsilon > 0$ and $R \geqslant R(\varepsilon)$. This condition is sufficient for uniform convergence of integrals $K_i^{p,q}(X_1)$ in every set W_1.

Lemma 2. *If the function* $f_2(s)$, $s \geqslant 0$, *is bounded and absolutely integrable, then the integrals* $H_i^{p,q}(X_1)$ *are uniformly convergent in every set* W_2.

Proof. The integrals $H_i^{p,q}(R, X_1)$ have common majorant

$$C \int\limits_{|s|\geqslant R} \frac{ds}{(s-x_i)^2+(as-y_i)^2}$$

Since

$$\tfrac{1}{4}s^2 \leqslant (s-x_i)^2+(as-y_i)^2 \quad \text{for} \quad s\geqslant R$$

thus

$$\int\limits_{|s|\geqslant R} \frac{ds}{(s-x_i)^2+(as-y_i)^2} \leqslant 2 \int\limits_{|s|\geqslant R} \frac{ds}{s^2} \leqslant \varepsilon$$

for every $\varepsilon>0$ and $R>R(\varepsilon)$. From the above inequality follows the uniform convergence of integrals $H_i^{p,q}(X_1)$ in every set W_2.

From lemmas 1, 2 follows

LEMMA 3. *The integrals* $K_i^{p,q}(X_1)$ *and* $H_i^{p,q}(X_1)$ *are uniformly convergent in every set* $W_3 = W_1 \cap W_2$.

From lemmas 1, 2, 3 follows

LEMMA 4. *If the functions* $f_1(s), f_2(s)$, $s\geqslant 0$ *are bounded and absolutely integrable, then exist in* W_3 *the functions* $D^{p,q}K_i(X_1)$, $D^{p,q}H_i(X_1)$, $i = 1, 2, \ldots, n; p, q = 0, 1, 2;$ $0\leqslant p+q\leqslant 2$
and

$$D^{p,q}K_i(X_1) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \bar{f}_1(s)\, D^{p,q}\left(\frac{y_i}{R_i^2}\right) ds$$

and

$$D^{p,q}H_i(X_1) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \bar{f}_2(s)\, D^{p,q}\left(\frac{y_i-ax_i}{\varrho_i^2}\right) ds\,.$$

LEMMA 5. *If the functions* $f_1(s), f_2(s)$, $s\geqslant 0$, *are bounded and absolutely integrable, then the function* $u(X_1)$ *defined by formulas* (8) *or* (9) *is of class* $C^{(2)}$ *in every set* $W_3 \subset D_n$ *and is harmonic in* W_3.

Proof. Since the transformation (6) is orthogonal in view of lemma 4, we have

$$\Delta u(X_1) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} f_1(s)\, D_t\big(\Delta_{x_1}(G(X, Y))\big)\big|_{t=0} ds +$$

$$+\frac{1}{\pi} \int\limits_{|s|\geqslant 0} f_2(s)\, D_N\big(\Delta_{x_1}(G(X, Y))\big)\big|_{t=as}\sqrt{1+a^2}\, ds = 0 \quad \text{for} \quad X_1 \in W_3$$

6. Now we shall verify the boundary conditions (3) and (4).

LEMMA 6. [2]. *If the function* $f_1(s)$ *is continuous, bounded and absolutely integrable for* $s\geqslant 0$, *then*

$$\lim K_1(X_1) = \lim \frac{1}{\pi} \int_{-\infty}^{+\infty} f_1(s) \frac{y\,ds}{(s-x_t)^2+y_2} = f_1(x_0)$$

as $X_1 \to (x_0, 0)$, $x_0 > 0$.

LEMMA 7. *If the function* $f_2(s)$ *is continuous, bounded and absolutely integrable for* $s \leqslant 0$,
then

$$\lim H_1(X_1) = \lim \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{f}_2(s) \frac{ax-y}{(s-x)^2+(as-y)^2}\,ds = f_2(x_0)$$

as $\qquad X_1 \to (x_0, ax_0),\ x_0 > 0\,.$

Proof. Let us consider the integral

$$I = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{ax-y}{(s-x)^2+(as-y)^2}\,ds$$

Introducing in integral I the substitution: $s-x = (ax-y)_t t$ we have

$$as-y = (ax-y)at+(ax-y) = (ax-y)(at+1)$$

and

$$I = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{t^2+(at+1)^2} = 1\,.$$

We can rewrite $H_1(X_1)$ in the form:

$$H_1(X_1) = \frac{1}{\pi} \int_{-\infty}^{+\infty} [\bar{f}_2(s)-f_2(x_0)] \frac{as-y}{(s-x)^2+(as-y)^2}\,ds + f_2(x_0)$$

If $|s-x_0| < \delta(\varepsilon)$ and $|\bar{f}_2(s)-f_2(x_0)| < \varepsilon$
then

$$\frac{1}{\pi} \int_{|s-x_0|<\delta} |\bar{f}_2(s)-f_2(x_0)| \frac{ax-y}{(s-x)^2+(as-y)^2}\,ds \leqslant \varepsilon$$

and

$$\frac{1}{\pi} \int_{|s-x_0|>\delta} |\bar{f}_2(s)-f_2(x_0)| \frac{as-y}{(s-x)^2+(as-y)^2}\,ds = \frac{M}{\pi} \int_{-\infty}^{-\frac{\delta}{ax-y}} \frac{dt}{t^2+(at-1)^2} +$$

$$+ \frac{M}{\pi} \int_{\frac{\delta}{ax-y}}^{+\infty} \frac{dt}{t^2+(at+1)^2} \to 0, \text{ as } (ax-y) \to (ax_0-ax_0) = 0^+$$

LEMMA 8. *If the functions $f_1(s)$, $f_2(s)$ are bounded and absolutely integrable for $s \geqslant 0$, then the integrals*

$$P_i(X_1) = \int_{-\infty}^{+\infty} \bar{f}_1(s) \frac{\ln r_i}{t}\bigg|_{t=0} ds, \quad Q_i(X_1) = \int_{-\infty}^{+\infty} \bar{f}_2(s) \frac{\ln r_i}{N}\bigg|_{t=as} ds\,,$$

for $i = 2, 3, \ldots, 2n$ are uniformly convergent in every set

$$W_4 = \{(x, y)\colon\ 0<c<x<A, 0<y<ax-ac\}\,,$$

where A, a, c denote arbitrary positive numbers.

The proof is analoguous to those of lemmas 1, 2.

LEMMA 9. *If the functions $f_1(s), f_2(s)$, $s \geqslant 0$ are bounded, continuous and absolutely integrable, then the function*

a) $u(X_1) \to f_1(x_0)$, *as* $(X_1) \to (x_0, 0^+)$, $x_0 > 0$.

b) $u(X_1) \to f_2(x_0)$, *as* $(X_1) \to (x_0, ax_0)$.

Proof. Ad a). Using the formula (6) we get: $\lim D_t(\ln r_{2n-i} + \ln r_{i+2}) = 0$ as $y \to 0^+$, $(i = 1, \ldots, n)$, $\lim D_N(\ln r_{2n-i} + \ln r_{i+2}) = 0$ as $y \to 0^+$, $(i = 1, \ldots, n)$

By lemmas 6 and 8 we obtain

$$\lim u(X_1) = f_1(x_0) + \lim \int_{-\infty}^{+\infty} \bar{f}_1(s) \sum_{i=1}^{n} (-1)^i (D_t(\ln r_{2n-i} + \ln r_{i+2}))|_{t=0} ds +$$

$$+ \lim \int_{-\infty}^{+\infty} \bar{f}_2(s) \sum_{i=1}^{n} D_N(\ln r_{2n-i} + \ln r_{i+2})|_{t=as} ds = f_1(x_0)$$

as $y \to 0^+$.

Ad b). By formula (7) we have:

$$\lim D_t(\ln r_{2n-i} + \ln r_i) = 0 \quad \text{as} \quad (X_1) \to (x_0, ax_0),\ (i = 1, \ldots, n)$$

and

$$\lim D_N(\ln r_{2n-i} + \ln r_{i+2}) = 0, \quad \text{as} \quad (X_1) \to (x_0, ax_0),\ (i = 1, 3, \ldots, 2n-1)$$

and

$$\lim D_N(\ln r_{2n-i} + \ln r_{i+4}) = 0 \quad \text{as} \quad (X_1) \to (x_0, ax_0),\ (i = 2, 4, \ldots, 2n)$$

From lemmas 7 and 8 we get:

$$\lim u(X_1) = \bar{f}_2(x_0) + \lim \int_{-\infty}^{+\infty} f_1(s) \sum_i (-1)^i D_t(\ln r_{2n-i} + \ln r_i)|_{t=0} ds +$$

$$+ \lim \int_{-\infty}^{+\infty} \bar{f}_2(s) \sum_i{}^{(1)} D_N(\ln r_{2n-i} + \ln r_{i+2})|_{t=as} ds +$$

$$+ \lim \int_{-\infty}^{+\infty} f_2(s) \sum_i{}^{(2)} D_N(\ln r_{2n-i} + \ln r_{i+4})|_{t=as} ds =$$

$$= f_2(x_0), \quad \text{as} \quad (X_1) \to (x_0, ax_0)\,,$$

where $\sum^{(1)}$ denotes the convenient sum for odd i and $\sum^{(2)}$ for even i. From lemma 9 follows

THEOREM 2. *If the functions* $f_1(s), f_2(s)$, $s \geq 0$ *are continuous, bounded and absolutely integrable, then the function* $u(x, y)$, *defined by formulas* (8) *or* (9) *is the solution of the problem* (2), (3), (4) *in every set* (1).

References

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