LUDWIKA ŻĄDŁO

On certain properties of the volume Green potential for the iterated Helmholtz equation and for the octant E_3^+

1. Let $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and $X_0 = (x_1^0, x_2^0, x_3^0)$ denote the points of the Euclidean space E_3.
Let

$$r^2 = \sum_{i=1}^{3} (y_i - x_i)^2, \quad r_0^2 = \sum_{i=1}^{3} (y_i - x_i^0)^2,$$

$\Delta = \sum_{i=1}^{3} D_{x_i}^2$, $\Delta^2 = \Delta(\Delta)$, $H = \Delta - C^2$, $H^2 = (\Delta - C^2)^{(2)} = \Delta^2 - 2C^2\Delta + C^4$, C being a positive constant.

Let $M = \sup_{E_3} |F(Y)|$ and let $K(X_0, \delta)$ denote ball with center at X_0 and radius δ.

2. Let $\varphi(X, Y)$ be the function defined and continuonus in the set $E_3 \times E_3^+$ and for $X \neq Y$, and uniformly continuous in every compact of this set. Let $F(Y)$ be the bounded and absolutely integrable function defined in E_3^+.

Definition 1. *We say that the integral*

$$I(X) = \iiint_{E_3^+} F(Y)\varphi(X, Y)dV_Y$$

is uniformly convergent at the point X_0 if 1^0 the integral $I(X_0)$ is convergent, 2^0 for arbitrary positive number ε, there exist the numbers δ and R such that for every $X \in K(X_0, \delta)$

$$\iiint_{K(X_0,\delta)} |F(Y)\varphi(X, Y)|dV_Y < \varepsilon, \quad \text{and} \quad \iiint_{E_3^+ \cap K(O,R)} |F(Y)\varphi(X, Y)|dV_Y < \varepsilon$$

Definition 2. *The integral*

$$I_0 = (8\pi C)^{-1} \iiint_{E_3^+} F(Y)\exp(-Cr)dV_Y$$

is called the volume Green potential for the iterated Helmholtz equation

(1) $$H^2 u(X) = F(X)$$

3. Now we shall prove some lemmas which we shall use in the sequel.

LEMMA 1. *If the function F is absolutely integrable in E_3^+, then the integrals*

$$I_m(X) = \iiint_{E_3^+} F(Y)((y_i-x_i)r^{-1})^m \exp(-Cr)dV_Y \quad (i = 1,2,3;\ m = 0,1,2,3).$$

are uniformly convergent in the set E_3^+.

Proof. Since $|((x_i-y_i)r^{-1})^m|<1$ and $\exp(-Cr)<1$ thus the common majorant of the integrals I_m is the convergent integral

$$\iiint_{E_3^+} F(Y)dV_Y$$

LEMMA 2. *If the function F is bounded and absolutely integrable in E_3^+, then the integrals*

$$P_i(X) = \iiint_{E_3^+} F(Y)r^{-i}\exp(-Cr)dV_Y \quad (i = 1,2)$$

are uniformly convergent at every point $X_0 \in E_3^+$.

Proof. We shall prove this lemma for the integral $P_1(X)$. The proof for the integral $P_2(X)$ is similar. Let $K(X_0,\delta)\subset E_3^+$ and let $P_1(X) = P_1^1(X)+P_1^2(X)$ where

$$P_1^1(X) = \iiint_{K(X_0,\delta)} F(Y)r^{-1}\exp(-Cr)dV_Y .$$

and

$$P_1^2(X) = \iiint_{E_3^+\setminus K(X_0,\delta)} F(Y)r^{-1}\exp(-Cr)dV_Y .$$

Applying in the integrals $P_1^i(X_0)$ $(i = 1,2)$ the change of variables $y_1-x_1^0 = \varrho\cos\varphi\cos\psi$, $y_2-x_2^0 = \varrho\sin\varphi\cos\psi$, $y_3-x_3^0 = \varrho\sin\psi$ we get the inequalities

$$|P_1^1(X_0)| \leqslant 4\pi M\int_0^\delta \varrho\exp(-C\varrho)d\varrho \leqslant 2\pi M\delta^2$$

and

$$|P_1^2(X_0)| \leqslant 4\pi M\int_\delta^\infty \varrho\exp(-C\varrho)d\varrho .$$

This implies that the integral $P_1(X)$ is convergent at the point X_0. Let $X\in K(X_0,\delta)$ and $|XY|<2\delta$. Then $K(X_0,\delta)\subset K(X,2\delta)$ and

$$|P_1^1(X)| \leqslant \iiint_{K(X,2\delta)} |F(Y)|r^{-1}\exp(-Cr)dV_Y$$

Introducing in above the integral the new variables $y_1-x_1 = \varrho\cos\varphi\cos\psi$, $y_2-x_2 = \varrho\sin\varphi\cos\psi$, $y_3-x_3 = \varrho\sin\psi$ we obtain the inequalities

$$|P_1^1(X)| < 4\pi M\int_0^{2\delta} \varrho\exp(-C\varrho)d\varrho \leqslant 8\pi M\delta^2 < \varepsilon$$

for

$$\delta < \left((8\pi M)^{-1}\varepsilon\right)^{\frac{1}{2}}$$

Let us consider the integral

$$P_3(X) = \iiint\limits_{E_3^+ \setminus K(X_0,\delta)} F(Y) r^{-1} \exp(-Cr)\, dV_Y$$

We shall prove that the integral $P_3(X)$ is arbitrary small for sufficiently great R. Namely, there exists a number R such that for every $X \in K(X_0, \delta)$ hold inequalities

$$\tfrac{1}{2}|0Y| \leqslant r \leqslant 2|0Y|$$

Moreover

$$\exp(-Cr) \leqslant (Cr)^{-n} n! \quad \text{for} \quad n \in N.$$

For the integral $P_3(X)$ we have following inequalities

$$|P_3(X)| \leqslant 2M \iiint\limits_{|0Y|>R} \exp(-\tfrac{1}{2}C|0Y|)(|0Y|)^{-1} dV_Y \leqslant$$

$$\leqslant 2M \iiint\limits_{|0Y|>R} (C|0Y|)^{-3} 3!(|0Y|)^{-1} dV_Y$$

Applying in the last integral the change of variables $y_1 = \varrho\cos\varphi\cos\psi$, $y_2 = \varrho\sin\varphi\cos\psi$, $y_3 = \varrho\sin\psi$ we get

$$|P_3(X)| \leqslant 3! M C^{-3}\pi \int_R^\infty \varrho^{-2} d\varrho = C_1 R^{-1} < \varepsilon$$

for

$$R > 6\pi M C^{-3}\varepsilon^{-1}$$

Consequently the integral $P_1(X)$ is uniformly convergent at every point $X_0 \in E_3^+$. By lemmas 1 and 2 we get

THEOREM 1. *If the function F is bounded and absolutely integrable in the set E_3^+, then there exist the derivatives $D_{x_i}^j w(X)$ $(i = 1, 2, 3; j = 1, 2)$ and exist the integrals*

$$\iiint\limits_{E_3^+} F(Y) D_{x_i}^j(\exp(-Cr))\, dV_Y \quad (i = 1, 2, 3; j = 1, 2)$$

and

$$D_{x_i}^j I_0(X) = (8\pi C)^{-1} \iiint\limits_{E_3^+} F(Y) D_{x_i}^j(\exp(-Cr))\, dV_Y \ (i = 1, 2, 3;\ j = 1, 2).$$

By lemma 2 we get

THEOREM 2. *If the function F is bounded and absolutely integrable in the set E_3^+, then the function $P_1(X)$ is of class $C^{(1)}(E_3^+)$*
and

$$D_{x_i} P_1(X) = \iiint\limits_{E_3^+} F(Y) D_{x_i}(r^{-1}\exp(-Cr))\, dV_Y \quad (i = 1, 2, 3).$$

4. Now we shall prove

LEMMA 3. *If the function* F *is of class* $C^1(E_3^+)$, *then there exist the derivatives* $D^2_{x_i}P_1(X)$ $(i = 1, 2, 3)$ *for every* $X \in K(X_0, \delta)$.

Proof. From theorem 2 we have

$$D_{x_i}P_1(X) = D_{x_i}P_1^1(X) + D_{x_i}P_1^2(X) =$$
$$= \iiint_{K(X_0,\delta)} F(Y)\, D_{x_i}(r^{-1}\exp(-Cr))\,dV_Y + \iiint_{E_3^+\setminus K(X_0,\delta)} F(Y)\, D_{x_i}(r^{-1}\exp(-Cr)\,dV_Y \qquad (i = 1, 2, 3)$$

By ([1], p. 326) and by the formulae

$$D_{x_i}(r^{-1}\exp(-Cr)) = D_{y_i}(r^{-1}\exp(-Cr)) \qquad (i = 1, 2, 3) \tag{2}$$

and

$$D_{y_i}(F(Y)r^{-1}\exp(-Cr)) = r^{-1}\exp(-Cr)\,D_{y_i}F(Y) + F(Y)\,D_{y_i}(r^{-1}\exp(-Cr) \qquad (i = 1, 2, 3) \tag{3}$$

we have

$$D_{x_i}P_1^1(X) = -\iint_{\partial K(X_0,\delta)} F(Y)r^{-1}\exp(-Cr)\cos(n, y_i)\,dS_Y + \iiint_{K(X_0,\delta)} r^{-1}\exp(-Cr)\,D_{y_i}F(Y)\,dV_Y \tag{4}$$

For every $X \in K(X_0, \delta)$ by (4) and (2) we have

$$D^2_{x_i}P_1^1(X) = \iint_{\partial K(X_0,\delta)} F(Y)\,D_{y_i}(r^{-1}\exp(-Cr)\cos(n, y_i))\,dS_Y + \iiint_{K(X_0,\delta)} D_{x_i}(r^{-1}\exp(-Cr))\,D_{y_i}F(Y)\,dV_Y \qquad (i = 1, 2, 3)$$

n being the exterior normal to $K(X_0, \delta)$.

Moreover for every $X \in K(X_0, \delta)$ we have

$$D^2_{x_i}P_1^2(X) = \iiint_{E_3^+\setminus K(X_0,\delta)} F(Y)\,D^2_{x_i}(r^{-1}\exp(-Cr))\,dV_Y \qquad (i = 1, 2, 3) \tag{5}$$

Now we shall prove the following

LEMMA 4. *If the function* F *is of class* $C^1(E_3^+)$, *then for every* $X \in K(X_0, \delta)$ *there exist the function* $\Delta P_1^1(X)$
and

$$\lim \Delta P_1^1(X) = F(X_0) \quad \textit{as} \quad X \to X_0\,.$$

Proof. By lemma 3 we have

$$\Delta P_1^1(X) = P_4(X) + P_5(X)\,,$$

where

$$P_4(X) = \iint\limits_{\partial K(X_0,\delta)} F(Y)\sum_{i=1}^{3}\cos(n, y_i)D_{y_i}(r^{-1}\exp(-Cr))dS_Y$$

and

$$P_5(X) = \iiint\limits_{K(X_0,\delta)} \sum_{i=1}^{3} D_{x_i}(r^{-1}\exp(-Cr))D_{y_i}F(Y)dV_Y$$

For $X = X_0$ we obtain

$$P_4(X_0) = \iint\limits_{\partial K(X_0,\delta)} F(Y)D_{n_Y}(r_0^{-1}\exp(-Cr_0))dS_Y$$

Applying to the integral $P_4(X_0)$ the mean value theorem we get

$$P_4(X_0) = 4\pi\delta^2F(Q)(\delta^{-2}-\delta^{-1})\exp(-C\delta),$$

Q being any point belonging to $\partial K(X_0,\delta)$.
By continuity of the integrals $P_i(X)$ $(i = 4, 5)$ at the point X_0 we get

$$\lim_{\delta\to 0} P_4(X) = \lim_{\delta\to 0}(4\pi\delta^2F(Q)(\delta^{-2}-\delta^{-1})\exp(-C\delta)) = 4\pi F(X_0)$$

Moreover by lemmas 1 and 2 we have

$$\lim P_5(X) = 0 \quad \text{as} \quad \delta\to 0$$

because

$$|P_5(X)| \leqslant M_1 \iiint\limits_{K(X_0,\delta)} \sum_{i=1}^{3} D_{x_i}(r^{-1}\exp(-Cr))dV_Y \to 0 \quad \text{when} \quad \delta\to 0.$$

Finally we get

$$\lim \Delta P_1^1(X) = F(X_0) \quad \text{as} \quad X\to X_0$$

5. Now let us consider the integral

$$\iiint\limits_{E_3^+\setminus K(X_0,\delta)} F(Y)H^2(\exp(-Cr))dV_Y.$$

This integral is uniformly convergent at the point X_0 and we get

(6) $$H^2\Big[\iiint\limits_{E_3^+\setminus K(X_0,\delta)} F(Y)\exp(-Cr)dV_Y\Big] = \iiint\limits_{E_3^+\setminus K(X_0,\delta)} F(Y)H^2(\exp(-Cr))dV_Y = 0$$

because

$$H^2\exp(-Cr) = \Delta^2\exp(-Cr) - 2C^2\Delta\exp(-Cr) + C^4\exp(-Cr) =$$
$$= \exp(-Cr)(C^4 - 4C^3r^{-1} - 2C^4 + 4C^3r^{-1} + C^4) \equiv 0 \quad \text{for} \quad X\in K(X_0,\delta).$$

We have prove the fundamental

THEOREM 3. *If the function F is bounded, absolutely integrable and of class $C^1(E_3^+)$, then the function I_0 satisfies the equation* (1) *for every $X\in E_3^+$.*

Proof. By (6) and lemmas 1, 2, 3 and theorems 1, 2 we get

$$H^2 I_0(X) = (-8\pi C)^{-1} H^2\Big(\iiint_{E_3^+} F(Y)\exp(-Cr)\Big)dV_Y =$$

$$= (-8\pi C)^{-1}[H^2 \iiint_{K(X_0,\delta)} F(Y)\exp(-Cr)dV_Y)+$$

$$+ \iiint_{E_3^+ \setminus K(X_0,\delta)} F(Y)H^2\exp(-Cr)dV_Y] =$$

$$= (-8\pi C)^{-1}[H(\iiint_{K(X_0,\delta)} F(Y)H\exp(-Cr)dV_Y)] =$$

$$= (-8\pi C)^{-1} H\Big(\iiint_{K(X_0,\delta)} F(Y)(-2Cr^{-1}\exp(-Cr))dV_Y\Big) =$$

$$= (4\pi)^{-1}[\Delta(\iiint_{K(X_0,\delta)} F(Y)r^{-1}\exp(-Cr)dV_Y)-$$

$$- C^2 \iiint_{K(X_0,\delta)} F(Y)r^{-1}\exp(-Cr)dV_Y] = (4\pi)^{-1}\big(\Delta P_1^1(X) - C^2 P_1^1(X)\big)$$

From lemma 4 we have

$$\Delta P_1^1(X) \to 4\pi F(X_0) \quad \text{as} \quad X \to X_0$$

and by lemma 2 we get

$$|P_1^1(X)| \to 0 \quad \text{as} \quad X \to X_0$$

Thus

$$H^2 I_0(X) = (4\pi)^{-1}\big(\Delta P_1^1(X) - C^2 P_1^1(X)\big) \to F(X_0) \quad \text{as} \quad X \to X_0$$

References

[1] M. Krzyżański: *Partial differential equations of second order*. Vol. I, Warszawa 1972.