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Asymptotical set stability for a functional equation of first order

1. Introduction. In this paper we investigate several notions of asymptotical set stability for the functional equation of iterative type

(1) $$\varphi[f(x)] = g(x, \varphi(x)).$$

The given functions f and g will be assumed to fulfil the following hypothesis:

(H_1) $f: I \to I$, $g: I \times H \to H$ are continuous in I and $I \times H$ respectively, where $I = (0,b)$, $b \in R$, H is an open and connected subset of a Banach space B. Moreover f is strictly increasing in I and $0 < f(x) < x$ for $x \in I$.

We shall be interested in solutions of (1) which are defined in the interval I and assume values in H.

G.A.Shanholt has proved in [2] stability theorems for a difference equation. Similar results for the equation (1) are presented in [3] by E.Turdza. In this paper we will use some theorems from [3].

2. Preliminaries. We adopt the following notation in this paper.

$R_+ := [0,\infty)$, B is a Banach space with a norm $\| \ \|$. For a set A in B, $d(x,A) := \inf\{\|x - y\| : y \in A\}$ and for any $\varepsilon > 0$, $N(A,\varepsilon) := \{x \in B : d(x,A) < \varepsilon\}$. For a function $\varphi : I \to H$ and a set $G \subset H$ and $\varepsilon > 0$, $d(\varphi, G) < \varepsilon$ denotes that for every x we have $d(\varphi(x), G) < \varepsilon$.

$K := \{\Phi | \Phi : R_+ \to R_+, \Phi$ is strictly increasing, continuous function, $\Phi(0) = 0\}$, $I_0 := [f(x_0), x_0]$ for $x_0 \in I$, $\varphi_0 : I_0 \to H$ will denote a continuous function such that $\varphi_0[f(x_0)] = g(x_0, y_0)$, where y_0 is an arbitrary point of the set H and $\varphi_0(x_0) = y_0$. Finally, $\varphi(x, x_0, y_0, \varphi_0)$ will denote the unique continuous solution of equation (1) defined on $(0, x_0]$ and such that $\varphi|_{I_0} = \varphi_0$.

In the sequel we will assume the following:

For a closed set $G \subset H$ and $\alpha > 0$ such that $N(G,\alpha) \subset H$

(H_2) there exists a $k > 0$ such that $\varepsilon \in (0,\alpha)$, $y_0 \in B$, $d(y_0, G) < \varepsilon$ imply $d(g(x,y_0), G) < k\varepsilon$.

R e m a r k 1. Observe that under the hypothesis (H_1) for given $x_0 \in I$, $y_0 \in H$ a solution $\varphi(x, x_0, y_0, \varphi_0)$ exists.

R e m a r k 2. If hypothesis (H_2) is satisfied and G is a connected set then $N(G,\varepsilon)$ is arcwise connected set, thus we may take φ_0 in $N(G,\varepsilon)$ and for such a φ_0 there exists $\varphi(x, x_0, y_0, \varphi_0)$ (see [3]).

Now we will adopt the following definitions of stability for equation (1) (see [3]).

DEFINITION 1. Let $G \subset H$ be a closed subset of H. We say:

(i) G is stable if for every $x_o \in I$ and $\varepsilon > o$ there exists a $\delta = \delta(x_o, \varepsilon) > o$ such that $d(\varphi_o, G) < \delta$ implies that $\varphi(x, x_o, y_o, \varphi_o)$ exists and $d(\varphi(x, x_o, y_o, \varphi_o), G) < \varepsilon$;

(ii) G is uniformly stable if it is stable and δ in (i) is independent of x_o;

(iii) G is asymptotically stable if it is stable and if for every $x_o \in I$ there exists $\eta = \eta(x_o) > o$ such that $d(\varphi_o, G) < \eta$ implies $\lim_{x \to o} d(\varphi(x, x_o, y_o, \varphi_o), G) = o$;

(iv) G is uniformly asymptotically stable if it is uniformly stable, and η in (iii) is independent of x_o and the limit is uniform in x_o, y_o, $\varphi_o(t)$ $(t \in I_o)$ for $(x_o, y_o, \varphi_o(t)) \in I \times N(G, \eta) \times N(G, \eta)$.

DEFINITION 2. Let $V: I \times N(G, \alpha) \to R_+$. We say that:

(i) V is positive definite with respect to the set G if there exists a $\Phi \in K$ such that $\Phi(d(y, G)) \leqslant V(x, y)$ for $(x, y) \in I \times N(G, \alpha)$;

(ii) V is decrescent with respect to the set G if there exists a $\psi \in K$ such that $\psi(d(y, G)) \geqslant V(x, y)$ for $(x, y) \in I \times N(G, \alpha)$;

(iii) V satisfies property (B) with respect to the set G if for each $\varepsilon > o$ and $x_o \in I$ there exists a $\delta(x_o, \varepsilon) \in (o, \alpha)$ such that $d(y, G) < \delta$ implies

$V(x,y) < \varepsilon$ for $x \in I_0$.

(iv) V is a Lyapunov function for equation (1) on $N(G,\alpha)$ if it satisfies property (B) with respect to G and $\Delta V(x,y) \leqslant 0$, where

$\Delta V(x,y) := V(f(x), g(x,y)) - V(x,y)$ for $(x,y) \in I \times [N(G,\alpha) \cap H]$.

DEFINITION 3. A Lyapunov function V for equation (1) on $N(G,\alpha)$ has a strongly negative difference along solutions of (1) if there exists a $\beta > 0$ such that

(2) $\Delta V(x,y) \leqslant -\beta \| g(x,y) - y \|$ for $(x,y) \in I \times [N(G,\alpha) \cap H]$.

The following theorems from [3] will be usefull in the sequel.

THEOREM 1. Let G be a closed and connected subset of H with $N(G,\alpha) \subset H$ for a $\alpha > 0$. If hypothesis (H_1) and (H_2) are satisfied and if there exists a Lyapunov function V for (1) on $N(G,\alpha)$ and it has strongly negative difference along solutions of (1), then G is stable. Moreover, for each $x_0 \in I$ there exists a $\vartheta > 0$ such that for $y_0 \in N(G,\vartheta)$ the solution $\varphi(x,x_0,y_0,\varphi_0)$ where $d(\varphi_0,G) < \vartheta$, is bounded.

THEOREM 2. Under assumptions of Theorem 1 and if moreover V is decrescent with respect to the set G, then G is uniformly stable.

Now we shall present a theorem about a linear functional inequality. This result is similar to theorem 2.8 from [1].

We start with following lemma:

LEMMA 1. Let $f: I \to R_+$ be a continuous function such that $0 < f(x) < x$ for $x \in I$, and let functions $g: I \to R$, $F: I \to R$ be bounded in I. Further, let $\varphi : I \to R_+$ be a solution of inequality

(3) $$\varphi[f(x)] \leqslant g(x)\varphi(x) + F(x).$$

Then

$$|\varphi[f^n(x)]| \leqslant M(x)\frac{1 - L^n}{1 - L} + L^n|\varphi(x)| \text{ for } x \in I \text{ and } n = 1,2\ldots$$

where

(4) $$M(x) := \sup_{t \in I_x}|F(t)|, \quad L := \sup_{t \in I}|g(t)| < 1 \text{ and } I_x = (0,x].$$

The inductive proof of this lemma is very simple (see Lemma 2.1 in [1]).

THEOREM 3. Let $f: I \to R_+$ be a continuous function such that $0 < f(x) < x$ for $x \in I$ and suppose that the function $F: I \to R$ fulfils the condition

(5) $$\lim_{x \to 0^+} F(x) = 0.$$

Suppose further that for $g: I \to R$ there exist $\delta > 0$ and $\vartheta \in (0,1)$ such that

(6) $$|g(x)| < \vartheta \quad \text{for} \quad x \in (0,\delta) \cap I.$$

Then every solution $\varphi : I \to R_+$ of inequality (3) in I which is bounded in a neighbourhood of zero fulfils the condition

(7) $$\lim_{x \to 0^+} \varphi(x) = 0.$$

P r o o f. We may assume that δ in (6) is chosen in such a manner that $\delta \in I$ and F and φ are bounded in $(0,\delta)$. Thus

(8) $$|\varphi(x)| \leqslant C \quad \text{for} \quad x \in (0, \delta).$$

We have by (5), for the function (4), $\lim_{x \to 0^+} M(x) = 0$. Consequently, given an $\varepsilon > 0$ we can find a δ_1, $0 < \delta_1 < \delta$, such that

(9) $$M(x) < \frac{1}{2}(1 - \vartheta)\varepsilon \quad \text{for} \quad x \in (0, \delta_1).$$

Further we can find an index N such that

(10) $$\vartheta^N < \frac{\varepsilon}{2C}.$$

We put $m(x) := \sup_{t \in (0,x]} f(t)$. Then $0 < m(x) < x$ and m is monotonic function. Set $\delta_2 = m^N(\delta_1)$. Since for every n, $f^n((0, \delta_1)) \supset (0, m^n(\delta_1))$ we have in particular $f^N((0, \delta_1)) \supset (0, \delta_2)$. Consequently for every $x \in (0, \delta_2)$ there exists an $x^* \in (0, \delta_1)$ such that $f^N(x^*) = x$. Hence by Lemma 1 and by (8), (9) and (10) we have for $x \in (0, \delta_2)$

$$|\varphi(x)| = |\varphi[f^N(x^*)]| < M(x^*)\frac{1}{1 - \vartheta} + \vartheta^N |\varphi(x^*)| < \varepsilon ,$$

which proves relation (7).

3. Sufficient conditions for asymptotical set stability

In this section we are going to present some theorems about asymptotical set stability for equation (1).

We will assume the following hypothesis:

(H_3) The function $g: I \times H \to H$ fulfils Lipschitz condition with constant $L \in (0,1)$ in $I \times H$ i.e.

$\|g(x,y_1) - g(x,y_2)\| \leq L\|y_1 - y_2\|$ for $x \in I$, $y_1, y_2 \in H$.

(H_4) The set $Z: = \{\lambda \in B: \lim_{x \to 0^+} g(x,\lambda) = \lambda\}$ is not empty.

THEOREM 4. Let G be a closed and connected subset of H with $N(G,\alpha) \subset H$ for an $\alpha > 0$. Suppose that hypothesis (H_1), (H_2), (H_3), (H_4) are fulfilled and there exists a Lyapunov function V for (1) on $N(G,\alpha)$ and it has strongly negative difference along solutions of (1).

Moreover, assume that for each $x_0 \in I$ there is a $\xi(x_0) \in (0,\alpha)$ such that $d(\varphi_0,G) < \xi$ implies
$\liminf_{n\to\infty} d(\varphi(f^n(x_0),x_0,y_0,\varphi_0),G) = 0.$

Then G is asymptotically stable.

P r o o f. According to Theorem 1 G is stable. For any $x_0 \in I$ choose $\eta(x_0) := \min[\xi(x_0), \gamma(x_0), \delta(x_0,\xi(x_0))]$ where δ satisfies part (i) of Definition 1 and γ is as in the last sentence of Theorem 1.

First we shall prove that $d(\varphi_0,G) < \eta$ implies $\lim_{n\to\infty} d(\varphi(f^n(x_0),x_0,y_0,\varphi_0),G) = 0$. Suppose this is false, that is exists $\varphi_0: I_0 \to H$ such that

$$d(\varphi_0,G) < \eta \text{ and } \limsup_{n\to\infty} d(\varphi(f^n(x_0),x_0,y_0,\varphi_0),G) \neq 0.$$

Consequently, there is an $\varepsilon_0 > 0$ and sequences $\{m_i\}$ and

$\{k_i\}$, $m_i \to \infty$, $k_i \to \infty$, $k_{i+1} > m_i > k_i$ for $i = 1,2,\dots,$ such that

(11) $$d(\varphi(f^{k_i}(x_o), x_o, y_o, \varphi_o), G) < \frac{\varepsilon_o}{2},$$

and

(12) $$d(\varphi(f^{m_i}(x_o), x_o, y_o, \varphi_o), G) \geq \varepsilon_o.$$

From assumption $o < f(x) < x$ for $x \in I$ we have

$$\lim_{n\to\infty} f^n(x_o) = o.$$

Define the integer valued function l by

$$l(n) = j \quad \text{whenever} \quad f^{m_{j+1}}(x_o) < f^n(x_o) \leq f^{m_j}(x_o) \quad \text{for}$$

$n \geq m_1$.

Since $d(\varphi(f^i(x_o), x_o, y_o, \varphi_o), G) < \alpha$, puting in (2) $y := \varphi(f^{i-1}(x_o))$ and $x := f^{i-1}(x_o)$ for $i = 1,2,\dots,n$ we obtain for the solution $\varphi(x) = \varphi(x, x_o, y_o, \varphi_o)$ the following inequalities

(13) $$V(f^i(x_o), \varphi(f^i(x_o))) - V(f^{i-1}(x_o), \varphi(f^{i-1}(x_o))) \leq -\beta\|\varphi(f^i(x_o)) - \varphi(f^{i-1}(x_o))\| \quad \text{for } i=1,2,\dots,n.$$

Then we have form (13)

$$V(f^n(x_o), \varphi(f^n(x_o))) - V(x_o, \varphi(x_o)) \leq -\beta \sum_{i=1}^{n} \|\varphi(f^i(x_o)) - \varphi(f^{i-1}(x_o))\|.$$

Combining this with the inequalities

$$\sum_{i=1}^{l(n)} \|\varphi(f^{m_i}(x_o)) - \varphi(f^{k_i}(x_o))\| \leq \sum_{i=1}^{n} \|\varphi(f^i(x_o)) - \varphi(f^{i-1}(x_o))\|,$$

$$|d(\varphi(f^{k_i}(x_o)), G) - d(\varphi(f^{m_i}(x_o)), G)| \leq \|\varphi(f^{k_i}(x_o)) - \varphi(f^{m_i}(x_o))\|,$$

and (11), (12) we arrive at the estimate

(14) $$V(f^n(x_o), \varphi(f^n(x_o))) \leq V(x_o, \varphi(x_o)) +$$

$$+\beta\sum_{i=1}^{l(n)}\left[d(\varphi(f^{k_i}(x_o)),G)-d(\varphi(f^{m_i}(x_o)),G)\right] \leqslant V(x_o,\varphi(x_o)) - \frac{\beta\varepsilon_o l(n)}{2}$$

for $n > m_1$.

Since $l(n)\to\infty$ as $n\to\infty$, (14) contradicts $V \geqslant o$.

Thus we have proved that

$$\lim_{n\to\infty} d(\varphi(f^n(x_o),x_o,y_o,\varphi_o),G) = o. \tag{15}$$

Let $\varphi_o: I_o \to H$ be such that $d(\varphi_o,G) < \eta$ and let $\lambda\in Z$.

Define the functions k, F and $\varkappa$ by

$$k(x)=\begin{cases}\dfrac{\|g(x,\varphi(x)) - g(x,\lambda)\|}{\|\varphi(x) - \lambda\|}, & \varphi(x)\neq\lambda, \quad F(x):= \|g(x,\lambda)-\lambda\|,\\ 0 & ,\ \varphi(x)=\lambda, \quad \varkappa(x):=\|\varphi(x) -\lambda\|, \text{ for } x\in I\end{cases} \tag{16}$$

From the definition of λ we have

$$\lim_{x\to o^+} F(x) = o \quad \text{and} \quad |k(x)| \leqslant L.$$

We have also the linear inequality

$$\varkappa[f(x)] \leqslant k(x)\,\varkappa(x) + F(x)$$

and from Theorem 1 the function $\varkappa$ is bounded. Consequently, Theorem 3 implies

$$\lim_{x\to o^+}\varkappa(x) = o \quad \text{i.e.} \quad \lim_{x\to o^+}\varphi(x) = \lambda. \tag{17}$$

Let $x_n\to o^+$. From inequality

$$d(\varphi(x_n,x_o,y_o,\varphi_o),G) \leqslant d(\varphi(f^n(x_o),x_o,y_o,\varphi_o),G) + \|\varphi(f^n(x_o)) -\lambda\| + \|\varphi(x_n) - \lambda\| \tag{18}$$

and (15), (17) we have $\lim_{n\to\infty} d(\varphi(x_n,x_o,y_o,\varphi_o),G) = o$ which is equivalent to the relation

$$\lim_{x\to o^+} d(\varphi(x,x_o,y_o,\varphi_o),G) = o.$$

The proof is complete.

THEOREM 5. Let G be a closed and connected subset of H with $N(G,\alpha) \subset H$ for an $\alpha > 0$. Suppose that hypothesis (H_1), (H_2), (H_3), (H_4) are fulfilled and there exists a Lyapunov function V for (1) on $N(G,\alpha)$ and it has strongly negative difference along solutions of (1).

Moreover, assume that for every $y \in (B - G) \cap N(G,\alpha)$ there exists a $\varsigma > 0$ and an $h: I \to R_+$ with $\sum_{n=1}^{\infty} h(f^n(x_0)) = \infty$ for $x_0 \in I$ such that $\varphi(f^n(x_0)) \in (B-G) \cap N(G,\alpha)$ and $\|\varphi(f^n(x_0)) - y\| < \zeta$ implies $\|\varphi(f^{n+1}(x_0)) - \varphi(f^n(x_0))\| \geqslant h(f^n(x_0))$. Then G is asymptotically stable.

P r o o f. By Theorem 1, G is stable. For any $x_0 \in I$ and a fixed $r \in (0,1)$ define $\eta(x_0) := \delta(x_0, r\alpha)$ where δ satisfies part (i) of Definition 1. We claim that $d(\varphi_0, G) < \eta$ guarantees that $\lim_{n\to\infty} d(\varphi(f^n(x_0), x_0, y_0, \varphi_0), G) = 0$. Suppose that this claim is false, that is, there is a φ_0 such that $d(\varphi_0, G) < \eta$ and $d(\varphi(f^n(x_0)), G) \not\to 0$, $n \to \infty$. Since φ is bounded, there is an increasing sequence $\{n_i\}$, $n_i \to \infty$ as $i \to \infty$, and $y \in N(G,\alpha)$ such that

$$\lim_{i\to\infty} d(\varphi(f^{n_i}(x_0)), y) = 0.$$

If $y \in G$ then $\lim_{i\to\infty} d(\varphi(f^{n_i}(x_0)), G) = 0$ and consequently exist sequences $\{k_i\}$ and $\{m_i\}$, $k_i \to \infty$, $m_i \to \infty$, $k_{i+1} > m_i > k_i$, such that inequalities (11) and (12) hold. Proceeding as in the proof of Theorem 4 from inequalities (13) and (14) we arrive at the contradiction $V < 0$.

If $y \in (B - G) \cap N(G,\alpha)$ and $\varphi(f^n(x_0))$ does not converge to y then there is an ε_o in (o,α) and sequences $\{m_i\}$ and $\{k_i\}$, $m_i \to \infty$, $k_i \to \infty$ as $i \to \infty$, $k_{i+1} > m_i > k_i$ for all i, $k_1 > n_o$, such that

$$\|\varphi(f^{k_i}(x_o)) - y\| < \frac{\varepsilon_o}{2} \quad \text{and} \quad \|\varphi(f^{m_i}(x_o)) - y\| \geqslant \varepsilon_o.$$

Proceeding as in the proof of Theorem 4, we arrive also at the contradiction. The remaining possibility is that

$$\lim_{n\to\infty} \|\varphi(f^n(x_o)) - y\| = o.$$

Let $\xi' \in (o,\xi)$ be small enough to ensure that $N(\{y\},\xi') \subset N(G,\alpha)$, and choose m so that $n \geqslant m$ implies $d(\varphi(f^n(x_o)), \{y\}) < \xi'$. By our hypothesis, we see that for $n > m$

$$V(f^n(x_o),\varphi(f^n(x_o))) \leqslant V(f^m(x_o),\varphi(f^m(x_o))) +$$

$$-\beta \sum_{i=m+1}^{n} \|\varphi(f^i(x_o)) - \varphi(f^{i-1}(x_o))\|$$

$$\leqslant V(f^m(x_o), \varphi(f^m(x_o))) - \beta \sum_{i=m+1}^{n} h(f^i(x_o)) .$$

From assumption concerning h this shows that $V(f^n(x_o), \varphi(f^n(x_o))) < o$ for n sufficiently large, and we have a contradiction.

Thus we have proved that

$$\lim_{n\to\infty} d(\varphi(f^n(x_o)), G) = o. \tag{19}$$

Define the functions k, F and $\mathcal{H}$ by (16), where $\lambda \in Z$. Then, making use of Theorem 3, we get (17).

Let $x_n \to o^+$. From inequality (17) and (18), (19) we have $\lim_{n\to\infty} d(\varphi(x_n), G) = o$. This implies $\lim_{n\to\infty} d(\varphi(x), G) = o$, and the proof is complete.

R e m a r k 3. Combining Theorems 2 and 4, and 2 and 5, we obtain two theorems which guarantee uniform asymptotical stability for G.

R e m a r k 4. The assumptions of Theorems 1 and 2 can be weakened. Namely, it is enough to consider a metric space B, closed subset G of H. We may also drop hypothesis (H_2). But then it is possible that equation (1) will have no continuous solution in $N(G,\alpha)$.

References

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