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On the Fatou problem for the iterated Helmholz equation in the half-plane

1. In the paper [1] F.Barański solved the Lauricelli problem for the equation $(\Delta - c^2)^2 u = 0$ in the half-plane. In the present paper we solve the same problem on the half-plane, with more general boundary conditions, namely those stated in terms of signed measures. The similar problems for the biharmonic equation has been investigated in [2] and [3].

2. Notations:

$X = (x,y)$, $Y = (s,t)$, $Z = (\xi, 0)$, $|X - Y|^2 = (x-s)^2 + (y-t)^2$,

$\varsigma^2 = (x-s)^2 + y^2$,

$\mathcal{B}(R)$ – denotes the class of Borel subset of R,

$J(\xi, r) = \{x \in R: |x - \xi| < r\}$,

$S(Z,a) = \{(x,y) \in R^2: |x - \xi| < ay, \ 0 < a = \operatorname{tg}\alpha\}$,

$L(\xi)$ – denotes the collection of intervals J containing a point $\xi \in R$.

Let μ be an extended signed measure defined on $\mathcal{B}(R)$, assuming at most one of the values $+\infty$ and $-\infty$.

We shall denote

$$(\ast) \qquad (D\mu)(\xi) = \lim_{\substack{J \in L(\xi) \\ |J| \to 0}} \frac{\mu(J)}{|J|},$$

where $|J|$ denotes one-dimensional Lebesgue measure of set J.

If the limit $(\ast)$ exists under the condition that J is interval with center at ξ it will be called the symmetric derivative of μ at ξ and will be denoted by $(D_s\mu)(\xi)$.

We shall also denote

$$(D_s^2\mu)(\xi) = \lim_{r \to 0} \frac{\mu(J(\xi, r))}{r^2}$$

and call it the second symmetric derivative of μ at ξ.

We suppose through this paper that μ and η are two signed measures on $\mathcal{B}(R)$ and that

A) μ, η are σ-finite,

B) for aby $\gamma > 0$ $|\mu|(J) = o(e^{\gamma r})$, $|\eta|(J) = O(e^{\gamma r})$ $(r \to \infty)$,

where J denotes the interval included in R (dist $J = 2r$) and $|\mu|$ denotes the absolute variation of the measure μ, i.e.

$$|\mu|(E) = \sup\{\mu(F): F \in \mathcal{B}(R),\ F \subset E,\ \mu(F) \geqslant 0\} + $$
$$+ \sup\{-\mu(F): F \in \mathcal{B}(R),\ F \subset E,\ \mu(F) \leqslant 0\}.$$

The radial limit $R - \lim_{X \to Z} u(X)$ at the point $Z = (\xi, 0)$ of a function u defined on $R_2^+ = \{(x,y) \in R^2:\ |x| < \infty,\ y > 0\}$ will be defined by the formula

$$R - \lim_{X \to Z} u(X) := \lim_{\substack{y \to 0^+ \\ x = \xi}} u(X).$$

If $\lim_{\substack{X \to Z \\ X \in S(Z,a)}} u(X)$ exists and is the same for every $a > 0$, then we denote it by $\Theta\text{-}\lim_{X \to Z} u(X)$ and we call it non-tangential limit at the point Z.

In the paper [1] was proved

THEOREM 1. If the functions f_1, f_2, f_1' are Lebesgue integrable in the interval $(-\infty, \infty)$ and continuous at the point x_0, then the function

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_1(s) y^3 c^2 \varrho^{-2} K_2(c\varrho)\,ds + \frac{1}{\pi}\int_{-\infty}^{\infty} f_2(s) c y^2 \varrho^{-1} K_1(c\varrho)\,ds$$

is a solution of the Lauricelli problem for the equation $(\Delta - c^2)^2\, u(x,y) = 0$ in the half-plane R_2^+; where $K_\nu(z)$ denote the Mac-Donald functions of the order ν [4].

Our aim is to solve the equation $(\Delta - c^2)^2\, u(x,y) = 0$ in the half-plane R_2^+ with the boundary condition

a) $\Theta\text{-}\lim_{X \to Z} u(X) = (D\mu)(\xi)$,

b) $\Theta\text{-}\lim_{X \to Z} \frac{\partial}{\partial y} u(X) = (D\eta)(\xi)$.

The obtained solution will be of the form

$$u(X) = P(\mu, \eta, R)(X) = \frac{1}{\pi}\int_R y^3 c^2 \varrho^{-2} K_2(c\varrho)\,d\mu(s) + \frac{1}{\pi}\int_R c y^2 \varrho^{-1} K_1(c\varrho)\,d\eta(s).$$

Let

$$P(\mu,\eta,A)(X) = \frac{1}{\pi}\int_A y^3 c^2 \varrho^{-2} K_2(c\varrho)\,d\mu(s) + \frac{1}{\pi}\int_A c y^2 \varrho^{-1} K_1(c\varrho)\,d\eta(s),$$

$$P_1(\mu,A)(X) = \frac{1}{\pi}\int_A y^3c^2\varrho^{-2}K_2(c\varrho)\,d\mu(s),$$

$$P_2(\eta,A)(X) = \frac{1}{\pi}\int_A y^2c\varrho^{-1}K_1(c\varrho)\,d\eta(s),$$

$$P_3(\mu,A)(X) = \frac{3}{\pi}\int_A y^2c^2\varrho^{-2}K_2(c\varrho)\,d\mu(s),$$

$$P_4(\mu,A)(X) = -\frac{1}{\pi}\int_A y^4c^3\varrho^{-3}K_3(c\varrho)\,d\mu(s),$$

$$P_5(\eta,A)(X) = \frac{2}{\pi}\int_A yc\varrho^{-1}K_1(c\varrho)\,d\eta(s),$$

$$P_6(\eta,A)(X) = -\frac{1}{\pi}\int_A y^3c^2\varrho^{-2}K_2(c\varrho)\,d\eta(s).$$

Now we shall give

LEMMA 1. Let μ and η be signed measures defined on $\mathcal{B}(R)$, satisfying the conditions A and B such that

$$(D_s\mu)(\xi) = (D_s\eta)(\xi) = 0,$$

then

$$R\text{-}\lim_{X\to Z} P_1(\mu,R)(X) = 0,\quad R\text{-}\lim_{X\to Z} P_i(\eta,R)(X) = 0 \quad \text{for} \quad i=2,5,6.$$

P r o o f. We shall show that

$$R\text{-}\lim_{X\to Z} P_1(\mu,R)(X) = 0.$$

Let ε be an arbitrary positive number. From the assumption $(D_s\mu)(\xi) = 0$ it follows that there exists a positive number $r_0 = r_0(\varepsilon)$, such that

$$(1)\qquad \left|\frac{\mu(J(\xi,r))}{2r}\right| \leqslant \varepsilon \quad \text{for} \quad 0 < r \leqslant r_0.$$

Let us denote

$$F(r) = \int_{J(\xi,r)} d\mu(s) = \mu(J(\xi,r)).$$

Let

$$P_1(\mu,R)(X) = P_1(\mu, \mathcal{J}(\xi, r_0))(X) + P_1(\mu, C\,\mathcal{J}(\xi, r_0))(X),$$

where

$$C\,\mathcal{J}(\xi, r_0) = R - \mathcal{J}(\xi, r_0).$$

We have

$$(2)\quad P_1(\mu, \mathcal{J}(\xi, r_0))(X) = \frac{1}{\pi}\int_0^{r_0} y^3 c^2 (y^2+r^2)^{-1} K_2(c(y^2+r^2)^{1/2})\,d\,F(r) =$$

$$= \frac{1}{\pi}\, y^3 c^2 (y^2 + r^2)^{-1}\, F(r) K_2(c(y^2 + r^2)^{1/2} \Big|_0^{r_0} +$$

$$+ \frac{1}{\pi}\int_0^{r_0} y^3 c^3\; r\, F(r)(y^2 + r^2)^{-3/2} K_3(c(y^2 + r^2)^{1/2})\,dr.$$

Observe that

$$(3)\quad R\text{-}\lim_{X \to Z} \frac{1}{\pi}\, y^3 c^2 (y^2 + r^2)^{-1} F(r) K_2(c(y^2 + r^2)^{1/2})\Big|_0^{r_0} = 0.$$

By (1) and by the formulas ([8] p.276), ([4] p.146,117)

$$\int_0^{\infty} \frac{K_\nu(\alpha\,(x^2+z^2)^{1/2})}{(x^2 + z^2)^{\nu/2}}\, x^{2q+1}\, dx = \frac{2^q\,\Gamma(q+1)}{\alpha^{q+1}\, z^{\nu-q-1}}\, K_{\nu-q-1}(\alpha z)$$

$(\alpha > 0,\ q > -1)$; $K_\nu(x) \approx 2^{\nu-1}\,\Gamma(\nu)\, x^{-\nu}$; $K_{-\nu}(x) = K_\nu(x)$,

$x \to 0,\quad x > 0,\quad \nu > 0$

we obtain the following estimate

$$\left| \frac{1}{\pi}\int_0^{r_0} y^3 c^3\; r\, F(r)(y^2+r^2)^{-3/2}\, K_3(c(y^2+r^2)^{1/2})\,dr \right| \leqslant$$

$$\leqslant \frac{1}{\pi} c^3 y^3 2\varepsilon \int_0^{\infty} r^2 (y^2+r^2)^{-3/2}\, K_3(c(y^2+r^2)^{1/2})\,dr =$$

$$= \frac{1}{\pi}\, c^{3/2}\, \varepsilon \sqrt{8}\, \Gamma\!\left(\tfrac{3}{2}\right) y^{3/2}\, K_{3/2}(cy) \leqslant M\varepsilon\,,$$

where M is a positive constant.

By (2) and (3) we get

$$(4) \qquad \underset{X \to Z}{\text{R-lim}}\ P_1(\mu, \mathfrak{I}(\xi, r_o))(X) = 0.$$

Now we shall prove that

$$\underset{X \to Z}{\text{R-lim}}\ P_1(\mu, C\,\mathfrak{I}(\xi, r_o))(X) = 0.$$

The function $\varrho \to \varrho^{-2} K_2(c\varrho)$ is decreasing in $(0, \infty)$ and

$$K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \quad (x \to \infty) \quad ([4]\ \text{p.}146),$$

whence

$$\left| \frac{1}{\pi} \int_{C\,\mathfrak{I}(\xi, r_o)} y^3 c^2 \varrho^{-2} K_2(c\varrho)\, d\mu(s) \right| =$$

$$= \frac{1}{\pi} \int_{r_0}^{\infty} y^3 c^2 (y^2+r^2)^{-1} K_2(c(y^2+r^2)^{1/2})\, dF(r) \leqslant$$

$$\leqslant \frac{1}{\pi} \int_{r_0}^{\infty} y^2 c^2 r^{-2} K_2(cr)\, d|F(r)| = \frac{1}{\pi} y^3 c^2 |F(r)|\, r^{-2} K_2(cr) \Big|_{r_0}^{\infty} +$$

$$+ \frac{1}{\pi} \int_{r_0}^{\infty} y^3 c^3 |F(r)|\, r^{-2} K_3(cr)\, dr \leqslant$$

$$\leqslant \frac{1}{\pi} y^3 c^2 |F(r)|\, r^{-2} K_2(cr) \Big|_{r_0}^{\infty} + \frac{1}{\pi} M_2 \int_{r_0}^{\infty} y^3 c^2 e^{\delta r} e^{-cr} r^{-5/2}\, dr \leqslant$$

$$\leqslant M_3\, y^3,$$

where M_2, M_3 are positive constants and $0 < \delta \leqslant c$.

This implies

$$(5) \qquad \underset{X \to Z}{\text{R-lim}}\ P_1(\mu, C\,\mathfrak{I}(\xi, r_o))(X) = 0.$$

From (4) and (5) it follows that

$$\underset{X \to Z}{\text{R-lim}}\ P_1(\mu, R)(X) = 0.$$

Similarly it can be proved that

$$\underset{X \to Z}{\text{R-lim}}\ P_i(\eta, R)(X) = 0 \quad \text{for} \quad i = 2,5,6,$$

which concludes the proof of Lemma 1.

Similarly we can prove

LEMMA 2. Let μ be a signed measure defined on $\mathcal{B}(R)$ satisfying assumptions A and B and $(D_s^2\mu)(\xi) = 0$, then

$$R\text{-}\lim_{X \to Z} P_i(\mu, R)(X) = 0 \quad \text{for} \quad i = 3,4.$$

Next we shall prove

LEMMA 3. Let μ and η be one-sign measures defined on $\mathcal{B}(R)$ satisfying assumptions A and B and let

$$(D_s\mu)(\xi) = (D_s\eta)(\xi) = 0.$$

Under these assumptions we obtain

1^0 $\Theta\text{-}\lim_{X \to Z} P_1(\mu,R)(X) = 0$, $\Theta\text{-}\lim_{X \to Z} P_i(\eta,R)(X) = 0$, for $i = 2,5,6$.

Moreover, if we also suppose that $(D_s^2|\mu|)(\xi) = 0$, then

2^0 $\Theta\text{-}\lim_{X \to Z} P_i(\mu,R)(X) = 0$ for $i = 3,4$.

P r o o f. We shall carry out the proof on the example of integral $P_1(\mu,R)(X)$.

Let measure μ be non-negative. Let us denote the axis of the cone $S(Z,a)$ by p and let w denote projection of the point X on the line p. We have

$$\frac{|w - X|}{|w - Z|} \leqslant a \quad \text{for} \quad X \in S(Z,a).$$

Hence

$$|w - Y| \leqslant |w - X| + |X - Y| \leqslant a|w - Z| + |X - Y| \leqslant (a+1)|X - Y|,$$

where $Y = (s,0)$.

Therefore, for every $X \in S(Z,a)$ and $Y = (s,0)$, we have

$$\varrho' = |w - Y| \leqslant (a+1)\varrho .$$

Because, $(c\varrho)^{-\alpha} K_\alpha(c\varrho)$ are decreasing functions for $\alpha > 0$, then

$$(6) \quad (c\varrho)^{-\alpha} K_\alpha(c\varrho) \leqslant \left(\frac{1}{a+1} c\varrho'\right)^{-\alpha} K_\alpha\left(\frac{1}{a+1} c\varrho'\right).$$

Hence by Lemma 1 we have

$$0 \leqslant \theta\text{-}\lim_{X \to Z} P_1(\mu, R)(X) \leqslant$$

$$\leqslant \theta\text{-}\lim_{X \to Z} \frac{1}{\pi} \int_R y^3 c^4 \left(c \frac{1}{a+1} \varrho'\right)^{-2} K_2\left(c \frac{1}{a+1}\varrho'\right) d\mu(s) =$$

$$= R - \lim_{w \to \xi} \frac{1}{\pi} \int_R y^3 c^4 \left(c \frac{1}{a+1} \varrho'\right)^{-2} K_2\left(c \frac{1}{a+1}\varrho'\right) d\mu(s) = 0.$$

For non-positive measure μ the proof is similar.

Similarly, by inequality (6) and by Lemmas 1, 2, we can prove that

$$\theta\text{-}\lim_{X \to Z} P_i(\mu, R)(X) = 0, \ (i=3,4), \ \theta\text{-}\lim_{X \to Z} P_i(\eta, R)(X) = 0 \ (i=2,5,6),$$

which ends the proof of Lemma 3.

We say that measure μ defined on $\mathcal{B}(R)$ satisfies the condition D on a point $\xi \in R$ if and only if there exists a function f defined on R such that

1^0 f is differentiable on R and its derivative is continuous at the point $\xi \in R$,

2^0 $f(s) = O(e^{\delta s})$ for every $\gamma > 0$, $s \to \infty$,

3^0 the measure

$$\lambda(B) := \mu(B) - \int_B f(s)ds , \quad B \in \mathcal{B}(R), \ |\mu(B)| < \infty$$

satisfies the condition $(D_s^2 |\lambda|)(\xi) = 0$.

We shall need

LEMMA 4. We assume that

1° the measure μ satisfies the condition D, at the point $\xi \in R$,

2° there exists the bounded symmetric derivative

$$(D_s \eta)(\xi) = b,$$

3° the measures μ and η satisfy assumption A and B,

4° $(D_s|\delta|)(\xi) = 0$, where $\delta(B) := \eta(B) - b\int_B ds$, $|\eta(B)| < \infty$.

Under these assumptions we obtain

$$\Theta\text{-}\lim_{X \to Z} \frac{\partial}{\partial y} P(\mu.\eta.R)(X) = b.$$

P r o o f. Observe that

$$\frac{\partial}{\partial y} P(\mu,\eta,R)(X) = P_3(\mu,R)(X) + P_4(\mu,R)(X) + P_5(\eta,R)(X) + P_6(\eta,R)(X).$$

Let f be function given by the condition D. Let us denote

$$I := \Theta\text{-}\lim_{X \to Z} \frac{\partial}{\partial y} P(\mu,\eta,R)(X) - b,$$

We shall prove that. $I = 0$.

Similarly as in [1] we can prove that

$$b = \lim_{X \to Z} \frac{\partial}{\partial y} P(f,b,R)(X),$$

where

$$P(f,b,R)(X) = \frac{1}{\pi}\int_R f(s) y^3 c^2 \varrho^{-2} K_2(c\varrho) ds + \frac{1}{\pi}\int_R bcy^2 \varrho^{-1} K_1(c\varrho) ds.$$

Hence

$$I = \Theta - \lim_{X \to Z} \frac{\partial}{\partial y} P(\mu,\eta,R)(X) - \Theta - \lim_{X \to Z} \frac{\partial}{\partial y} P(f,b,R)(X).$$

Let

$$I = \sum_{i=3}^{6} I_i,$$

where

$$I_3 = \theta - \lim_{X \to Z} \left(P_3(\mu,R)(X) - \frac{3}{\pi} \int_R f(s) y^2 c^2 \varrho^{-2} K_2(c\varrho) ds \right),$$

$$I_4 = -\theta - \lim_{X \to Z} \left(P_4(\mu,R)(X) - \frac{1}{\pi} \int_R f(s) y^4 c^3 \varrho^{-3} K_3(c\varrho) ds \right),$$

$$I_5 = \theta - \lim_{X \to Z} \left(P_5(\eta,R)(X) - \frac{2}{\pi} \int_R bcy \varrho^{-1} K_1(c\varrho) ds \right),$$

$$I_6 = -\theta - \lim_{X \to Z} \left(P_6(\eta,R)(X) - \frac{1}{\pi} \int_R bc^2 y^3 \varrho^{-2} K_2(c\varrho) ds \right).$$

Let r_0 denote a positive number and let

$$I_1 = \theta - \lim_{X \to Z} \frac{\partial}{\partial y} P(\lambda, \delta, J(\xi, r_0))(X),$$

$$I_2 = \theta - \lim_{X \to Z} \frac{\partial}{\partial y} P(\mu, \eta, C J(\xi, r_0))(X) - \frac{\partial}{\partial y} P(f, b, C J(\xi, r_0))(X).$$

Observe that

$$I_1 + I_2 = I.$$

Because the measures $|\lambda|$ and $|\delta|$ satisfy the assumption Lemma 3,

$$I_1 = 0.$$

The proof of the fact, that $I_2 = 0$ is similar to the proof of Lemma 1.

We need the following

LEMMA 5. ([2], [7] Theorem 1, Chapter VII).

If f is integrable with respect to the Lebesgue measure on R, then

a) $\Theta - \lim\limits_{X \to Z} \frac{y}{\pi} \int_R \frac{f(s)}{(x-s)^2 + y^2}\, ds = f(\xi) \quad \text{(a.e.)}^{*},$

b) $\Theta - \lim\limits_{X \to Z} \frac{2}{\pi} y^3 \int_R \frac{f(s)}{((x-s)^2 + y^2)^2}\, ds = f(\xi) \quad \text{(a.e.)}.$

We can write ([4] p.117)

$$c\varrho^{-1} K_1(c\varrho) = \varrho^{-2} + O(1),$$
$$c^2 \varrho^{-2} K_2(c\varrho) = 2\varrho^{-4} + O(1).$$

Hence we obtain the following

LEMMA 6. If f is integrable with respect to the Lebesgue measure on R, then

a) $\Theta - \lim\limits_{X \to Z} \frac{1}{\pi} \int_R f(s) c y \varrho^{-1} K_1(c\varrho)\, ds = f(\xi) \quad$ a.e. ,

b) $\Theta - \lim\limits_{X \to Z} \frac{1}{\pi} \int_R f(s) c^2 y^3 \varrho^{-2} K_2(c\varrho)\, ds = f(\xi) \quad$ a.e. .

We shall state our result in

THEOREM 2. Let μ and η be signed measures defined on $\mathcal{B}(R)$ and satisfying assumptions A and B. Let the measure μ satisfy the condition D, then

a) $\Theta - \lim\limits_{X \to Z} P(\mu, \eta, R)(X) = (D\mu)(\xi) \quad$ (a.e.),

b) $\Theta - \lim\limits_{X \to Z} \frac{\partial}{\partial y} P(\mu, \eta, R)(X) = (D\eta)(\xi) \quad$ (a.e.).

P r o o f. From Lebesgue theorem on decomposition (see [5] p.215-217) we have one

$$\mu = \mu_a + \mu_s, \quad \eta = \eta_a + \eta_s,$$

into absolutely continuous post and the singular post.

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* Further on instead of "a.e. with respect to the Lebesgue measure on R" we shall write shortly (a.e.).

We obtain

$$P(\mu,\eta,R)(X) = P_1(\mu_a,R)(X) + P_1(\mu_s,R)(X) + P_2(\eta_a,R)(X) + P_2(\eta_s,R)(X).$$

It is a well-known fact (see [6] p.155), that

$$(D\mu_s)(\xi) = (D\eta_s)(\xi) = 0 \quad (\text{a.e.}).$$

Hence $(D_s\mu_s)(\xi) = (D_s\eta_s)(\xi) = 0$ (a.e.), so from Lemma 3 it follows that

(7) $$\Theta - \lim_{X\to Z} P(\mu_s, \eta_s, R)(X) = 0 \ (\text{a.e.}).$$

From Radon-Nikodym theorem ([5] p.209) there follows the existence of the functions f and g defined and integrable on R such that for every $B \in \mathcal{B}(R)$ we have

$$\mu_a(B) = \int_B f(s)\, ds, \quad \eta_a(B) = \int_B g(s)\, ds.$$

It is well-known that

$$(D\mu_a)(\xi) = f(\xi) \quad (\text{a.e.}).$$

Hence from Lemma 6 we have

(8) $$\Theta - \lim_{X\to Z} P(\mu_a,\eta_a,R)(X) = \Theta - \lim_{X\to Z} P(f,g,R)(X) = f(\xi) \quad (\text{a.e.}).$$

By (7) and (8) we get the first part of Theorem 2.

Now we shall prove the second part.

Since

$$(D\eta_s^+)(\xi) = (D\eta_s^-)(\xi) = 0 \quad \text{a.e.}$$

and from Lemma 4, we have

$$\Theta - \lim_{X\to Z} P_5(\eta_s^+,R)(X) = \Theta - \lim_{X\to Z} P_5(\eta_s^-,R)(X) =$$
$$= \Theta - \lim_{X\to Z} P_6(\eta_s^+,R)(X) = \Theta - \lim_{X\to Z} P_6(\eta_s^-,R)(X) = 0 \quad (\text{a.e.}).$$

Let ξ be a point such that $(D\eta_a)(\xi) = g(\xi)$ (it holds true almost everywhere).

From Lemma 6 we get

$$\Theta - \lim_{X\to Z} P_5(\eta_a, R)(X) + \Theta - \lim_{X\to Z} P_6(\eta_a, R)(X) =$$

$$= \Theta - \lim_{X\to Z} \frac{2}{\pi}\int_R cy\varrho^{-1} K_1(c\varrho)\, g(s)\, ds -$$

$$- \Theta - \lim_{X\to Z} \frac{1}{\pi}\int_R c^2y^3\varrho^{-2} K_2(c\varrho) g(s) ds = 2g(\xi) - g(\xi) =$$

$$= g(\xi) = (D\eta_a)(\xi) \quad \text{(a.e.)}.$$

Let us define the measure δ by the formula

$$\delta(B) = \eta(B) - g(\xi)\int_B ds \quad \text{for } B \in \mathcal{B}(R).$$

From the Lebesgue theorem (see [6] p.158) we have

$$D|\delta|(\xi) = 0 \quad \text{(a.e.)}.$$

Hence by Lemma 4 we obtain

$$\Theta - \lim_{X\to Z} \frac{\partial}{\partial y} P(\mu, \eta, R)(X) = g(\xi) \quad \text{(a.e.)}.$$

This ends the proof of the second part of Theorem 2.

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